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# Homotopy splittings involving $G$ and $G/O$

STEWART PRIDDY<sup>1</sup>

## Introduction

In this note we show that in a strong sense  $SG$  and  $G/O$  are factors in the spaces  $QBD_8$  and  $QBO_2$  respectively, where  $D_8$  is the dihedral group of order 8. All spaces (throughout the note) are localized at 2. These results can be thought of as analogous to the theorem of D. S. Khan and the author [KP] which states that  $Q_0S^0$  is a factor in  $QRP^\infty$ . In particular, here, as in [KP], the transfer is used to construct the required splittings. Additional difficulties arise in the present work, however, because the infinite loop space structure of  $SG$  is markedly more complicated than that of  $Q_0S^0$ . Also, in the case of  $G/O$  we must use the Becker-Gottlieb transfer [BG].

To state our results precisely, we recall that  $QS^0 = \lim \Omega^n S^n$  has components  $Q_k S^0$ ,  $k \in \mathbb{Z}$ , and that  $SG = Q_1 S^0$ . We shall denote by  $*$  and  $\#$  the loop and composition products of  $QS^0$ . If  $\mathcal{S}_n$  is the  $n$ -th symmetric group then there is a well-known map  $\varphi_n: B\mathcal{S}_n \rightarrow Q_n S^0$  [BKP, P1]. Since  $D_8 \approx \mathcal{S}_2 \wr \mathcal{S}_2 \subset \mathcal{S}_4$  one has two natural maps  $BD_8 \rightarrow SG$ , namely the composites

$$\delta_1: BD_8 \rightarrow B\mathcal{S}_4 \xrightarrow{\varphi_4} Q_4 S^0 \xrightarrow{*[-3]} SG$$

and

$$\delta_2: BD_8 \rightarrow B\mathcal{S}_4 \xrightarrow{\varphi_4} Q_4 S^0 \xrightarrow{*[-1]} Q_3 S^0 \xrightarrow{(\#[3])^{-1}} SG$$

where  $[n]$  denotes the basepoint of  $Q_n S^0$  ( $\#[3]$  is an equivalence at 2).

Let  $\delta = \delta_1$  or  $\delta_2$  and let  $Q(\delta): QBD_8 \rightarrow SG$  denote the induced infinite loop map.

**THEOREM**    *A. There is a map  $t: SG \rightarrow QBD_8$  such that*  

$$SG \xrightarrow{t} QBD_8 \xrightarrow{Q(\delta)} SG \text{ is an equivalence at 2.}$$

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The affirmative solution of the Adams' conjecture [Q], [S] provides a map  $\gamma: BO \rightarrow G/O$  such that

$$\begin{array}{ccccc} & G/O & & & \\ \gamma \nearrow & & \searrow \tau & & \\ BO & \xrightarrow{\psi^3-1} & BSO & \xrightarrow{BJ} & BSG \end{array}$$

commutes up to homotopy, where  $\tau$  is the homotopy fibre of  $BJ$ . By abuse of notation, we shall let  $Q(\gamma): QBO_2 \rightarrow G/O$  denote the restriction of the induced infinite loop map.

**THEOREM B.** *There is a map  $T: G/O \rightarrow QBO_2$  such that the composite  $G/O \xrightarrow{T} QBO_2 \xrightarrow{Q(\gamma)} G/O$  is an equivalence at 2.*

The paper is organized as follows: In Sections 1 and 2 we recall the necessary preliminaries on symmetric groups, the transfer and  $H_*SG$  (throughout all (co-)homology groups are taken with simple coefficients in  $\mathbf{Z}/2$ ). The proof of Theorems A and B are given in Sections 3 and 4 respectively.

By way of background we mention other splittings derived from the transfer. Segal [Sg] has shown that  $BU$  is a factor in  $QBU_1$ . Becker and Gottlieb [BG2] have shown that  $BO$  and  $BSp$  are factors in  $QBO_2$  and  $QBSp_1$  respectively.

## §1. Preliminaries on symmetric groups and the transfer

Consider the symmetric group  $\mathcal{S}_{2^k}$  and 2-Sylow subgroup  $\mathcal{S}(2^k, 2) = \mathcal{S}_2 \wr \cdots \wr \mathcal{S}_2$ , the  $k$ -fold wreath product. The transfer homomorphism

$$tr_*: H_*(B\mathcal{S}_{2^k}) \rightarrow H_*(B\mathcal{S}(2^k, 2))$$

in mod-2 homology was studied in [KP2]. We shall recall those results needed for our work.

Two basic operations useful in describing the homology of symmetric groups are the wreath product  $\mathcal{S}_k \wr \mathcal{S}_l$  ( $\mathcal{S}_k \wr G = \mathcal{S}_k \circ G^k$ , the semi-direct product with  $\mathcal{S}_k$  acting by permuting factors) and the ordinary product  $\mathcal{S}_k \times \mathcal{S}_l$ . One has inclusions of subgroups

$$\mathcal{S}_k \wr \mathcal{S}_l \rightarrow \mathcal{S}_{kl} \tag{1.1}$$

$$\mathcal{S}_k \times \mathcal{S}_l \rightarrow \mathcal{S}_{k+l}. \tag{1.2}$$

Now let  $e_i \in H_i B\mathcal{S}_2 = \mathbf{Z}/2$  denote the non-zero element. If  $H_*(BG)$  has as  $\mathbf{Z}/2$ -vector space basis  $x_0 = 1, x_1, x_2, \dots$  then  $H_*(B\mathcal{S}_2 \wr G)$  has as basis

$$x_i | x_j = e_0 \otimes x_i \otimes x_j \quad i < j$$

$$e_i \wr x_j = e_i \otimes x_j \otimes x_j \quad i > 0$$

If  $I = (i_1, i_2, \dots, i_k)$  is a sequence of non-negative integers let  $\hat{e}_I = e_{i_1} \wr \dots \wr e_{i_k} \in H_* B\mathcal{S}(2^k, 2)$ . Let  $s: \mathcal{S}(2^k, 2) \rightarrow \mathcal{S}_{2^k}$  denote inclusion and let  $e_I = s_* \hat{e}_I$ . The length  $l(I)$  of  $I$  is defined to be  $k$ .  $I$  is said to be *allowable* if  $0 < i_1 \leq i_2 \leq \dots \leq i_k$ .

Nakaoka [N] has shown that  $H_*(B\mathcal{S}_{2m})$  is spanned by

$$\{e_{I_1} * \dots * e_{I_l} * e_0^p \mid 2m = \sum 2^{l(I_j)} + 2p\}$$

where  $*$  is the commutative pairing induced by (1.2). Furthermore these monomials form a basis if the sequences  $I_j$  are required to be allowable.

**THEOREM 1.3 [KP2]** *Let  $x = e_{i_1} * \dots * e_{i_p} * e_{I_1} * \dots * e_{I_l} \in H_* B\mathcal{S}_{2^k}$  with  $l(I_j) \geq 2$  then*

- i)  $tr_*(x) = \hat{e}_{i_1} | \hat{e}_{i_2} | \dots | \hat{e}_{i_p} | \hat{e}_{I_1} | \dots | \hat{e}_{I_l} + \hat{e}_x$  where  $\hat{e}_x = \sum \hat{e}_{i_1} | \dots | \hat{e}_{i_p} | \hat{e}_{I'_1} | \dots | \hat{e}_{I'_l}$ , the summation being taken over certain elements of the form indicated (or permutations thereof) with  $l(I'_j) = l(I_j)$ . Furthermore
- ii)  $s_*(\hat{e}_x) = 0$ .

**Remark 1.4.** The  $\hat{e}_i$ 's occurring in  $\hat{e}_x$  can be rearranged into successive even groupings, e.g.  $\hat{e}_{i_1} | \hat{e}_{i_2} | \hat{e}_{I'_1} | \hat{e}_{i_3} | \dots | \hat{e}_{i_6} | \hat{e}_{I'_2} | \dots | \hat{e}_{I'_l}$ . This fact is obvious for  $k = 2$ , for a general  $k$  it follows from an easy induction argument using the commutative law  $x | y = y | x$  in  $H_*(B\mathcal{S} \wr G)$ .

## §2. Preliminaries on $H_* SG$

The structure of  $H_* SG$  as an algebra over the Dyer-Lashof algebra is quite complicated. In this section we shall recall several results of Madsen [Md], May [M1], and Milgram [Mg] needed for our work.

Let  $Q_i: H_k QS^0 \rightarrow H_{2k+i} QS^0$  denote the Dyer-Lashof operations derived from the loop product  $*$ . Then

$$H_* QS^0 = \mathbf{Z}/2[[-1], [1], Q_I[1] \mid I \text{ allowable}]$$



The weight function  $\omega : H_*QS^0 \rightarrow \mathbf{Z}^+$  is defined by

$$\begin{aligned} w(Q_I[1]) &= 2^{l(I)} & w([i]) &= 0 \\ w(x * y) &= wx + wy & w\left(\sum x_i\right) &= \min\{w x_i\}. \end{aligned}$$

It is known that  $\#$  does not decrease weight [M1; 5.6], i.e.

$$w(xy) \geq wx + wy$$

(on the level of homology we denote the  $\#$  product by juxtaposition). Let  $u_i = Q_i[1] * [-1]$ ,  $x_I = Q_I[1] * [1 - 2^{l(I)}]$  where  $l(I) \geq 2$  then the fundamental result of Milgram [Mg] states

$$H_*SG = E[u_1, u_2, \dots] \otimes \mathbf{Z}/2[x_{(0,a)}, x_I \mid a > 0, I \text{ allowable}] \quad (2.1)$$

There are several connections between  $*$ -decomposable elements of  $H_*QS^0$  and  $\#$ -decomposable elements of  $H_*SG$ . Let  $I_k$  be the set of positive dimensional elements of  $H_*Q_kS^0$ ,  $I = \sum I_k$ . If  $x, y, z \in I$  then by [M1; 6.6ii and p. 137]

$$\text{i) } x * y * z * [1 - w] \in I_1 \# I_1 \quad \text{where} \quad w = w(x * y * z) \quad (2.2)$$

$$\text{ii) } Q_a[1] * Q_b[1] * [-3] + Q_a[1]Q_b[1] * [-3] \in I_1 \# I_1$$

also

$$Q_a[1]Q_b[1] = \sum_{l(I)=2} Q_I[1] \quad (2.3)$$

where the sum is taken over certain  $I$  with  $l(I) = 2$  [Mg, 6.2].

Let  $A$  be the subalgebra of  $H_*Q_0S^0$  generated by  $Q_I[1] * [-2^{l(I)}]$ ,  $l(I) \geq 2$  and let  $B$  be the subalgebra of  $H_*SG$  generated by  $x_I$ ,  $l(I) \geq 2$  then  $B = A * [1]$ . Further if  $\bar{A}$ ,  $\bar{B}$  denote the augmentation ideals then

$$H_*Q_0S^0 * \bar{A} * [1] = H_*SG \# \bar{B} \quad (2.4)$$

(see [Mg; 6.1]) and

$$Q_a[-1] = Q_a[1] * [-4] + \alpha \quad (2.5)$$

where  $\alpha$  is a  $*$ -decomposable element of  $H_*Q_0S^0 * \bar{A} * [-2]$  (see [Mg; §4], [P2; 2.1]).

Let  $\tilde{Q}_i : H_k SG \rightarrow H_{2k+i} SG$  denote the Dyer-Lashof operations associated with the composition product  $\#$ . The following result is due to Madsen [Md; 4.13] (see also [M1, 6.12]): let  $I = (J, K)$ ,  $l(K) = 2$  then

$$\tilde{Q}_J(x_k) \equiv x_I + \sum_{2 \leq l(M) < l(I)} x_M \pmod{I_1 \# I_1} \quad (2.6)$$

Finally we recall

$$(x * [i])(y * [j]) = \sum x' y' * x''[j] * y''[i] * [ij] \quad (2.7)$$

(see [Mg; 2.2], [M1; 1.5]).

LEMMA 2.8.  $Q_a[1] * Q_b[1] * [-3] \equiv u_a u_b + \sum_{l(I)=2} x_I \pmod{I_1 \# \bar{B}}$

*Proof.* By (2.7),

$$u_a u_b = \sum_{\substack{i+j=a \\ k+l=b}} Q_i[1] Q_k[1] * Q_j[-1] * Q_l[-1] * [1]$$

Thus by (2.5),  $u_a u_b = \sum Q_i[1] Q_k[1] * (Q_j[1] * [-4] + \alpha_j) * (Q_l[1] * [-4] + \beta_l) * [1]$  where  $\alpha_j, \beta_l$  are  $*$ -decomposable elements of  $H_* Q_0 S^0 * \bar{A} * [-2]$ . Thus  $u_a u_b = Q_a[1] * Q_b[1] * [-3] + Q_a[1] Q_b[1] * [-3] + \gamma$  where  $\gamma \in I_0 * \bar{A} * [1]$ . By (2.2) (ii)  $\gamma \in I_1 \# I_1$ , and by (2.4)  $\gamma \in H_* SG \# \bar{B}$  and so  $\gamma \in I_1 \# \bar{B}$ . This completes the proof by (2.3).

LEMMA 2.9. If  $x \in I_k$ ,  $w(x) = l$  then  $x[3] = x * [2k] + \alpha$  where  $w(\alpha) = 2l$ ,  $\alpha \in (I_k * I_{2k}) \cap (\bar{A} * [3k])$

*Proof.* By the distributive law, we have  $x[3] = x([1] * [2]) = \sum x'_i[1] * x''_i[2] = x * [2k] + \alpha$  where

$$\alpha = \sum_{\deg x'_i > 0} x'_i * x''_i[2] \in (I_k * I_{2k}) \cap (\bar{A} * [3k]), w(\alpha) = 2l.$$

LEMMA 2.10. If  $x, y, z \in I$  and  $x * y * z \in H_* Q_3 S^0 * \bar{A}$  then  $x * y * z \in I_1 \# \bar{B} \# [3]$

*Proof.* The proof proceeds by downward induction on weight. Let  $l = w(x * y * z)$ ,  $x \in I_k$ ,  $y \in I_m$ ,  $z \in I_n$ . By (2.2) (i) and (2.4)  $x * y * z * [-2] \in I_1 \# \bar{B}$

hence multiplying by  $[3]$

$$x[3] * y[3] * z[3] * [-6] \in I_1 \# \bar{B} \# [3].$$

Using Lemma 2.9 to evaluate this term we have  $(x * [2k] + \alpha) * (y * [2m] + \beta) * (z * [2n] + \gamma) * [-6] = x * y * z + 3\text{-fold } *- \text{decomposable terms in } H_* Q_3 S^0 * \bar{A}$  of weight greater than  $l$ . Thus by induction  $x * y * z \in I_1 \# \bar{B} \# [3]$ . Q.E.D.

LEMMA 2.11.

- i)  $(Q_a[1] * [1])[\frac{1}{3}] \equiv u_a + x_{(0,a/2)} \text{ modulo } I_1 \# \bar{B}$
- ii)  $(Q_a Q_b[1] * [-1])[\frac{1}{3}] \equiv x_{(a,b)} \text{ modulo } I_1 \# \bar{B}$
- iii)  $(Q_a[1] * Q_b[1] * [-1])[\frac{1}{3}] \equiv u_a u_b + \sum_{l(I)=2} x_l \text{ modulo } I_1 \# \bar{B}$

*Proof.* Since  $[\frac{1}{3}]$  has inverse  $[3]$  we can establish these equations by applying  $[3]$  to both sides ( $\xi(x) = x * x$ )

$$\begin{aligned} \text{i) } u_a[3] + x_{(0,a/2)}[3] &= (Q_a[1] * [-1])[3] + (\xi Q_{a/2}[1] * [-3])[3] \\ &= Q_a[3] * [-3] + \xi Q_{a/2}[3] * [-9] \\ &= Q_a[1] * [1] + \sum_{0 < i < a/2} Q_{a-2i}[1] * \xi Q_i[1] * [-3] \\ &\quad + \xi Q_{a/2}[1] * [-1] + \xi \{ Q_{a/2}[1] * [4] \\ &\quad \quad \quad + \sum_{j > 0} Q_{(a/2)-2j}[1] * \xi Q_j[1] \} * [-9] \\ &= Q_a[1] * [1] + \sum_{0 < i < a/2} Q_{a-2i}[1] * \xi Q_i[1] * [-3] \\ &\quad \quad \quad + \sum_{j > 0} \xi Q_{(a/2)-2j}[1] * \xi^2 Q_j[1] * [-9] \end{aligned}$$

By Lemma 2.10 all of these terms except the leading one belong to  $I_1 \# \bar{B} \# [3]$

ii) Using the Cartan formula we have

$$\begin{aligned} Q_a Q_b[3] &= Q_a Q_b([1] * [2]) = Q_a \left( \sum_{i \geq 0} \xi Q_i[1] * Q_{b-2i}[1] \right) \\ &= \sum_{i,j} \xi(Q_j Q_i[1]) * Q_{a-2j} Q_{b-2i}[1]. \end{aligned}$$

Thus

$$\begin{aligned}
 x_{(a,b)}[3] &= (Q_a Q_b[1] * [-3])[3] = Q_a Q_b[3] * [-9] \\
 &= \sum_{i,j \geq 0} \xi(Q_j Q_i[1]) * Q_{a-2j} Q_{b-2i}[1] * [-9] \\
 &= Q_a Q_b[1] * [-1] + \sum_{\substack{i > 0 \\ \text{or } j > 0}} \xi(Q_j Q_i[1]) * Q_{a-2j} Q_{b-2i}[1] * [-9].
 \end{aligned}$$

Each of the trailing terms belongs to  $I_1 \# \bar{B} \# [3]$  by Lemma 2.10 and (2.4).

iii) From Lemma 2.8 we have

$$(Q_a[1] * Q_b[1] * [-3])[3] \equiv u_a u_b[3] + \sum x_i[3] \bmod I_1 \# \bar{B} \# [3]$$

However  $Q_a[3] * Q_b[3] * [-9] = (Q_a[1] * [4] + \alpha) * (Q_b[1] * [4] + \beta) * [-9] = Q_a[1] * Q_b[1] * [-1] + 3\text{-fold } * \text{-decomposable elements in } H_* Q_3 S^0 * \bar{A}$  which belong to  $I_1 \# \bar{B} \# [3]$  by Lemma 2.10. Thus  $Q_a[1] * Q_b[1] * [-1] \equiv u_a u_b[3] + \sum x_i[3] \bmod I_1 \# \bar{B} \# [3]$ . Q.E.D.

### §3. Proof of Theorem A

Consider the composite

$$\sum^\infty B\mathcal{S}_{2^k} \xrightarrow{\tau} \sum^\infty QBD \xrightarrow{d} \sum^\infty SG \quad (D = D_8)$$

where  $\tau = \sum^\infty \beta \circ \sum^\infty u \circ tr'$ ,  $d = \sum^\infty Q(\delta)$  and  $tr': \sum^\infty B\mathcal{S}_{2^k} \rightarrow \sum^\infty B\mathcal{S}(2^k, 2)$  is the stable transfer [KP].  $u: B\mathcal{S}(2^k, 2) \rightarrow B\mathcal{S}_{2^{k-2}} \wr \mathcal{S}_2 \wr \mathcal{S}_2$  is inclusion.  $\beta: B\mathcal{S}_{2^{k-2}} \wr \mathcal{S}_2 \wr \mathcal{S}_2 = E\mathcal{S}_{2^{k-2}} \times_{\mathcal{S}_2^{k-2}} (BD)^{2^{k-2}} \rightarrow QBD$  is the restriction of the Dyer-Lashof map

$$E\mathcal{S}_{2^{k-2}} \times_{\mathcal{S}_2^{k-2}} (QBD)^{2^{k-2}} \rightarrow QBD.$$

Recall that in homology  $tr'$  is equivalent to  $tr$  [KP; 1.7]

**LEMMA 3.1.**  *$d \circ \tau$  is a homotopy equivalence at 2 in a range of dimensions which increases with  $k$ .*

We can now obtain Theorem A in the following manner: Lemma 3.1 implies that  $d_*: \pi_* \sum^\infty QBD \rightarrow {}_2 \pi_* \sum^\infty SG$  is a surjection. Now arguing as in Adams

[A, p. 50] one shows that

$$\left\{ \sum^{\infty} X, \sum^{\infty} BD^n \right\} \xrightarrow{\text{adj } \delta} \left\{ \sum^{\infty} X, B^{\infty} SG \right\}$$

is surjective for any CW-complex  $X$  of dimension  $< 2n$ , where  $\delta$  is defined in the Introduction and superscript  $n$  denotes the  $n$ -skeleton. Now applying this to  $X = SG^n$  we see that the composite

$$\sum^{\infty} SG^n \rightarrow \sum^{\infty} SG \xrightarrow{\alpha} B^{\infty} SG$$

(where  $\alpha$  is the stable adjoint of  $SG \xrightarrow{\text{id}} SG$ ) factors as

$$\sum^{\infty} SG^n \rightarrow \sum^{\infty} BD^n \xrightarrow{\text{adj } \delta} B^{\infty} SG$$

Thus upon applying  $\Omega^{\infty}$  and including  $SG^n \subset \Omega^{\infty} \sum^{\infty} SG^n$  we obtain the homotopy commutative diagram

$$\begin{array}{ccc} & QBD^n & \\ \nearrow & & \searrow Q(\delta) \\ SG^n & \xrightarrow{\quad} & SG \end{array}$$

Although there is no (obvious) compatibility in these diagrams with increasing  $n$ , the use of inverse limits [A] shows (since all homotopy groups in sight are finite) that there is a homotopy commutative diagram

$$\begin{array}{ccc} & QBD & \\ \nearrow t & & \searrow Q(\delta) \\ SG & \xrightarrow{\text{id}} & SG \end{array}$$

(3.2)

which completes the proof. It remains to consider the

*Proof of Lemma 3.1.* There is a well-known homology equivalence  $H_* B\mathcal{S}_{\infty} \approx H_* Q_0 S^0$  [BKP] also  $H_* B\mathcal{S}_{2^k} \approx H_* B\mathcal{S}_{\infty}$  in a range  $[N]$ . These facts, together with the obvious equivalence  $Q_0 S^0 \simeq SG$  as spaces, show that it is enough to prove that  $d_* \circ \tau_*$  is surjective in a range. We do this first for  $\delta = \delta_1$ . Because Theorem 1.3 is

our main tool we shall re-express  $d \circ \tau$  as

$$\sum_{\infty} B\mathcal{S}_{2^k} \xrightarrow{\text{tr}'} \sum_{\infty} B\mathcal{S}(2^k, 2) \xrightarrow{d'} \sum_{\infty} SG$$

where  $d'$  is the composite  $d \circ \sum_{\infty} \beta \circ \sum_{\infty} u$ .

If  $x = u_{i_1} u_{i_2}, \dots, u_{i_m} x_{I_1} x_{I_2}, \dots, x_{I_n}$  we shall write

$$a(x) = m + n, b(x) = k \text{ (} k \text{ is the number of terms } x_{I_j} \text{ with } l(I_j) = 2\text{)}$$

$$c(x) = n \text{ (} n \text{ is the number of terms } x_{I_j} \text{ with } l(I_j) \geq 2\text{)}.$$

As usual we extend these definitions to sums by setting

$$a(x + y) = \min \{a(x), a(y)\} \quad c(x + y) = \min \{c(x), c(y)\}$$

$$b(x + y) = \min \{b(x), b(y)\}$$

Let  $I_1^v = I_1 \# \dots \# I_1$  ( $v$ -factors).

*Step 1.*  $d'_* \text{tr}'_*$  is surjective modulo  $I_1^2$

i) Consider  $x = u_a$  and let  $2N = 2^k - 2$  then by Th. 1.3

$$d'_* \text{tr}'_*(e_a * e_0^N) = d'_*(\hat{e}_a \mid \hat{e}_0^N) = u_a$$

ii) Consider  $x = x_{(a,b)}$  and let  $2N = 2^k - 4$  then by Th. 1.3

$$\begin{aligned} d'_* \text{tr}'_*(e_{(a,b)} * e_0^N) &= d'_*(\hat{e}_{(a,b)} \mid \hat{e}_0^N + \sum \hat{e}_{(a',b')} \mid \hat{e}_0^N) \\ &= x_{(a,b)} + \sum_{\substack{\parallel \\ 0}} x_{(a',b')} = x_{(a,b)} \end{aligned}$$

iii) Consider  $x = x_I$ ,  $I = (J, K)$ ,  $l(K) = 2$ . Let  $2p = 2^k - 2^{l(I)}$  then by Th. 1.3

$$\begin{aligned} d'_* \text{tr}'_*(e_I * e_0^p) &= d'_*(\hat{e}_I \mid \hat{e}_0^p + \sum \hat{e}_{I'} \mid \hat{e}_0^p) \quad (I' = (J', K')) \\ &= \tilde{Q}_J(x_K) + \sum \tilde{Q}_{J'}(x_{K'}) \\ &\equiv x_I + \sum_{\substack{\parallel \\ 0}} x_{I'} + \sum_{2 \leq l(M) < l(I)} x_M \pmod{I_1^2} \quad (\text{by 2.6}) \end{aligned}$$

The terms  $x_M \in \text{Im}(d'_* \text{tr}'_*) \pmod{I_1^2}$  by induction on length starting with length 2 which is covered by ii).

Taken together i), ii), and iii) prove Step 1.

*Step 2.*  $d'_*tr'_*$  is surjective: Assume by induction that  $x \in \text{Im}(d'_*tr'_*) \bmod I_1^v$  for all  $x$  such that  $a(x) < v$ . Now consider  $x$  such that  $a(x) = v$  say  $x = u_{i_1}, \dots, u_{i_{2p}} x_{I_1}, \dots, x_{I_k} x_{I_{k+1}}, \dots, x_{I_{k+n}}$  where  $i_1 < \dots < i_{2p}$ ,  $v = 2p + k + n$ ,  $l(I_j) = 2$  for  $1 \leq j \leq k$  and  $l(I_j) > 2$  for  $k < j < k + n$ . Let  $s = wx$  and set  $e = e_{i_1} * \dots * e_{i_{2p}} * e_{I_1} * \dots * e_{I_{k+n}}$ , by Theorem 1.3 we have (with  $I_j = (J_j, K_j)$ ,  $l(K_j) = 2$ )

$$\begin{aligned}
 d'_*tr'_*(e) &= d_*(\hat{e}_{i_1} | \dots | \hat{e}_{i_{2p}} | \hat{e}_{I_1} | \dots | \hat{e}_{I_k} | \hat{e}_{I_{k+1}} | \dots | \hat{e}_{I_{k+n}}) \\
 &\quad + \sum \hat{e}_{i_1} | \dots | \hat{e}_{i_{2p}} | \hat{e}_{I'_1} | \dots | \hat{e}_{I'_k} | \hat{e}_{I'_{k+1}} | \dots | \hat{e}_{I'_{k+n}} \\
 &= (Q_{i_1}[1] * Q_{i_2}[1] * [-3]) \cdots (Q_{i_{2p-1}}[1] * Q_{i_{2p}}[1] * [-3]) \cdot \\
 &\quad x_{I_1} \cdots x_{I_k} \cdot \tilde{Q}_{J_{k+1}}(x_{K_{k+1}}) \cdots \tilde{Q}_{J_{k+n}}(x_{K_{k+n}}) \\
 &\quad + \sum (Q_{i_1}[1] * Q_{i_2}[1] * [-3]) \cdots (Q_{i_{2p-1}}[1] * Q_{i_{2p}}[1] * [-3]) \cdot \\
 &\quad x_{I'_1} \cdots x_{I'_k} \cdot \tilde{Q}_{J'_{k+1}}(x_{K'_{k+1}}) \cdots \tilde{Q}_{J'_{k+n}}(x_{K'_{k+n}}) \\
 &= u_{i_1} u_{i_2} \cdots u_{i_{2p-1}} u_{i_{2p}} x_{I_1} \cdots x_{I_k} x_{I_{k+1}} \cdots x_{I_{k+n}} \\
 &\quad + \sum u_{i_1} \cdots u_{i_{2p}} x_{I'_1} \cdots x_{I'_k} x_{I'_{k+1}} \cdots x_{I'_{k+n}} \\
 &\quad + \alpha_e + \beta_e + \gamma_e + \delta_e
 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
 a(\alpha_e) &\geq v \\
 \alpha(\beta_e) &\leq v, b(\beta_e) > k \\
 \alpha(\gamma_e) &= v, b(\gamma_e) = k, c(\gamma_e) > k + n \\
 \alpha(\delta_e) &= v, b(\gamma_e) = k, c(\gamma_e) = k + n, w(\gamma_e) < s.
 \end{aligned}$$

The third equality of (3.3) results from (2.6) and Lemma 2.8: The term  $\alpha_e$  occurs because of the  $\#$ -decomposable elements introduced by (2.6) and Lemma 2.8; the term  $\beta_e$  occurs because the factors  $Q_a[1] * Q_b[1] * [-3]$  can give rise (by Lemma 2.8) to monomials of lesser  $a$ -value but higher  $b$ -value; the term  $\gamma_e$  occurs because the  $\#$ -decomposable terms introduced from Lemma 2.8 can increase the  $c$ -value without changing (by 2.6) the  $a$  or  $b$ -values; the term  $\delta_e$  occurs because the factors  $\tilde{Q}_J(x_k)$  can give rise (from 2.6) to monomials of lesser weight.

From our analysis of (3.3) we have

LEMMA 3.4.  $b(d'_*tr'_*(e)) \geq k$ , i.e.  $d'_*tr'_*$  does not decrease the number of factors of length 2.

Finally we claim  $\sum u_{i_1} \cdots u_{i_{2p}} x_{I'_1} \cdots x_{I'_{k+n}} = 0$ . By Theorem 1.3(ii)

$$\begin{aligned} & s_* \left( \sum \hat{e}_{i_1} | \cdots | \hat{e}_{i_{2p}} | \hat{e}_{I'_1} | \cdots | \hat{e}_{I'_{k+n}} \right) \\ &= \sum e_{i_1} * \cdots * e_{i_{2p}} * e_{I'_1} * \cdots * e_{I'_{k+n}} = 0. \end{aligned}$$

There are no relations in the  $*$ -product except commutativity and  $e_M * e_M = e_{(0,M)}$ . Since commutativity also holds in  $H_*SG$  and  $\tilde{Q}_0(x_M) = x_M \cdot x_M$  the claim follows. We need not consider the relation  $u_i u_j = 0$  since we are assuming  $i_1 < \cdots < i_{2p}$ .

Now among those  $x$  with  $a(x) = v$  consider those with maximum  $b$ -value and among those ones with maximum  $c$ -value and among those ones with minimum  $w$ -value. Such  $x \in \text{Im}(d'_*tr'_*) \bmod I_1^{v+1}$  by 3.3 (we observe that no terms  $\beta_e$  can occur by induction and Lemma 3.4). Now proceed by upward induction on the  $w$ -value and then downward induction on the  $c$ -value. We now must consider lowering the value of  $b$  which will introduce terms of the form  $\beta_e$ . However by Lemma 3.4 and induction we may assume such elements are in  $\text{Im}(d'_*tr'_*) \bmod I_1^{v+1}$ . Thus we may proceed by downward induction on  $b$  until we have  $x \in \text{Im}(d'_*tr'_*) \bmod I_1^{v+1}$  for all  $x$  with  $a(x) = v$ . This completes the induction. To complete Step 2 we must also consider elements  $x = u_{i_1}, \dots, u_{i_{2p-1}} x_{I_1}, \dots, x_{I_{k+n}}$  but the proof is entirely analogous.

It remains to consider  $\delta = \delta_2$ , however by Lemma 2.11 we can use the same argument. Q.E.D.

#### §4. Proof of Theorem B

From the affirmative solution of the Adams' conjecture we have a homotopy commutative diagram

$$\begin{array}{ccccc} SG & \xrightarrow{\pi} & G/O & \xrightarrow{\tau} & BSO & \xrightarrow{BJ} & BSG \\ & & \nwarrow \gamma & & \nearrow \psi^3 - 1 & & \\ & & BSO & & & & \end{array} \quad (4.0)$$



where the horizontal maps from the usual fibre sequence. Let  $e: G/O \rightarrow BSO$  denote the map obtained from the  $KO$ -orientation of Spin bundles [ABS]. Madsen-Tornehave-Snaith [MST] have shown that  $e$  is an infinite loop map (the range of  $e$  is actually  $BSO_\infty$  but by the theorem of Adams and the author [AP] we may ignore this point). Further  $e\gamma \simeq \rho^3$  an equivalence at 2. Let  $C \xrightarrow{\varphi} G/O$  be the homotopy fibre of  $e$ ,  $C$  is usually called the cokernel of  $J$ . We recall the splitting of Sullivan [S], [MST; 5.5], [M2; V.4.7]

$$g: C \times BSO \xrightarrow{\approx} G/O, \quad g = \varphi \cdot \gamma.$$

Since  $\tilde{K}O^*(C) = 0$  [H, S1] there is a lifting  $\psi$  (unique up to homotopy)

$$\begin{array}{ccc} & SG & \\ \psi \nearrow & \downarrow \pi & \\ C & \xrightarrow{\varphi} & G/O \end{array}$$

Now let  $T_G$  be the composite

$$T_G: SG \xrightarrow{t} QBD \xrightarrow{i} QBO_2$$

where  $t$  is the transfer of (3.2) and  $i$  is induced by the standard orthogonal representation of  $D$  on  $\mathbf{R}^2$ .

Set  $t_C = T_G \circ \psi: C \rightarrow QBO_2$ . Let

$$T_B: BO \rightarrow QBO_2$$

be the map induced by Becker and Gottlieb transfer [S2; I(3.5)] and set  $t_B = T_B j: BSO \rightarrow QBO_2$  where  $j: BSO \rightarrow BO$  is inclusion. Finally let  $T = u \circ (t_C \times t_B) \circ g^{-1}: G/O \rightarrow QBO_2$  where  $u: QBO_2 \times QBO_2 \rightarrow QBO_2$  is the loop product.

Theorem B is equivalent to

**THEOREM 4.1.**  $G/O \xrightarrow{T} QBO_2 \xrightarrow{Q(\gamma)} G/O$  is an equivalence at 2.

Before giving the proof of Theorem 4.1 we prepare some necessary lemmas. Brumfiel and Madsen [BM, Lemma A.1] have shown that the following diagram

is homotopy commutative

$$\begin{array}{ccc}
 QBD & \xrightarrow{Q(\delta_2)} & SG \\
 \downarrow i & & \downarrow \pi \\
 QBO_2 & \xrightarrow{Q(\gamma)} & G/O
 \end{array} \tag{4.2}$$

Let

$$\chi = p_1 \circ g^{-1}: G/O \rightarrow C \times BSO \rightarrow C$$

where  $p_1$  is projection.

LEMMA 4.3.  $\chi Q(\gamma)t_C \simeq id_C$ .

*Proof.*

$$\begin{aligned}
 \chi Q(\gamma)t_C &= \chi Q(\gamma)T_G\psi \\
 &= \chi Q(\gamma)i\psi \\
 &\simeq \chi\pi Q(\delta_2)t\psi \quad (\text{by 4.2}) \\
 &= \chi\pi\psi \quad (\text{by 3.2}) \\
 &= \chi\varphi = id_C \quad \text{Q.E.D.}
 \end{aligned}$$

LEMMA 4.4.  $eQ(\gamma)t_B$  is an equivalence.

*Proof.* We will show that in mod-2 cohomology  $(eQ(\gamma)t_B)^*(w_2) \neq 0$ . From this and the action of the Steenrod algebra it follows that  $(eQ(\gamma)t_B)^*(w_i) = w_i + \text{decomposables}$  and thus that  $eQ(\gamma)t_B$  is an equivalence. Snaith [S2,] has observed that if  $k: BO_2 \rightarrow BO$  denotes inclusion then

$$BO_2 \xrightarrow{k} BO \xrightarrow{T_B} QBO_2$$

is the standard inclusion  $BO_2 \rightarrow QBO_2$ . Hence  $T_B^*(w_2) = w_2$ . It is well-known (and easy to prove from 4.0 or 4.2) that  $\gamma^*$  is non-zero on the bottom (2-dimensional) class in  $H^*G/O$ . Since  $e\gamma$  is an equivalence  $e^*(w_2) \neq 0$ . Thus  $(eQ(\gamma)T_B)^*(w_2) \neq 0$  and the result follows. Q.E.D.

Let  $R_B = eQ(\gamma)(t_C \cdot t_B): C \times BSO \rightarrow BSO$

LEMMA 4.5 i)  $\chi \times e: G/O \rightarrow C \times BSO$  is an equivalence.

ii)  $R_B \simeq eQ(\gamma)t_B p_2$ .

*Proof.* i)  $(\chi \times e)g = \chi g \times eg$  where we recall  $g = \varphi \cdot \gamma$  is an equivalence.  $eg = e(\varphi \cdot \gamma) \simeq e\varphi \cdot e\gamma \simeq e\gamma p_2$  since  $\tilde{K}O^*(C) = 0$  implies  $e\varphi \simeq 0$ .  $\chi g = p_1 g^{-1} g = p_1$ . This completes the proof of i) since  $e\gamma$  is an equivalence ii).  $eQ(\gamma)(t_C \cdot t_B) \simeq eQ(\gamma)t_C \cdot eQ(\gamma)t_B \simeq eQ(\gamma)t_B p_2$  since  $\tilde{K}O^*(C) = 0$  implies  $eQ(\gamma)t_C \simeq 0$ . Q.E.D.

*Proof of Theorem 4.1.* Let  $R = (\chi \times e)Q(\gamma)(t_C \cdot t_B)$ ,  $R_C = \chi Q(\gamma)(t_C \cdot t_B)$  then  $R = R_C \times R_B$ . Let  $x \oplus y \in \pi_k C \oplus \pi_k BSO$  then  $R(x \oplus y) = R_C(x \oplus y) \oplus R_B(x \oplus y)$ . By Lemma 4.5ii)  $R_B(x \oplus y) = eQ(\gamma)t_B(y)$ . By Lemma 4.3  $R_C(x) = x$ . Hence  $R(x \oplus y) = x + \chi Q(\gamma)t_C(y) \oplus eQ(\gamma)t_B(y)$  and so  $R$  is an isomorphism since  $eQ(\gamma)t_B$  is an equivalence by Lemma 4.4. Thus  $R$  and hence  $Rg^{-1} = (\chi \times e)Q(\gamma)T$  is an equivalence. This completes the proof by Lemma 4.5i).

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