Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 53 (1978)

Artikel: Homotopy splittings involving G and G/O.
Autor: Priddy, Stewart
DOI: https://doi.org/10.5169/seals-40780

Comment. Math. Helvetici 53 (1978) 470–484 Birkhäuser Verlag, Basel

Homotopy splittings involving G and G/O

STEWART PRIDDY[1]

Introduction

In this note we show that in a strong sense SG and G/O are factors in the spaces QBD_8 and QBO_2 respectively, where D_8 is the dihedral group of order 8. All spaces (throughout the note) are localized at 2. These results can be thought of as analogous to the theorem of D. S. Khan and the author [KP] which states that Q_0S^0 is a factor in QRP^∞. In particular, here, as in [KP], the transfer is used to construct the required splittings. Additional difficulties arise in the present work, however, because the infinite loop space structure of SG is markedly more complicated than that of Q_0S^0. Also, in the case of G/O we must use the Becker-Gottlieb transfer [BG].

To state our results precisely, we recall that $QS^0 = \lim\limits_{\to} \Omega^n S^n$ has components Q_kS^0, $k \in \mathbf{Z}$, and that $SG = Q_1S^0$. We shall denote by $*$ and $\#$ the loop and composition products of QS^0. If $\mathcal{S}_n$ is the n-th symmetric group then there is a well-known map $\varphi_n : B\mathcal{S}_n \to Q_nS^0$ [BKP, P1]. Since $D_8 \approx \mathcal{S}_2 \int \mathcal{S}_2 \subset \mathcal{S}_4$ one has two natural maps $BD_8 \to SG$, namely the composites

$$\delta_1 : BD_8 \to B\mathcal{S}_4 \xrightarrow{\varphi_4} Q_4S^0 \xrightarrow{*[-3]} SG$$

and

$$\delta_2 : BD_8 \to B\mathcal{S}_4 \xrightarrow{\varphi_4} Q_4S^0 \xrightarrow{*[-1]} Q_3S^0 \xrightarrow{(\#[3])^{-1}} SG$$

where $[n]$ denotes the basepoint of Q_nS^0 ($\#[3]$ is an equivalence at 2).

Let $\delta = \delta_1$ or δ_2 and let $Q(\delta) : QBD_8 \to SG$ denote the induced infinite loop map.

THEOREM A. *There is a map $t : SG \to QBD_8$ such that $SG \xrightarrow{t} QBD_8 \xrightarrow{Q(\delta)} SG$ is an equivalence at* 2.

[1] Supported in part by NSF Grant MPS76-07051

The affirmative solution of the Adams' conjecture [Q], [S] provides a map $\gamma : BO \to G/O$ such that

$$\begin{tikzcd} & G/O \arrow[d, "\tau"] & \\ BO \arrow[ur, "\gamma"] \arrow[r, "\psi^3-1"] & BSO \arrow[r, "BJ"] & BSG \end{tikzcd}$$

commutes up to homotopy, where τ is the homotopy fibre of BJ. By abuse of notation, we shall let $Q(\gamma): QBO_2 \to G/O$ denote the restriction of the induced infinite loop map.

THEOREM B. *There is a map $T: G/O \to QBO_2$ such that the composite $G/O \xrightarrow{T} QBO_2 \xrightarrow{Q(\gamma)} G/O$ is an equivalence at* 2.

The paper is organized as follows: In Sections 1 and 2 we recall the necessary preliminaries on symmetric groups, the transfer and H_*SG (throughout all (co-) homology groups are taken with simple coefficients in $\mathbf{Z}/2$). The proof of Theorems A and B are given in Sections 3 and 4 respectively.

By way of background we mention other splittings derived from the transfer. Segal [Sg] has shown that BU is a factor in QBU_1. Becker and Gottlieb [BG2] have shown that BO and BSp are factors in QBO_2 and $QBSp_1$ respectively.

§1. Preliminaries on symmetric groups and the transfer

Consider the symmetric group $\mathscr{S}_{2^k}$ and 2-Sylow subgroup $\mathscr{S}(2^k, 2) = \mathscr{S}_2 \wr \cdots \wr \mathscr{S}_2$, the k-fold wreath product. The transfer homomorphism

$$tr_*: H_*(B\mathscr{S}_{2^k}) \to H_*(B\mathscr{S}(2^k, 2))$$

in mod-2 homology was studied in [KP2]. We shall recall those results needed for our work.

Two basic operations useful in describing the homology of symmetric groups are the wreath product $\mathscr{S}_k \wr \mathscr{S}_l (\mathscr{S}_k \wr G = \mathscr{S}_k \circ G^k$, the semi-direct product with $\mathscr{S}_k$ acting by permuting factors) and the ordinary product $\mathscr{S}_k \times \mathscr{S}_l$. One has inclusions of subgroups

$$\mathscr{S}_k \wr \mathscr{S}_l \to \mathscr{S}_{kl} \tag{1.1}$$

$$\mathscr{S}_k \times \mathscr{S}_l \to \mathscr{S}_{k+l}. \tag{1.2}$$

Now let $e_i \in H_i B\mathcal{S}_2 = \mathbf{Z}/2$ denote the non-zero element. If $H_*(BG)$ has as $\mathbf{Z}/2$-vector space basis $x_0 = 1, x_1, x_2, \ldots$ then $H_*(B\mathcal{S}_2 \wr G)$ has as basis

$$x_i \mid x_j = e_0 \otimes x_i \otimes x_j \qquad i<j$$

$$e_i \wr x_j = e_i \otimes x_j \otimes x_j \qquad i>0$$

If $I = (i_1, i_2, \ldots, i_k)$ is a sequence of non-negative integers let $\hat{e}_I = e_{i_1} \wr \cdots \wr e_{i_k} \in H_* B\mathcal{S}(2^k, 2)$. Let $\varsigma : \mathcal{S}(2^k, 2) \to \mathcal{S}_{2^k}$ denote inclusion and let $e_I = \varsigma_* \hat{e}_I$. The *length* $l(I)$ of I is defined to be k. I is said to be *allowable* if $0 < i_1 \leq i_2 \leq \cdots \leq i_k$.

Nakaoka [N] has shown that $H_*(B\mathcal{S}_{2m})$ is spanned by

$$\{e_{I_1} * \cdots * e_{I_l} * e_0^p \mid 2m = \sum 2^{l(I_j)} + 2p\}$$

where $*$ is the commutative pairing induced by (1.2). Furthermore these monomials form a basis if the sequences I_j are required to be allowable.

THEOREM 1.3 [KP2] *Let* $x = e_{i_1} * \cdots * e_{i_p} * e_{I_1} * \cdots * e_{I_l} \in H_* B\mathcal{S}_{2^k}$ *with* $l(I_j) \geq 2$ *then*

i) $tr_*(x) = \hat{e}_{i_1} \mid \hat{e}_{i_2} \mid \cdots \mid \hat{e}_{i_p} \mid \hat{e}_{I_1} \mid \cdots \mid \hat{e}_{I_l} + \hat{e}_x$ *where* $\hat{e}_x = \sum \hat{e}_{i_1} \mid \cdots \mid \hat{e}_{i_p} \mid \hat{e}_{I'_1} \mid \cdots \mid \hat{e}_{I'_l}$, *the summation being taken over certain elements of the form indicated (or permutations thereof) with* $l(I'_j) = l(I_j)$. *Furthermore*
ii) $\varsigma_*(\hat{e}_x) = 0$.

Remark 1.4. The $\hat{e}_i$'s occurring in $\hat{e}_x$ can be rearranged into successive even groupings, e.g. $\hat{e}_{i_1} \mid \hat{e}_{i_2} \mid \hat{e}_{I'_1} \mid \hat{e}_{i_3} \mid \cdots \mid \hat{e}_{i_6} \mid \hat{e}_{I'_2} \mid \cdots \mid \hat{e}_{I'_l}$. This fact is obvious for $k=2$, for a general k it follows from an easy induction argument using the commutative law $x \mid y = y \mid x$ in $H_*(B\mathcal{S} \wr G)$.

§2. Preliminaries on H_*SG

The structure of H_*SG as an algebra over the Dyer-Lashof algebra is quite complicated. In this section we shall recall several results of Madsen [Md], May [M1], and Milgram [Mg] needed for our work.

Let $Q_i : H_k QS^0 \to H_{2k+i} QS^0$ denote the Dyer-Lashof operations derived from the loop product $*$. Then

$$H_* QS^0 = \mathbf{Z}/2[[-1], [1], Q_I[1] \mid I \text{ allowable}]$$

The weight function $\omega: H_*QS^0 \to \mathbf{Z}^+$ is defined by

$$w(Q_I[1]) = 2^{l(I)} \qquad w([i]) = 0$$

$$w(x * y) = wx + wy \qquad w\left(\sum x_i\right) = \min\{w\, x_i\}.$$

It is known that # does not decrease weight [M1; 5.6], i.e.

$$w(xy) \geq wx + wy$$

(on the level of homology we denote the # product by juxaposition). Let $u_i = Q_i[1] * [-1]$, $x_I = Q_I[1] * [1 - 2^{l(I)}]$ where $l(I) \geq 2$ then the fundamental result of Milgram [Mg] states

$$H_*SG = E[u_1, u_2, \ldots] \otimes \mathbf{Z}/2[x_{(0,a)}, x_I \mid a > 0, I \text{ allowable}] \tag{2.1}$$

There are several connections between $*$-decomposable elements of H_*QS^0 and #-decomposable elements of H_*SG. Let I_k be the set of positive dimensional elements of $H_*Q_kS^0$, $I = \sum I_k$. If $x, y, z \in I$ then by [M1; 6.6ii and p. 137]

$$\text{i) } x * y * z * [1 - w] \in I_1 \# I_1 \quad \text{where} \quad w = w(x * y * z) \tag{2.2}$$

$$\text{ii) } Q_a[1] * Q_b[1] * [-3] + Q_a[1]Q_b[1] * [-3] \in I_1 \# I_1$$

also

$$Q_a[1]Q_b[1] = \sum_{l(I)=2} Q_I[1] \tag{2.3}$$

where the sum is taken over certain I with $l(I) = 2$ [Mg, 6.2].

Let A be the subalgebra of $H_*Q_0S^0$ generated by $Q_I[1] * [-2^{l(I)}]$, $l(I) \geq 2$ and let B be the subalgebra of H_*SG generated by x_I, $l(I) \geq 2$ then $B = A * [1]$. Further if $\bar{A}$, $\bar{B}$ denote the augmentation ideals then

$$H_*Q_0S^0 * \bar{A} * [1] = H_*SG \# \bar{B} \tag{2.4}$$

(see [Mg; 6.1]) and

$$Q_a[-1] = Q_a[1] * [-4] + \alpha \tag{2.5}$$

where α is a $*$-decomposable element of $H_*Q_0S^0 * \bar{A} * [-2]$ (see [Mg; §4], [P2; 2.1]).

Let $\tilde{Q}_i : H_k SG \to H_{2k+i} SG$ denote the Dyer-Lashof operations associated with the composition product $\#$. The following result is due to Madsen [Md; 4.13] (see also [M1, 6.12]): let $I=(J, K)$, $l(K)=2$ then

$$\tilde{Q}_J(x_k) \equiv x_I + \sum_{2 \le l(M) < l(I)}{}' x_M \mod I_1 \# I_1 \tag{2.6}$$

Finally we recall

$$(x * [i])(y * [j]) = \sum x'y' * x''[j] * y''[i] * [ij] \tag{2.7}$$

(see [Mg; 2.2], [M1; 1.5]).

LEMMA 2.8. $Q_a[1] * Q_b[1] * [-3] \equiv u_a u_b + \sum_{l(I)=2} x_I$ modulo $I_1 \# \bar{B}$

Proof. By (2.7),

$$u_a u_b = \sum_{\substack{i+j=a \\ k+l=b}} Q_i[1]Q_k[1] * Q_j[-1] * Q_l[-1] * [1]$$

Thus by (2.5), $u_a u_b = \sum Q_i[1]Q_k[1] * (Q_j[1] * [-4] + \alpha_j) * (Q_l[1] * [-4] + \beta_l) * [1]$ where α_j, β_l are $*$-decomposable elements of $H_* Q_0 S^0 * \bar{A} * [-2]$. Thus $u_a u_b = Q_a[1] * Q_b[1] * [-3] + Q_a[1]Q_b[1] * [-3] + \gamma$ where $\gamma \in I_0 * \bar{A} * [1]$. By (2.2) (ii) $\gamma \in I_1 \# I_1$, and by (2.4) $\gamma \in H_* SG \# \bar{B}$ and so $\gamma \in I_1 \# \bar{B}$. This completes the proof by (2.3).

LEMMA 2.9. *If* $x \in I_k$, $w(x) = l$ *then* $x[3] = x * [2k] + \alpha$ *where* $w(\alpha) = 2l$, $\alpha \in (I_k * I_{2k}) \cap (\bar{A} * [3k])$

Proof. By the distributive law, we have $x[3] = x([1] * [2]) = \sum x_i'[1] * x_i''[2] = x * [2k] + \alpha$ where

$$\alpha = \sum_{\deg x_i'' > 0} x_1' * x_i''[2] \in (I_k * I_{2k}) \cap (\bar{A} * [3k]),\ w(\alpha) = 2l.$$

LEMMA 2.10. *If* $x, y, z \in I$ *and* $x * y * z \in H_* Q_3 S^0 * \bar{A}$ *then* $x * y * z \in I_1 \# \bar{B} \# [3]$

Proof. The proof proceeds by downward induction on weight. Let $l = w(x * y * z)$, $x \in I_k$, $y \in I_m$, $z \in I_n$. By (2.2) (i) and (2.4) $x * y * z * [-2] \in I_1 \# \bar{B}$

hence multiplying by [3]

$$x[3] * y[3] * z[3] * [-6] \in I_1 \# \bar{B} \# [3].$$

Using Lemma 2.9 to evaluate this term we have $(x * [2k]+\alpha) * (y * [2m]+\beta) * (z * [2n]+\gamma) * [-6] = x * y * z + 3$-fold $*$-decomposable terms in $H_* Q_3 S^0 * \bar{A}$ of weight greater than l. Thus by induction $x * y * z \in I_1 \# \bar{B} \# [3]$. Q.E.D.

LEMMA 2.11.

i) $(Q_a[1] * [1])[\tfrac{1}{3}] \equiv u_a + x_{(0,a/2)}$ modulo $I_1 \# \bar{B}$

ii) $(Q_a Q_b[1] * [-1])[\tfrac{1}{3}] \equiv x_{(a,b)}$ modulo $I_1 \# \bar{B}$

iii) $(Q_a[1] * Q_b[1] * [-1])[\tfrac{1}{3}] \equiv u_a u_b + \sum_{l(I)=2} x_I$ modulo $I_1 \# \bar{B}$

Proof. Since $[\tfrac{1}{3}]$ has inverse [3] we can establish these equations by applying [3] to both sides ($\xi(x) = x * x$)

$$\begin{aligned}
\text{i)}\ u_a[3] + x_{(0,a/2)}[3] &= (Q_a[1] * [-1])[3] + (\xi Q_{a/2}[1] * [-3])[3] \\
&= Q_a[3] * [-3] + \xi Q_{a/2}[3] * [-9] \\
&= Q_a[1] * [1] + \sum_{0<i<a/2} Q_{a-2i}[1] * \xi Q_i[1] * [-3] \\
&\quad + \xi Q_{a/2}[1] * [-1] + \xi\{Q_{a/2}[1] * [4] \\
&\qquad\qquad + \sum_{j>0} Q_{(a/2)-2j}[1] * \xi Q_j[1]\} * [-9] \\
&= Q_a[1] * [1] + \sum_{0<i<a/2} Q_{a-2i}[1] * \xi Q_i[1] * [-3] \\
&\qquad\qquad + \sum_{j>0} \xi Q_{(a/2)-2j}[1] * \xi^2 Q_j[1] * [-9]
\end{aligned}$$

By Lemma 2.10 all of these terms except the leading one belong to $I_1 \# \bar{B} \# [3]$

ii) Using the Cartan formula we have

$$\begin{aligned}
Q_a Q_b[3] = Q_a Q_b([1] * [2]) &= Q_a\Big(\sum_{i \ge 0} \xi Q_i[1] * Q_{b-2i}[1]\Big) \\
&= \sum_{i,j} \xi(Q_j Q_i[1]) * Q_{a-2j} Q_{b-2i}[1].
\end{aligned}$$

Thus

$$x_{(a,b)}[3]=(Q_aQ_b[1]*[-3])[3]=Q_aQ_b[3]*[-9]$$
$$=\sum_{i,j\geq 0}\xi(Q_jQ_i[1])*Q_{a-2j}Q_{b-2i}[1]*[-9]$$
$$=Q_aQ_b[1]*[-1]+\sum_{\substack{i>0\\ \text{or } j>0}}\xi(Q_jQ_i[1])*Q_{a-2j}Q_{b-2i}[1]*[-9].$$

Each of the trailing terms belongs to $I_1\#\bar{B}\#[3]$ by Lemma 2.10 and (2.4).

iii) From Lemma 2.8 we have

$$(Q_a[1]*Q_b[1]*[-3])[3]\equiv u_au_b[3]+\sum x_I[3] \bmod I_1\#\bar{B}\#[3]$$

However $Q_a[3]*Q_b[3]*[-9]=(Q_a[1]*[4]+\alpha)*(Q_b[1]*[4]+\beta)*[-9]=Q_a[1]*Q_b[1]*[-1]+$3-fold $*$-decomposable elements in $H_*Q_3S^0*\bar{A}$ which belong to $I_1\#\bar{B}\#[3]$ by Lemma 2.10. Thus $Q_a[1]*Q_b[1]*[-1]\equiv u_au_b[3]+\sum x_I[3] \bmod I_1\#\bar{B}\#[3]$. Q.E.D.

§3. Proof of Theorem A

Consider the composite

$$\sum^\infty B\mathcal{S}_{2^k}\xrightarrow{\tau}\sum^\infty QBD\xrightarrow{d}\sum^\infty SG \qquad (D=D_8)$$

where $\tau=\sum^\infty\beta\circ\sum^\infty u\circ tr'$, $d=\sum^\infty Q(\delta)$ and $tr'\colon\sum^\infty B\mathcal{S}_{2^k}\to\sum^\infty B\mathcal{S}(2^k,2)$ is the stable transfer [KP]. $u\colon B\mathcal{S}(2^k,2)\to B\mathcal{S}_{2^{k-2}}\wr\mathcal{S}_2\wr\mathcal{S}_2$ is inclusion. $\beta\colon B\mathcal{S}_{2^{k-2}}\wr\mathcal{S}_2\wr\mathcal{S}_2=E\mathcal{S}_{2^{k-2}}\times_{\mathcal{S}_2^{k-2}}(BD)^{2^{k-2}}\to QBD$ is the restriction of the Dyer-Lashof map

$$E\mathcal{S}_{2^{k-2}}\times_{\mathcal{S}_{2^{k-2}}}(QBD)^{2^{k-2}}\to QBD.$$

Recall that in homology tr' is equivalent to tr [KP; 1.7]

LEMMA 3.1. *$d\circ\tau$ is a homotopy equivalence at 2 in a range of dimensions which increases with k.*

We can now obtain Theorem A in the following manner: Lemma 3.1 implies that $d_*\colon\pi_*\sum^\infty QBD\to{}_2\pi_*\sum^\infty SG$ is a surjection. Now arguing as in Adams

[A, p. 50] one shows that

$$\left\{\sum^{\infty} X, \sum^{\infty} BD^n\right\} \xrightarrow{adj\,\delta} \left\{\sum^{\infty} X, B^{\infty}SG\right\}$$

is surjective for any CW-complex X of dimension $<2n$, where δ is defined in the Introduction and superscript n denotes the n-skeleton. Now applying this to $X = SG^n$ we see that the composite

$$\sum^{\infty} SG^n \to \sum^{\infty} SG \xrightarrow{\alpha} B^{\infty}SG$$

(where α is the stable adjoint of $SG \xrightarrow{\text{id}} SG$) factors as

$$\sum^{\infty} SG^n \to \sum^{\infty} BD^n \xrightarrow{\text{adj}\,\delta} B^{\infty} SG$$

Thus upon applying Ω^{∞} and including $SG^n \subset \Omega^{\infty} \sum^{\infty} SG^n$ we obtain the homotopy commutative diagram

$$\begin{tikzcd} & QBD^n \arrow[dr, "Q(\delta)"] & \\ SG^n \arrow[ur] \arrow[rr, hook] & & SG \end{tikzcd}$$

Although there is no (obvious) compatibility in these diagrams with increasing n, the use of inverse limits [A] shows (since all homotopy groups in sight are finite) that there is a homotopy commutative diagram

$$\begin{tikzcd} & QBD \arrow[dr, "Q(\delta)"] & \\ SG \arrow[ur, "t"] \arrow[rr, "\text{id}"] & & SG \end{tikzcd} \tag{3.2}$$

which completes the proof. It remains to consider the

Proof of Lemma 3.1. There is a well-known homology equivalence $H_*B\mathcal{S}_\infty \approx H_*Q_0S^0$ [BKP] also $H_*B\mathcal{S}_{2^k} \approx H_*B\mathcal{S}_\infty$ in a range [N]. These facts, together with the obvious equivalence $Q_0S^0 \simeq SG$ as spaces, show that it is enough to prove that $d_* \circ \tau_*$ is surjective in a range. We do this first for $\delta = \delta_1$. Because Theorem 1.3 is our main tool we shall re-express $d \circ \tau$ as

$$\sum^{\infty} B\mathcal{S}_{2^k} \xrightarrow{\text{tr}'} \sum^{\infty} B\mathcal{S}(2^k, 2) \xrightarrow{\text{d}'} \sum^{\infty} SG$$

where d′ is the composite $d \circ \sum^{\infty} \beta \circ \sum^{\infty} u$.

If $x = u_{i_1} u_{i_2}, \ldots, u_{i_m} x_{I_1} x_{I_2}, \ldots, x_{I_n}$ we shall write

$$a(x) = m + n,\ b(x) = k \ (k \text{ is the number of terms } x_{I_j} \text{ with } l(I_j) = 2)$$
$$c(x) = n \ (n \text{ is the number of terms } x_{I_j} \text{ with } l(I_j) \geq 2).$$

As usual we extend these definitions to sums by setting

$$a(x+y) = \min\{a(x), a(y)\} \qquad c(x+y) = \min\{c(x), c(y)\}$$
$$b(x+y) = \min\{b(x), b(y)\}$$

Let $I_1^v = I_1 \# \cdots \# I_1$ (v-factors).

Step 1. $d'_* tr'_*$ is surjective modulo I_1^2

i) Consider $x = u_a$ and let $2N = 2^k - 2$ then by Th. 1.3

$$d'_* tr'_*(e_a * e_0^N) = d'_*(\hat{e}_a \mid \hat{e}_0^N) = u_a$$

ii) Consider $x = x_{(a,b)}$ and let $2N = 2^k - 4$ then by Th. 1.3

$$d'_* tr'_*(e_{(a,b)} * e_0^N) = d'_*(\hat{e}_{(a,b)} \mid \hat{e}_0^N + \sum \hat{e}_{(a',b')} \mid \hat{e}_0^N)$$
$$= x_{(a,b)} + \underset{\substack{\| \\ 0}}{\sum x_{(a',b')}} = x_{(a,b)}$$

iii) Consider $x = x_I$, $I = (J, K)$, $l(K) = 2$. Let $2p = 2^k - 2^{l(I)}$ then by Th. 1.3

$$d'_* tr'_*(e_I * e_0^p) = d'_*(\hat{e}_I \mid \hat{e}_0^p + \sum \hat{e}_{I'} \mid \hat{e}_0^p) \qquad (I' = (J', K'))$$
$$= \tilde{Q}_J(x_K) + \sum \tilde{Q}_{J'}(x_{K'})$$
$$\equiv x_I + \underset{\substack{\| \\ 0}}{\sum x_{I'}} + \sum_{2 \leq l(M) < l(I)} x_M \quad \text{mod} \quad I_1^2 \quad (\text{by } 2.6)$$

The terms $x_M \in \text{Im}(d'_* tr'_*)$ mod I_1^2 by induction on length starting with length 2 which is covered by ii).

Taken together i), ii), and iii) prove Step 1.

Step 2. $d'_* tr'_*$ is surjective: Assume by induction that $x \in \mathrm{Im}\,(d'_* tr'_*) \bmod I_1^v$ for all x such that $a(x) < v$. Now consider x such that $a(x) = v$ say $x = u_{i_1}, \ldots, u_{i_{2p}} x_{I_1}, \ldots, x_{I_k} x_{I_{k+1}}, \ldots, x_{I_{k+n}}$ where $i_1 < \cdots < i_{2p}$, $v = 2p + k + n$, $l(I_j) = 2$ for $1 \le j \le k$ and $l(I_j) > 2$ for $k < j < k+n$. Let $s = wx$ and set $e = e_{i_1} * \cdots * e_{i_{2p}} * e_{I_1} * \cdots * e_{I_{k+n}}$, by Theorem 1.3 we have (with $I_j = (J_j, K_j)$, $l(K_j) = 2$)

$$d'_* tr'_*(e) = d_*(\hat{e}_{i_1} | \cdots | \hat{e}_{i_{2p}} | \hat{e}_{I_1} | \cdots | \hat{e}_{I_k} | \hat{e}_{I_{k+1}} | \cdots | \hat{e}_{I_{k+n}} \tag{3.3}$$

$$+ \sum \hat{e}_{i_1} | \cdots | \hat{e}_{i_{2p}} | \hat{e}_{I'_1} | \cdots | \hat{e}_{I'_k} | \hat{e}_{I'_{k+1}} | \cdots | \hat{e}_{I'_{k+n}}$$

$$= (Q_{i_1}[1] * Q_{i_2}[1] * [-3]) \cdots (Q_{i_{2p-1}}[1] * Q_{i_{2p}}[1] * [-3]) \cdot$$

$$x_{I_1} \cdots x_{I_k} \cdot \tilde{Q}_{J_{k+1}}(x_{K_{k+1}}) \cdots \tilde{Q}_{J_{k+n}}(x_{K_{k+n}})$$

$$+ \sum (Q_{i_1}[1] * Q_{i_2}[1] * [-3]) \cdots (Q_{i_{2p-1}}[1] * Q_{i_{2p}}[1] * [-3]) \cdot$$

$$x_{I'_1} \cdots x_{I'_k} \cdot \tilde{Q}_{J'_{k+1}}(x_{K'_{k+1}}) \cdots \tilde{Q}_{J'_{k+n}}(x_{K_{k+n}})$$

$$= u_{i_1} u_{i_2} \cdots u_{i_{2p-1}} u_{i_{2p}} x_{I_1} \cdots x_{I_k} x_{I_{k+1}} \cdots x_{I_{k+n}}$$

$$+ \sum u_{i_1} \cdots u_{i_{2p}} x_{I'_1} \cdots x_{I'_k} x_{I'_{k+1}} \cdots x_{I'_{k+n}}$$

$$+ \alpha_e + \beta_e + \gamma_e + \delta_e$$

where

$$a(\alpha_e) \ge v$$
$$a(\beta_e) \le v,\ b(\beta_e) > k$$
$$a(\gamma_e) = v,\ b(\gamma_e) = k,\ c(\gamma_e) > k+n$$
$$a(\delta_e) = v,\ b(\gamma_e) = k,\ c(\gamma_e) = k+n,\ w(\gamma_e) < s.$$

The third equality of (3.3) results from (2.6) and Lemma 2.8: The term α_e occurs because of the #-decomposable elements introduced by (2.6) and Lemma 2.8; the term β_e occurs because the factors $Q_a[1] * Q_b[1] * [-3]$ can give rise (by Lemma 2.8) to monomials of lesser a-value but higher b-value; the term γ_e occurs because the #-decomposable terms introduced from Lemma 2.8 can increase the c-value without changing (by 2.6) the a or b-values; the term δ_e occurs because the factors $\tilde{Q}_J(x_k)$ can give rise (from 2.6) to monomials of lesser weight.

From our analysis of (3.3) we have

LEMMA 3.4. $b(d'_* tr'_*(e)) \geq k$, i.e. *$d'_* tr'_*$ does not decrease the number of factors of length* 2.

Finally we claim $\sum u_{i_1} \cdots u_{i_{2p}} x_{I'_1} \cdots x_{I'_{k+n}} = 0$. By Theorem 1.3(ii)

$$s_*\left(\sum \hat{e}_{i_1} | \cdots | \hat{e}_{i_{2p}} | \hat{e}_{I'_1} | \cdots | \hat{e}_{I'_{k+n}}\right)$$
$$= \sum e_{i_1} * \cdots * e_{i_{2p}} * e_{I'_1} * \cdots * e_{I'_{k+n}} = 0.$$

There are no relations in the $*$-product except commutativity and $e_M * e_M = e_{(0,M)}$. Since commutativity also holds in H_*SG and $\tilde{Q}_0(x_M) = x_M \cdot x_M$ the claim follows. We need not consider the relation $u_j u_j = 0$ since we are assuming $i_1 < \cdots < i_{2p}$.

Now among those x with $a(x) = v$ consider those with maximum b-value and among those ones with maximum c-value and among those ones with minimum w-value. Such $x \in \operatorname{Im}(d'_* tr'_*) \bmod I_1^{v+1}$ by 3.3 (we observe that no terms β_e can occur by induction and Lemma 3.4). Now proceed by upward induction on the w-value and then downward induction on the c-value. We now must consider lowering the value of b which will introduce terms of the form β_e. However by Lemma 3.4 and induction we may assume such elements are in $\operatorname{Im}(d'_* tr'_*) \bmod I_1^{v+1}$. Thus we may proceed by downward induction on b until we have $x \in \operatorname{Im}(d'_* tr'_*) \bmod I_1^{v+1}$ for all x with $a(x) = v$. This completes the induction. To complete Step 2 we must also consider elements $x = u_{i_1}, \ldots, u_{i_{2p-1}} x_{I_1}, \ldots, x_{I_{k+n}}$ but the proof is entirely analogous.

It remains to consider $\delta = \delta_2$, however by Lemma 2.11 we can use the same argument. Q.E.D.

§4. Proof of Theorem B

From the affirmative solution of the Adams' conjecture we have a homotopy commutative diagram

$$\begin{tikzcd} SG \arrow[r, "\pi"] & G/O \arrow[r, "\tau"] & BSO \arrow[r, "BJ"] & BSG \\ & BSO \arrow[u, "\gamma"] \arrow[ur, "\psi^3 - 1"'] & & \end{tikzcd} \tag{4.0}$$

where the horizontal maps from the usual fibre sequence. Let $e: G/O \to BSO$ denote the map obtained from the KO-orientation of Spin bundles [ABS]. Madsen-Tornehave-Snaith [MST] have shown that e is an infinite loop map (the range of e is actually $BSO_\otimes$ but by the theorem of Adams and the author [AP] we may ignore this point). Further $e\gamma \simeq \rho^3$ an equivalence at 2. Let $C \xrightarrow{\varphi} G/O$ be the homotopy fibre of e, C is usually called the cokernel of J. We recall the splitting of Sullivan [S], [MST; 5.5], [M2; V.4.7]

$$g: C \times BSO \xrightarrow{\simeq} G/O, \qquad g = \varphi \cdot \gamma.$$

Since $\tilde{KO}^*(C) = 0$ [H, S1] there is a lifting ψ (unique up to homotopy)

$$\begin{tikzcd} & SG \arrow[d, "\pi"] \\ C \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & G/O \end{tikzcd}$$

Now let T_G be the composite

$$T_G: SG \xrightarrow{t} QBD \xrightarrow{i} QBO_2$$

where t is the transfer of (3.2) and i is induced by the standard orthogonal representation of D on $\mathbf{R}^2$.

Set $t_C = T_G \circ \psi: C \to QBO_2$. Let

$$T_B: BO \to QBO_2$$

be the map induced by Becker and Gottlieb transfer [S2; I(3.5)] and set $t_B = T_B j: BSO \to QBO_2$ where $j: BSO \to BO$ is inclusion. Finally let $T = u \circ (t_C \times t_B) \circ g^{-1}$: $G/O \to QBO_2$ where $u: QBO_2 \times QBO_2 \to QBO_2$ is the loop product.

Theorem B is equivalent to

THEOREM 4.1. *$G/O \xrightarrow{T} QBO_2 \xrightarrow{Q(\gamma)} G/O$ is an equivalence at 2.*

Before giving the proof of Theorem 4.1 we prepare some necessary lemmas. Brumfiel and Madsen [BM, Lemma A.1] have shown that the following diagram is homotopy commutative

$$\begin{array}{ccc} QBD & \xrightarrow{Q(\delta_2)} & SG \\ \cap\big\downarrow i & & \big\downarrow \pi \\ QBO_2 & \xrightarrow{Q(\gamma)} & G/O \end{array} \tag{4.2}$$

Let

$$\chi = p_1 \circ g^{-1} : G/O \to C \times BSO \to C$$

where p_1 is projection.

LEMMA 4.3. $\chi Q(\gamma) t_C \simeq id_C$.

Proof.

$$\begin{aligned} \chi Q(\gamma) t_C &= \chi Q(\gamma) T_G \psi \\ &= \chi Q(\gamma) i\, t\psi \\ &\simeq \chi \pi Q(\delta_2) t\psi \quad \text{(by 4.2)} \\ &= \chi \pi \psi \quad \text{(by 3.2)} \\ &= \chi \varphi = id_C \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 4.4. *$eQ(\gamma)t_B$ is an equivalence.*

Proof. We will show that in mod-2 cohomology $(eQ(\gamma)t_B)^*(w_2) \neq 0$. From this and the action of the Steenrod algebra it follows that $(eQ(\gamma)t_B)^*(w_i) = w_i +$decomposables and thus that $eQ(\gamma)t_B$ is an equivalence. Snaith [S2,] has observed that if $k : BO_2 \to BO$ denotes inclusion then

$$BO_2 \xrightarrow{k} BO \xrightarrow{T_B} QBO_2$$

is the standard inclusion $BO_2 \to QBO_2$. Hence $T_B^*(w_2) = w_2$. It is well-known (and easy to prove from 4.0 or 4.2) that γ^* is non-zero on the bottom (2-dimensional) class in H^*G/O. Since $e\chi$ is an equivalence $e^*(w_2) \neq 0$. Thus $(eQ(\gamma)T_B)^*(w_2) \neq 0$ and the result follows. Q.E.D.

Let $R_B = e\,Q(\gamma)(t_C \cdot t_B) : C \times BSO \to BSO$

LEMMA 4.5 i) *$\chi\times e$: $G/O\to C\times BSO$ is an equivalence.*
ii) *$R_B\simeq e\,Q(\gamma)t_Bp_2$.*

Proof. i) $(\chi\times e)g=\chi g\times eg$ where we recall $g=\varphi\cdot\gamma$ is an equivalence. $eg=e(\varphi\cdot\gamma)\simeq e\varphi\cdot e\gamma\simeq e\gamma p_2$ since $\tilde{KO}^*(C)=0$ implies $e\varphi\simeq 0$. $\chi g=p_1g^{-1}g=p_1$. This completes the proof of i) since $e\gamma$ is an equivalence ii). $eQ(\gamma)(t_C\cdot t_B)\simeq eQ(\gamma)t_C\cdot eQ(\gamma)t_B\simeq eQ(\gamma)t_Bp_2$ since $\tilde{KO}^*(C)=0$ implies $eQ(\gamma)t_C\simeq 0$. Q.E.D.

Proof of Theorem 4.1. Let $R=(\chi\times e)Q(\gamma)(t_C\cdot t_B)$, $R_C=\chi Q(\gamma)(t_C\cdot t_B)$ then $R=R_C\times R_B$. Let $x\oplus y\in\pi_kC\oplus\pi_kBSO$ then $R(x\oplus y)=R_C(x\oplus y)\oplus R_B(x\oplus y)$. By Lemma 4.5ii) $R_B(x\oplus y)=eQ(\gamma)t_B(y)$. By Lemma 4.3 $R_C(x)=x$. Hence $R(x\oplus y)=x+\chi Q(\gamma)t_C(y)\oplus eQ(\gamma)t_B(y)$ and so R is an isomorphism since $eQ(\gamma)t_B$ is an equivalence by Lemma 4.4. Thus R and hence $Rg^{-1}=(\chi\times e)Q(\gamma)T$ is an equivalence. This completes the proof by Lemma 4.5i).

BIBLIOGRAPHY

[A] J. F. Adams, *The Kahn-Priddy theorem*, Proc. Camb. Phil. Soc. (1973), 45–55.

[AP] —— and S. B. Priddy, *Uniqueness of BSO*, Proc. Camb. Phil. Soc. *80* (1976), 475–509.

[ABS] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology 3 (Suppl. 1) (1964), 3–38.

[BG] J. C. Becker and D. H. Gottlieb, *The transfer map and fibre bundles*, Topology 14 (1975), 1–12.

[BG2] ——, *Characteristic classes and K-theory*, Lecture Notes in Math., Vol. 428, Springer-Verlag, 1974.

[BKP] M. G. Barratt, D. S. Kahn, and S. B. Priddy, *On $\Omega^\infty S^\infty$ and the infinite symmetric group*, Proc. Symp. Pure Math., Vol. 22, A.M.S. 1971.

[BM] G. Brumfiel and I. Madsen, *Evaluation of the transfer and the universal surgery classes*, Inventiones Math. *32* (1976) 133–169.

[H] L. Hodgkin, *The K-theory of some well-known spaces-II* Coker J. (to appear).

[KP] D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory*, Bull. A.M.S. *76*(6) (1972), 981–987.

[KP2] ——, *On the transfer in the homology of symmetric groups*, Math. Proc. Camb. Phil. Soc. *83* (1978), 91–101.

[Md] I. Madsen, *On the action of the Dyer-Lashof algebra in H_*G*, Pacific J. Math. *60* (1975), 235–275.

[MST] ——, V. Snaith, and J. Tornehave, *Infinite loop maps in geometric topology*, Math. Proc. Camb. Phil. Soc. *81* (1977), 399–429.

[M1] F. Cohen, T. Lada, J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math. vol. 533, Springer-Verlag, Berlin and New York, 1977.

[M2] —— (with contributions by F. Quinn, N. Ray and J. Tornehave), *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math. vol. 577, Springer-Verlag, Berlin and New York, 1977.

[Mg] R. J. Milgram, *The mod 2 spherical characteristic classes*, Annals Math. *92* (1970), 238–261.

[N] M. Nakaoka, *Homology of the infinite symmetric group*, Annals Math. *72* (1961), 229–257.

[P1] S. B. Priddy, *Transfer, symmetric groups, and stable homology*, Algebraic K-theory I, Lecture Notes in Math., Vol. 341, Springer-Verlag, 1973.

[P2] ——, *Dyer-Lashof operations for the classifying spaces of certain matrix groups*, Quart. J. Math. *26* (1975), 179–193.

[Q] D. G. Quillen, *The Adams' conjecture*, Topology 10 (1971), 67–80.
[Sg] G. Segal, *The stable homotopy of complex projective spaces*, Quart. J. Math. *24* (1973), 1–5.
[S1] V. Snaith, *Splitting of the image of the J-homomorphism*, (to appear).
[S2] ——, *Algebraic Cobordism and K-theory*, (to appear).
[S] D. Sullivan, *Geometric topology, part I. Localization, periodicity, and Galois symmetry.* Mimeographed Notes, M.I.T.

Northwestern University

Received April 25, 1977.