A note on the realization of distances within sets in euclidean space

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Dedicated to Professor H. Hadwiger on his seventieth birthday

In 1944 and 1945 H. Hadwiger [1, 2] proved the well known theorem.

THEOREM 1. *Let E^n be covered by $n+1$ closed sets. Then there is one of the sets within which all distances are realized.*

In 1972, D. G. Larman and C. A. Rogers [3] introduced the concept of critical distance and a critical number for a finite configuration and used it to give a considerable improvement of Theorem 1. The principal result of [3] was

THEOREM 2. *If E^n is covered by less than $\frac{1}{6}n(n-1)$ sets then there is a set of the covering within which all distances are realized.*

The purpose of this note is to give a configuration which leads to

THEOREM 3. *If E^n is covered by less than $\frac{1}{178200}(n-1)(n-2)(n-3)$ sets then there is a set of the covering within which all distances are realized.*

A considerable generalization of this configuration leads me to make the conjecture:

CONJECTURE. *If E^n is covered by less than $\frac{1}{3}(\frac{4}{3})^{3n/4}$ sets then there is a set of the covering within which all distances are realized.*

Using the theory of configurations developed in [3], Theorem 3 follows from the following theorem.

THEOREM 4. *Let A be the $\binom{n}{5}$ distinct 5-tuples chosen from n objects $1,\ldots,n$. Let B be a subset of A such that no two 5-tuples in B overlap in exactly two objects. Then the cardinality $|B|$ of B is at most $1485n(n-1)$.*

We require the following three lemmas.

LEMMA 1 (Hilton and Milner). *Let* $A_1, \ldots, A_r$ *be sets, each with* k *distinct elements chosen from the set* $1, 2, \ldots, n$. *Suppose that*

$$A_i \cap A_j \neq \emptyset, \quad 1 \leq i < j \leq r$$

but that

$$\bigcap_{i=1}^{r} A_i = \emptyset.$$

Then, provided $2k \leq n$,

$$r \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

Proof. See A. J. W. Hilton and E. C. Milner [4].

LEMMA 2. *Let* $A_1, \ldots, A_r$; $B_1, \ldots, B_s$ *be sets, each with* 2 *elements, chosen from the set* $1, \ldots, n$ *such that*

$$A_i \cap B_j \neq \emptyset, \quad 1 \leq i \leq r, \quad 1 \leq j \leq s.$$

Then either

$$\min(r, s) \leq 3$$

or

$$\bigcap_{i=1}^{r} A_i \cap \bigcap_{j=1}^{s} B_j \neq \emptyset.$$

Proof. We assume that $\min(r, s) \geq 4$. Suppose first that there are two non-overlapping members of $A_1, \ldots, A_r$, say A_1, A_2. Since each of $B_1, \ldots, B_s$ must meet each of A_1, A_2, A_3; $s \leq 3$. Consequently, every two members of $A_1, \ldots, A_r$ overlap and similarly every two members of $B_1, \ldots, B_s$ overlap. So, using lemma 1 with $k = 2$, and noting that $r + s > 3$,

$$\bigcap_{i=1}^{r} A_i \cap \bigcap_{j=1}^{s} B_j \neq \emptyset$$

as required.

LEMMA 3. *Let* 12345 *be a* 5 *tuple and let abcd be four distinct numbers amongst* 12345. *Let* $C(a, b, c)$, $C(a, b, d)$ *be two families of 5-tuples, each with at least four members, chosen from the n numbers* $1, \ldots, n$. *If each member of* $C(a, b, c)$ *meets each member of* $C(a, b, d)$ *in at least three numbers and each member of* $C(a, b, c)$, $C(a, b, d)$ *meets* 12345 *in precisely* (a, b, c), (a, b, d) *respectively then there exists* $e \neq 1, 2, 3, 4, 5$ *such that e belongs to each member of* $C(a, b, c) \cup C(a, b, d)$.

Proof. This is an immediate consequence of Lemma 2.

Proof of Theorem 4. Let $\mathbf{b} = b_1 b_2 b_3 b_4 b_5$ be a member of B. We shall say that $\mathbf{b}$ is *good* (with respect to B) if there exists a two tuple within $\mathbf{b}$ which is contained in at most 54 members of B. Otherwise $\mathbf{b}$ is *bad* (with respect to B).

The strategy in proving the theorem is to associate every member $\mathbf{b}$ of B with a good member $\phi(\mathbf{b})$ of B in such a way that no good member of B has more than 55 members of B associated with it.

In defining the mapping ϕ it will be enough to suppose that the 5-tuple $\mathbf{a} = 12345$ is a member of B and define $\phi(\mathbf{a})$.

If $\mathbf{a}$ *is good then* $\phi(\mathbf{a}) = \mathbf{a}$. $\qquad\qquad$ (1)

Otherwise $\mathbf{a}$ is bad.

Suppose first that there are at least 46 5-tuples which overlap $\mathbf{a}$ in 4 numbers. Then there are at least ten 5-tuples which (say) have the numbers 1234 in common with $\mathbf{a}$. We list ten such 5-tuples 1234k with k as close to 5 in the ordering of $1, \ldots, n$ as possible. Assume, without loss of generality, that these ten 5-tuples are 12346, 12347, $\ldots$, 1234(15).

If one of these 5-tuples is good then we choose one such 5-tuple say 1234k to be $\phi(\mathbf{a})$.

The 5-tuple 1234k *receives at most* $10\binom{5}{4} = 50$

associations in this way. $\qquad\qquad$ (2)

Otherwise each of

$$12345, 12346, \ldots, 1234(15)$$

are bad. For $5 \leq k \leq 15$, consider the 5-tuple 1234k. There are at least 54 5-tuples of B which contain the two tuple 4k. Since each of these 5-tuples must overlap 1234k, and hence each of

$$12345, 12346, \ldots, 1234(15)$$

in at least three numbers, they each must contain at least two of the numbers 123. So there exists at least 18 of these 5-tuples, forming a set C_1^k and numbers $\alpha(k), \beta(k)$, chosen from 123, such that each member of C_1^k contains $\alpha(k)\beta(k)4k$. We may suppose $\alpha(k)\beta(k) = 12$ for four values of k. Similarly, working with the two tuple $3k$, there exists a set C_2^k and numbers $\gamma(k)\delta(k)$, chosen from 124, such that each member of C_2^k contains $\gamma(k)\delta(k)3k$. Consequently there exists two values of k, say $5, 6$ with $\alpha(5)\beta(b) = \alpha(6)\beta(6) = 12$ and $\gamma(5)\delta(5) = \gamma(6)\delta(6)$.

Suppose, without loss of generality that every member of C_1^k contains $124k$ and every member of C_2^k contains $123k$, $k = 5, 6$.

The 4 tuples $1245, 1236$ only have two numbers 12 in common. Apart from $12345, 12456, 12356$ the members of C_1^5 and C_2^6 contain one number chosen from $7, \ldots, n$. Further for these members the numbers in $7, \ldots, n$ must be the same throughout. Consequently C_1^5 and C_2^6 have cardinality at most 4 which is impossible.

So now we may suppose that there are at most 45 5-tuples of B which overlap **a** in 4 numbers. Since **a** is bad there will exist, for each two tuple ij, $1 \le i < j \le 5$, at least ten 5 tuples in B which contain ij and which overlap **a** in exactly three numbers.

Therefore, there are at least four such 5-tuples containing the two tuple 12 and a fixed third number of **a**, 3 say. Let C_1 be the set of all 5-tuples in B which meet **a** in exactly 123. Similarly there are at least four such 5-tuples containing the two tuple 45 and a fixed third number, 3 say. Let C_2 be the set of all 5-tuples in B which meet **a** in exactly 345.

Notice that no members of the families C_1, C_2 contain any of the two tuples $14, 15, 24, 25$. The two tuple 15 can be accounted for in three different ways i.e. there exists a collection of at least four 5-tuples in B which meet **a** in precisely one of

(i) 125 (ii) 135 (iii) 145.

We analyse these three cases in some detail.

(i) 125. Let C_3 denote all the 5-tuples of B which contain 125 and which overlap **a** in precisely 125. Then C_3 has at least four members. In this case the triples 123 and 125 share two numbers 12 and so, using Lemma 3, there must be another number, 6 say, such that each of the 5-tuples in C_1 and C_3 also contain 6.

No member of C_1, C_2, C_3 contains either of the two tuples $14, 24$. The two tuple 14 can be accounted for in three different ways, i.e. there exists a collection $C_4(C_5$ or $C_6)$ of at least four 5-tuples in B which meet **a** in precisely one of the triples (a) 124, (b) 134, (c) 145.

(a) 124. Applying Lemma 3 to C_1 and C_4, it follows that each member of C_4 contains 6.

(b) 134. Applying Lemma 3 to C_1 and C_5 it follows that each member of C_5 also contains 6. Applying Lemma 3 to C_2 and C_5 it follows that every member of the families C_2 and C_5 must share a common number outside **a**. As C_5 contains at least four members, this number must be 6. Hence every member of C_1 contains 1236 and every member of C_2 contains 3456. So there must exist a member of C_1 and a member of C_2 which meet precisely in two tuple 36, which is impossible. So case (b) cannot arise.

(c) 145. Applying Lemma 3 to C_3 and C_6 it follows that each member of C_6 contains 6. Applying Lemma 3 to C_2 and C_6 it follows that every member of the families C_2 and C_6 must share a common number outside **a**. As C_6 contains at least four members, this number must be 6. So again there must exist a member of C_1 and a member of C_2 which meet precisely in the two tuple 36, which is impossible. So case (c) cannot arise.

(ii) 135. Let C_7 denote all the 5-tuples in B which contain 135 and which overlap **a** in precisely 135. Applying Lemma 3 to C_1 and C_7, it follows that every member of C_1 and C_7 share a common number, 6 say, outside **a**. Applying Lemma 3 to C_2 and C_7 it follows that every member of C_2 and C_7 share a common number outside **a**. As C_7 has at least four members, this number must be 6. So again there must exist a member of C_1 and a member of C_2 which overlap in precisely 36, which is impossible. So case (ii) cannot arise.

(iii) This case is exactly the same as (i) with 1 and 5, 2 and 4 interchanged. Consequently the only possible configuration is as in (i) a. i.e. there exists a number 6 say so that every 5-tuple in C_2 and C_8 contains 6, where C_8 is the family of 5-tuples in B, with at least four members, which meet **a** in precisely 145. Also there exists a family C_9 of 5-tuples in B, with at least four members, each of which contains 6 and meets **a** in precisely 245.

From these considerations it follows that, up to a permutation of the numbers 12345, there is only one possible configuration which can arise, namely that of case (i) *a*.

Hence we may assume that there exists four families C_1, C_2, C_3, C_4 in B, each with at least four members, and each meeting **a** in precisely three numbers. There is also a number, 6 say, such that

1236 belongs to **x** for each **x** in C_1
345 belongs to **x** for each **x** in C_2
1256 belongs to **x** for each **x** in C_3
1246 belongs to **x** for each **x** in C_4.

We also suppose that C_1, C_2, C_3, C_4 are maximal.

We shall show that every member of C_1, C_3 and C_4 is good. Because of the

symmetry, it suffices to show that a member of C_1, say 12367 is good. Since C_1 has at least four members, we suppose that the four membered set D

12367, 12368, 12369, 1236(10)

is in C_1.

If another 5-tuple $\mathbf{x} = x_1x_2x_3x_4x_5$ in B contains the two tuple 17 then, because of D, $\mathbf{x}$ must contain at least two of the numbers 236. This yields three cases according to whether $\mathbf{x}$ contains

(a) 23 (b) 26 (c) 36.

(a) Here $\mathbf{x}$ contains 1237 and so, considering the families C_3, C_4, $\mathbf{x}$ must be 12367. So the two tuple 17 is in only one member of B and hence 12367 is good.

(b) Here $\mathbf{x}$ contains 1267 and so, considering $\mathbf{a}$, $\mathbf{x}$ must contain one of 345. So the two tuple 17 is in at most four members of B and hence 12367 is good.

(c) Here $\mathbf{x}$ contains 1367 and so, considering the families C_2, C_4, $\mathbf{x}$ must be 12367. So the two tuple 17 is in only one member of B and hence 12367 is good.

Hence 12367 is good as are all the members of C_1, C_3, C_4. Define $\phi(\mathbf{a})$ to be one of the members of C_1, C_3, C_4, $\phi(\mathbf{a}) = 12367$ say. This completes the definition of ϕ.

We next look at the number of members of B which could be assigned to 12367 in this way.

Suppose that $\mathbf{b} = b_1b_2b_3b_4b_5$ is such a 5-tuple. Then it may be supposed that $b_1b_2b_3$ are amongst 12367 and that there exists another number b_6 amongst 12367 but different from $b_1b_2b_3b_4b_5$ so that there exists four families D_1, D_2, D_3, D_4, in B, each with at least four members, and each meeting $\mathbf{b}$ in precisely three numbers. Further

$b_1b_2b_3b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_1
$b_3b_4b_5$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_2
$b_1b_2b_4b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_3
$b_1b_2b_4b_6$ belongs to $\mathbf{x}$ for each $\mathbf{x}$ in D_4.

If $\mathbf{b}$ contains only one of 123 then $\mathbf{b}$ contains 67. However, $\mathbf{b}$ would then meet some member of C_1 in exactly two numbers, which is impossible.

If $\mathbf{b}$ contains 123 but not 6 then, using C_3 and C_4, $\mathbf{b} = \mathbf{a}$. If $\mathbf{b}$ contains 1236 then $\mathbf{b}$ is in C_1 and hence $\mathbf{b}$ is good. So $\phi(\mathbf{b}) = \mathbf{b} \neq 12367$.

If $\mathbf{b}$ contains 12 but not 3 then, using C_1, $\mathbf{b}$ contains 126. Also, using $\mathbf{a}$, $\mathbf{b}$ contains at least one of 4 and 5. If $\mathbf{b}$ contains 4 and 5 then $\mathbf{b} = 12456$. If $\mathbf{b}$ contains 4 but not 5 then $\mathbf{b}$ is in C_4 and if $\mathbf{b}$ contains 5 but not 4 then $\mathbf{b}$ is in C_3. In either case $\mathbf{b}$ is good and so $\phi(\mathbf{b}) = \mathbf{b} \neq 12367$.

If **b** contains 13 but not 2 then, using C_1, **b** contains 136. Using C_3, C_4, it follows that **b** = 13456.

If **b** contains 23 but not 1 then, using C_1, **b** contains 236. Using C_3, C_4, **b** = 23456.

Consequently at most four 5-tuples of B are associated
with 12367 in this manner.   (3)

Combining (1), (2) and (3), it follows that for each good 5-tuple **b** of B, $\phi^{-1}(\mathbf{b})$ has at most 55 members. Each good 5-tuple **b** of B contains a two tuple which occurs in at most 54 members of B. Since it is only possible to choose $\binom{n}{2}$ two tuples from the numbers $12 \cdots n$ it follows that

$|B| \le 1485n(n-1)$ as required.

Remarks. We may construct a suitable B in Theorem 4 by insisting that each member of B contains 123 and the other two numbers making up the 5-tuple are chosen in $4, \ldots, n$. This yields a set B with $|B| = \frac{1}{2}(n-4)(n-5)$ members which, using Theorem 4, is essentially best possible.

Generalizing this situation, take $4k$ numbers $1, 2, \ldots, 4k$ and consider all $2k$ tuples A chosen from these $4k$ numbers. Consider a subset B of A such that no two members of B overlap in exactly k numbers. It seems reasonable to suppose that B will have as many members as possible when B is constructed by insisting that every member of B contains $12 \cdots k+1$ and the other $k-1$ numbers are chosen amongst the numbers $k+2, \ldots, 4k$. If this were so then an application of Stirling's formula would prove the conjecture mentioned in the introduction.

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