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A note on the realization of distances within sets in euclidean space

D. G. LARMAN

Dedicated to Professor H. Hadwiger on his seventieth birthday

In 1944 and 1945 H. Hadwiger [1, 2] proved the well known theorem.

THEOREM 1. *Let E^n be covered by $n+1$ closed sets. Then there is one of the sets within which all distances are realized.*

In 1972, D. G. Larman and C. A. Rogers [3] introduced the concept of critical distance and a critical number for a finite configuration and used it to give a considerable improvement of Theorem 1. The principal result of [3] was

THEOREM 2. *If E^n is covered by less than $\frac{1}{6}n(n-1)$ sets then there is a set of the covering within which all distances are realized.*

The purpose of this note is to give a configuration which leads to

THEOREM 3. *If E^n is covered by less than $\frac{1}{178200}(n-1)(n-2)(n-3)$ sets then there is a set of the covering within which all distances are realized.*

A considerable generalization of this configuration leads me to make the conjecture:

CONJECTURE. *If E^n is covered by less than $\frac{1}{3}(\frac{4}{3})^{3n/4}$ sets then there is a set of the covering within which all distances are realized.*

Using the theory of configurations developed in [3], Theorem 3 follows from the following theorem.

THEOREM 4. *Let A be the $\binom{n}{5}$ distinct 5-tuples chosen from n objects $1, \ldots, n$. Let B be a subset of A such that no two 5-tuples in B overlap in exactly two objects. Then the cardinality $|B|$ of B is at most $1485n(n-1)$.*

We require the following three lemmas.

LEMMA 1 (Hilton and Milner). *Let* $A_1, \ldots, A_r$ *be sets, each with* k *distinct elements chosen from the set* $1, 2, \ldots, n$. *Suppose that*

$$A_i \cap A_j \neq \varnothing, \qquad 1 \leq i < j \leq r$$

but that

$$\bigcap_{i=1}^{r} A_i = \varnothing.$$

Then, provided $2k \leq n$,

$$r \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

Proof. See A. J. W. Hilton and E. C. Milner [4].

LEMMA 2. *Let* $A_1, \ldots, A_r$; $B_1, \ldots, B_s$ *be sets, each with* 2 *elements, chosen from the set* $1, \ldots, n$ *such that*

$$A_i \cap B_j \neq \varnothing, \qquad 1 \leq i \leq r, \qquad 1 \leq j \leq s.$$

Then either

$$\min(r, s) \leq 3$$

or

$$\bigcap_{i=1}^{r} A_i \cap \bigcap_{j=1}^{s} B_j \neq \varnothing.$$

Proof. We assume that $\min(r, s) \geq 4$. Suppose first that there are two non-overlapping members of $A_1, \ldots, A_r$, say A_1, A_2. Since each of $B_1, \ldots, B_s$ must meet each of A_1, A_2, A_3; $s \leq 3$. Consequently, every two members of $A_1, \ldots, A_r$ overlap and similarly every two members of $B_1, \ldots, B_s$ overlap. So, using lemma 1 with $k = 2$, and noting that $r + s > 3$,

$$\bigcap_{i=1}^{r} A_i \cap \bigcap_{j=1}^{s} B_j \neq \varnothing$$

as required.

LEMMA 3. *Let* 12345 *be a* 5 *tuple and let abcd be four distinct numbers amongst* 12345. *Let* $C(a, b, c)$, $C(a, b, d)$ *be two families of* 5*-tuples, each with at least four members, chosen from the n numbers* $1, \ldots, n$. *If each member of* $C(a, b, c)$ *meets each member of* $C(a, b, d)$ *in at least three numbers and each member of* $C(a, b, c)$, $C(a, b, d)$ *meets* 12345 *in precisely* (a, b, c), (a, b, d) *respectively then there exists* $e \neq 1, 2, 3, 4, 5$ *such that e belongs to each member of* $C(a, b, c) \cup C(a, b, d)$.

Proof. This is an immediate consequence of Lemma 2.

Proof of Theorem 4. Let $\mathbf{b} = b_1b_2b_3b_4b_5$ be a member of B. We shall say that $\mathbf{b}$ is *good* (with respect to B) if there exists a two tuple within $\mathbf{b}$ which is contained in at most 54 members of B. Otherwise $\mathbf{b}$ is *bad* (with respect to B).

The strategy in proving the theorem is to associate every member $\mathbf{b}$ of B with a good member $\phi(\mathbf{b})$ of B in such a way that no good member of B has more than 55 members of B associated with it.

In defining the mapping ϕ it will be enough to suppose that the 5-tuple $\mathbf{a} = 12345$ is a member of B and define $\phi(\mathbf{a})$.

If $\mathbf{a}$ *is good then* $\phi(\mathbf{a}) = \mathbf{a}$. (1)

Otherwise $\mathbf{a}$ is bad.

Suppose first that there are at least 46 5-tuples which overlap $\mathbf{a}$ in 4 numbers. Then there are at least ten 5-tuples which (say) have the numbers 1234 in common with $\mathbf{a}$. We list ten such 5-tuples $1234k$ with k as close to 5 in the ordering of $1, \ldots, n$ as possible. Assume, without loss of generality, that these ten 5-tuples are 12346, 12347, ..., 1234(15).

If one of these 5-tuples is good then we choose one such 5-tuple say $1234k$ to be $\phi(\mathbf{a})$.

The 5*-tuple* $1234k$ *receives at most* $10\binom{5}{4} = 50$

associations in this way. (2)

Otherwise each of

12345, 12346, ..., 1234(15)

are bad. For $5 \leq k \leq 15$, consider the 5-tuple $1234k$. There are at least 54 5-tuples of B which contain the two tuple $4k$. Since each of these 5-tuples must overlap $1234k$, and hence each of

12345, 12346, ..., 1234(15)

in at least three numbers, they each must contain at least two of the numbers 123. So there exists at least 18 of these 5-tuples, forming a set C_1^k and numbers $\alpha(k)$, $\beta(k)$, chosen from 123, such that each member of C_1^k contains $\alpha(k)\beta(k)4k$. We may suppose $\alpha(k)\beta(k)=12$ for four values of k. Similarly, working with the two tuple $3k$, there exists a set C_2^k and numbers $\gamma(k)\delta(k)$, chosen from 124, such that each member of C_2^k contains $\gamma(k)\delta(k)3k$. Consequently there exists two values of k, say 5, 6 with $\alpha(5)\beta(b)=\alpha(6)\beta(6)=12$ and $\gamma(5)\delta(5)=\gamma(6)\delta(6)$.

Suppose, without loss of generality that every member of C_1^k contains $124k$ and every member of C_2^k contains $123k$, $k=5, 6$.

The 4 tuples 1245, 1236 only have two numbers 12 in common. Apart from 12345, 12456, 12356 the members of C_1^5 and C_2^6 contain one number chosen from $7, \ldots, n$. Further for these members the numbers in $7, \ldots, n$ must be the same throughout. Consequently C_1^5 and C_2^6 have cardinality at most 4 which is impossible.

So now we may suppose that there are at most 45 5-tuples of B which overlap **a** in 4 numbers. Since **a** is bad there will exist, for each two tuple ij, $1\leq i<j\leq 5$, at least ten 5 tuples in B which contain ij and which overlap **a** in exactly three numbers.

Therefore, there are at least four such 5-tuples containing the two tuple 12 and a fixed third number of **a**, 3 say. Let C_1 be the set of all 5-tuples in B which meet **a** in exactly 123. Similarly there are at least four such 5-tuples containing the two tuple 45 and a fixed third number, 3 say. Let C_2 be the set of all 5-tuples in B which meet **a** in exactly 345.

Notice that no members of the families C_1, C_2 contain any of the two tuples 14, 15, 24, 25. The two tuple 15 can be accounted for in three different ways i.e. there exists a collection of at least four 5-tuples in B which meet **a** in precisely one of

(i) 125 (ii) 135 (iii) 145.

We analyse these three cases in some detail.

(i) 125. Let C_3 denote all the 5-tuples of B which contain 125 and which overlap **a** in precisely 125. Then C_3 has at least four members. In this case the triples 123 and 125 share two numbers 12 and so, using Lemma 3, there must be another number, 6 say, such that each of the 5-tuples in C_1 and C_3 also contain 6.

No member of C_1, C_2, C_3 contains either of the two tuples 14, 24. The two tuple 14 can be accounted for in three different ways, i.e. there exists a collection C_4(C_5 or C_6) of at least four 5-tuples in B which meet **a** in precisely one of the triples (a) 124, (b) 134, (c) 145.

(a) 124. Applying Lemma 3 to C_1 and C_4, it follows that each member of C_4 contains 6.

(b) 134. Applying Lemma 3 to C_1 and C_5 it follows that each member of C_5 also contains 6. Applying Lemma 3 to C_2 and C_5 it follows that every member of the families C_2 and C_5 must share a common number outside **a**. As C_5 contains at least four members, this number must be 6. Hence every member of C_1 contains 1236 and every member of C_2 contains 3456. So there must exist a member of C_1 and a member of C_2 which meet precisely in two tuple 36, which is impossible. So case (b) cannot arise.

(c) 145. Applying Lemma 3 to C_3 and C_6 it follows that each member of C_6 contains 6. Applying Lemma 3 to C_2 and C_6 it follows that every member of the families C_2 and C_6 must share a common number outside **a**. As C_6 contains at least four members, this number must be 6. So again there must exist a member of C_1 and a member of C_2 which meet precisely in the two tuple 36, which is impossible. So case (c) cannot arise.

(ii) 135. Let C_7 denote all the 5-tuples in B which contain 135 and which overlap **a** in precisely 135. Applying Lemma 3 to C_1 and C_7, it follows that every member of C_1 and C_7 share a common number, 6 say, outside **a**. Applying Lemma 3 to C_2 and C_7 it follows that every member of C_2 and C_7 share a common number outside **a**. As C_7 has at least four members, this number must be 6. So again there must exist a member of C_1 and a member of C_2 which overlap in precisely 36, which is impossible. So case (ii) cannot arise.

(iii) This case is exactly the same as (i) with 1 and 5, 2 and 4 interchanged. Consequently the only possible configuration is as in (i) a. i.e. there exists a number 6 say so that every 5-tuple in C_2 and C_8 contains 6, where C_8 is the family of 5-tuples in B, with at least four members, which meet **a** in precisely 145. Also there exists a family C_9 of 5-tuples in B, with at least four members, each of which contains 6 and meets **a** in precisely 245.

From these considerations it follows that, up to a permutation of the numbers 12345, there is only one possible configuration which can arise, namely that of case (i) *a*.

Hence we may assume that there exists four families C_1, C_2, C_3, C_4 in B, each with at least four members, and each meeting **a** in precisely three numbers. There is also a number, 6 say, such that

1236 belongs to **x** for each **x** in C_1
345 belongs to **x** for each **x** in C_2
1256 belongs to **x** for each **x** in C_3
1246 belongs to **x** for each **x** in C_4.

We also suppose that C_1, C_2, C_3, C_4 are maximal.

We shall show that every member of C_1, C_3 and C_4 is good. Because of the symmetry, it suffices to show that a member of C_1, say 12367 is good. Since C_1 has at least four members, we suppose that the four membered set D

12367, 12368, 12369, 1236(10)

is in C_1.

If another 5-tuple $\mathbf{x}=x_1x_2x_3x_4x_5$ in B contains the two tuple 17 then, because of D, $\mathbf{x}$ must contain at least two of the numbers 236. This yields three cases according to whether $\mathbf{x}$ contains

(a) 23 (b) 26 (c) 36.

(a) Here $\mathbf{x}$ contains 1237 and so, considering the families C_3, C_4, $\mathbf{x}$ must be 12367. So the two tuple 17 is in only one member of B and hence 12367 is good.

(b) Here $\mathbf{x}$ contains 1267 and so, considering $\mathbf{a}$, $\mathbf{x}$ must contain one of 345. So the two tuple 17 is in at most four members of B and hence 12367 is good.

(c) Here $\mathbf{x}$ contains 1367 and so, considering the families C_2, C_4, $\mathbf{x}$ must be 12367. So the two tuple 17 is in only one member of B and hence 12367 is good.

Hence 12367 is good as are all the members of C_1, C_3, C_4. Define $\phi(\mathbf{a})$ to be one of the members of C_1, C_3, C_4, $\phi(\mathbf{a})=12367$ say. This completes the definition of ϕ.

We next look at the number of members of B which could be assigned to 12367 in this way.

Suppose that $\mathbf{b}=b_1b_2b_3b_4b_5$ is such a 5-tuple. Then it may be supposed that $b_1b_2b_3$ are amongst 12367 and that there exists another number b_6 amongst 12367 but different from $b_1b_2b_3b_4b_5$ so that there exists four families D_1, D_2, D_3, D_4, in B, each with at least four members, and each meeting $\mathbf{b}$ in precisely three numbers. Further

$b_1b_2b_3b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_1
$b_3b_4b_5$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_2
$b_1b_2b_4b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_3
$b_1b_2b_4b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_4.

If $\mathbf{b}$ contains only one of 123 then $\mathbf{b}$ contains 67. However, $\mathbf{b}$ would then meet some member of C_1 in exactly two numbers, which is impossible.

If $\mathbf{b}$ contains 123 but not 6 then, using C_3 and C_4, $\mathbf{b}=\mathbf{a}$. If $\mathbf{b}$ contains 1236 then $\mathbf{b}$ is in C_1 and hence $\mathbf{b}$ is good. So $\phi(\mathbf{b})=\mathbf{b}\neq 12367$.

If $\mathbf{b}$ contains 12 but not 3 then, using C_1, $\mathbf{b}$ contains 126. Also, using $\mathbf{a}$, $\mathbf{b}$ contains at least one of 4 and 5. If $\mathbf{b}$ contains 4 and 5 then $\mathbf{b}=12456$. If $\mathbf{b}$ contains 4 but not 5 then $\mathbf{b}$ is in C_4 and if $\mathbf{b}$ contains 5 but not 4 then $\mathbf{b}$ is in C_3. In either case $\mathbf{b}$ is good and so $\phi(\mathbf{b})=\mathbf{b}\neq 12367$.

If **b** contains 13 but not 2 then, using C_1, **b** contains 136. Using C_3, C_4, it follows that $\mathbf{b} = 13456$.

If **b** contains 23 but not 1 then, using C_1, **b** contains 236. Using C_3, C_4, $\mathbf{b} = 23456$.

Consequently at most four 5-tuples of B are associated with 12367 in this manner. (3)

Combining (1), (2) and (3), it follows that for each good 5-tuple **b** of B, $\phi^{-1}(\mathbf{b})$ has at most 55 members. Each good 5-tuple **b** of B contains a two tuple which occurs in at most 54 members of B. Since it is only possible to choose $\binom{n}{2}$ two tuples from the numbers $12\cdots n$ it follows that

$$|B| \le 1485n(n-1) \text{ as required.}$$

Remarks. We may construct a suitable B in Theorem 4 by insisting that each member of B contains 123 and the other two numbers making up the 5-tuple are chosen in $4, \ldots, n$. This yields a set B with $|B| = \frac{1}{2}(n-4)(n-5)$ members which, using Theorem 4, is essentially best possible.

Generalizing this situation, take $4k$ numbers $1, 2, \ldots, 4k$ and consider all $2k$ tuples A chosen from these $4k$ numbers. Consider a subset B of A such that no two members of B overlap in exactly k numbers. It seems reasonable to suppose that B will have as many members as possible when B is constructed by insisting that every member of B contains $12\cdots k+1$ and the other $k-1$ numbers are chosen amongst the numbers $k+2, \ldots, 4k$. If this were so then an application of Stirling's formula would prove the conjecture mentioned in the introduction.

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