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Cylinders on surfaces

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In [2] B. Randol has shown that if M is a compact Rieman surface with metric of constant curvature -1, and γ is a simple closed geodesic on M of length L_γ, then the area, A_γ, of the largest topological cylinder swept out by geodesics of identical length perpendicular to and centered on γ, satisfies

$$A_\gamma \geq 2L_\gamma \operatorname{csch}(L_\gamma/2) \tag{1}$$

In Remark 4 Randol asked if there is a corrresponding result for surfaces of variable curvature. We point out in this note that the answer is yes, viz., if M is a compact orientable surface whose Gauss curvature function K satisfies the inequalities

$$-1 \leq K \leq -\kappa^2 < 0 \tag{2}$$

where κ is a positive constant, then A_γ satisfies the inequality

$$\text{(R)} \quad A_\gamma \geq (2L_\gamma/\kappa) \sinh\{\kappa \operatorname{arccosh}((\tanh(L_\gamma/2))^{-1})\}$$

(Note that when $\kappa = 1$ the two inequalities coincide.)

The proof will consist of two parts: (i) we show the validity in the universal covering of M, $\bar{M}$, of the construction given in Figure 3 in [2] (without the symmetry about the vertical geodesic) and then show, as in [2], that the top lateral geodesic in Figure 3 can intersect at most one of the side geodesics; (ii) will then consist of a comparison argument in the universal covering $\hat{M}$.

1. The Sturmian estimates

For the moment M will be any orientable complete 2-dimensional Riemannian manifold. For $p \in M$ we will denote the tangent space to M at p by M_p, and

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the tangent bundle by TM. For $\xi, \xi_1, \xi_2, \in M_p$, $\langle \xi_1, \xi_2 \rangle$ will denote the inner product of ξ_1 and ξ_2, and $|\xi|$ the norm of ξ. For any differentiable path $\gamma : \mathbf{R} \to M$, γ' will denote the velocity vector field along γ. The exponential map of TM to M will be denoted by exp. The map is defined by the property that for any $\xi \in TM$, the path

$$\gamma_\xi(t) = \exp t\xi$$

is the geodesic for which $\gamma_\xi(0)$ is the point in whose tangent space ξ is found, and $\gamma'_\xi(0) = \xi$. We assume a fixed orientation of M is chosen and define $\iota : TM \to TM$ to be the rotation in each tangent space of $\pi/2$ radians.

Let $\gamma : \mathbf{R} \to M$, $|\gamma'| = 1$ be a geodesic in M, and define $v : \mathbf{R}^2 \to M$ by

$$v(x, y) = \exp y\iota\gamma'(x). \tag{3}$$

We denote the coordinate tangent vector fields along v by $\partial_x v$, $\partial_y v$, and invariant differentiation (in the Levi–Civita connection of the Riemannian metric) with respect to x and y by ∇_x and ∇_y respectively. The standard arguments yield

$$|\partial_y v| = 1, \qquad \nabla_y \partial_y v = 0, \qquad (4)$$
$$\langle \partial_x v, \partial_y v \rangle = 0.$$

If we set

$$\eta = \langle \partial_x v, -\iota\, \partial_y v \rangle = \sqrt{\langle \partial_x, \partial_x \rangle} = \sqrt{E(x, y)}$$

then Jacobi's equation of geodesic deviation reads as

$$\partial_y^2 \eta + K\eta = 0$$

with initial conditions

$$\eta(x, 0) = 1, \qquad \partial_y \eta(x, 0) = 0$$

for all $x \in \mathbf{R}$. The standard Sturmian arguments verify the following

LEMMA. *If the Gauss curvature K of M satisfies* (2) *on M for some given* $\kappa > 0$, *then*

$$\cosh \kappa y \leq \eta(x, y) \leq \cosh y \tag{5}$$

for all $(x, y) \in \mathbf{R}^2$. *For all* $x \in \mathbf{R}$, $y > 0$ *we have*

$$\kappa \sinh \kappa y \leq \partial_y \eta(x, y) \leq \sinh y, \tag{6}$$

and for all $x \in \mathbf{R}$, $y < 0$ *we have*

$$\kappa \sinh \kappa y \geq \partial_y \eta(x, y) \geq \sinh y. \tag{7}$$

In particular v is of maximal rank on all of $\mathbf{R}^2$. Furthermore if γ is a covering of its image in M then v is a covering of M by $\mathbf{R}^2$.

2. The picture in the universal covering of M

We now let M be our compact orientable surface (thus complete) satisfying the inequalities (2) for some given $\kappa > 0$. Note that the Gauss-Bonnet theorem implies that M has genus ≥ 2. Let $\gamma : \mathbf{R} \to M$, $|\gamma'| = 1$ be a simple closed geodesic in M of length L_γ, i.e., $\gamma(x_1) = \gamma(x_2)$ if and only if $x_2 - x_1$ is an integral multiple of L_γ. Then γ is a covering of its image $\gamma(\mathbf{R})$ in M and the map v defined by (3) is periodic in x with period L_γ, and is a covering of M – the universal covering.

Now for sufficiently small $d > 0$, $v \mid \mathbf{R} \times (-d, d)$ is a covering of its image, a cylinder in M, with deck transformation group $L_\gamma \mathbf{Z}$ – the group of $\gamma : \mathbf{R} \to \gamma(\mathbf{R})$. Let d_0 be the largest such $d > 0$, i.e., d_0 is the distance from $\gamma(\mathbf{R})$ to its focal cut locus. The left inequality of (5) then implies

$$A_\gamma = A(v(\mathbf{R} \times (-d_0, d_0))) \geq (2L_\gamma/\kappa) \sinh \kappa d_0. \tag{8}$$

So our job is to estimate d_0 from below.

We note that since v is of maximal rank on all of $\mathbf{R}^2$ there must exist x_1, x_2 such that either

$$v(x_1, d_0) = v(x_2, d_0), \ \gamma(x_1) \neq \gamma(x_2) \tag{a}$$

or

$$v(x_1, -d_0) = v(x_2, -d_0), \ \gamma(x_1) \neq \gamma(x_2) \tag{b}$$

or

$$v(x_1, d_0) = v(x_2, -d_0), \tag{c}$$

i.e., there exist two distinct geodesics emanating from points of γ, orthogonal to γ, which meet at distance d_0 along the geodesics. In the first two cases they emanate from the same side of the geodesic and in the third from opposite sides. By an argument of W. Klingenberg [1, Lemma 1] they meet smoothly, i.e.,

$$\partial_y v(x_1, d_0) = -\partial_y v(x_2, d_0), \tag{a$'$}$$

$$\partial_y v(x_1, -d_0) = -\partial_y v(x_1, -d_0), \tag{b$'$}$$

$$\partial_y v(x_1, d_0) = \partial_y v(x_2, -d_0), \tag{c$'$}$$

respectively, The first two cases are geometrically the same so we shall only consider (a) and (c).

We now endow $\mathbf{R}^2$ with the Riemannian metric for which v is a Riemannian covering. Then the translation

$$(x, y) \to (x + L_\gamma, y) \tag{9}$$

is a deck transformation of v and an isometry of $\mathbf{R}^2$ in its new metric. When referring to $\mathbf{R}^2$ with the metric lifted from M via v we shall denote $\mathbf{R}^2$ by $\bar{M}$.

For convenience assume $x_1 = 0$, and let Γ be the geodesic in $\bar{M}$ given by $\Gamma(x) = (x, 0)$, let $\omega_1, \omega, \omega_2$ be the geodesics in $\bar{M}$ given by

$$\omega_1(y) = (-L_\gamma/2, y), \qquad \omega_2(y) = (L_\gamma/2, y), \qquad \omega(y) = (0, y),$$

and let σ be the geodesic in $\bar{M}$ through $(0, d_0)$, orthogonal to ω at $(0, d_0)$ and oriented from left to right through $(0, d_0)$. Then there exist maximal $\alpha, \beta > 0$ and a smooth function $f: (-\alpha, \beta) \to \mathbf{R}$ such that $\sigma(x) = (x, y(x))$. From the Lemma and Section 3 we have y strictly convex, i.e., $y'' > 0$.

We now claim that it is impossible that both $\alpha, \beta > L_\gamma/2$, i.e., that σ intersects both ω_1 and ω_2. We start with case (a).

Assume that σ intersects ω_1 at $\omega(y_1)$ and ω_2 at $\omega_2(y_2)$. Let σ_1 be the path in $\bar{M}$ consisting of σ composed, if $y_1 \neq y_2$, with ω_2 from $\omega_2(y_2)$ to $\omega_2(y_1)$. Then the projection of σ_1, $v(\sigma_1)$, is a piecewise smooth geodesic loop in M homotopic to γ, with 1 or 2 corners, depending on whether $y_1 = y_2$ or $y_1 \neq y_2$ respectively.

At $\Gamma(x_2)$ draw $\bar{\omega}(y) = (x_2, y)$ and lift $v(\sigma_1)$ to $\bar{\sigma}_1$ in $\bar{M}$ through $\bar{\omega}(d_0)$. Then the velocity vector of $\bar{\sigma}_1$ at $\bar{\omega}(d_0)$ is orthogonal to $\bar{\omega}$ and, by (a'), oriented from right to left. The smooth segment of $\bar{\sigma}_1$, containing $\bar{\omega}(d_0)$ is, of course, geodesic in $\bar{M}$ and remains transverse to the foliation $\{x = \text{const}\}$ in $\bar{M}$ including the limit of the velocity vector field at the endpoints of the segment.

Let p_1 be the lift of $\omega_1(y_1)$, p_2 the lift of $\omega_2(y_2)$, and p_3 the lift of $\omega_2(y_1)$; and

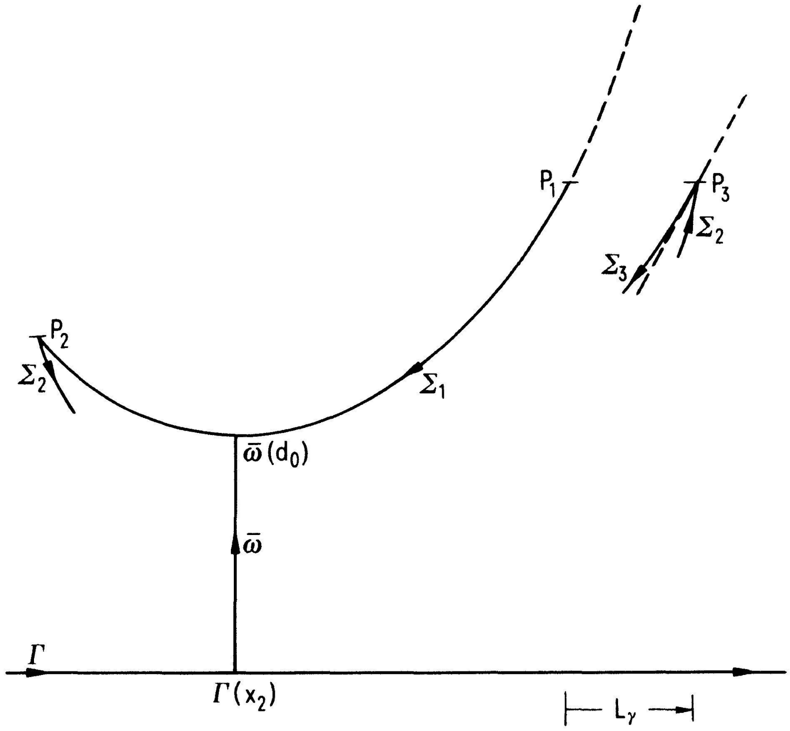

Figure 1

for $p \in \bar{M}$ let $x(p), y(p)$ denote its coordinates. Next let Σ_1 be the geodesic segment of $\bar{\sigma}_1$, containing $\bar{\omega}(d_0)$, i.e., connecting p_1 to p_2, Σ_2 the segment connecting p_2 to p_3, and Σ_3 the translate of Σ_1, L_γ units to the right (i.e., via (9)).

We now start our argument. Since Σ_1 is oriented from right to left, we have $x(p_2) < x(p_1)$. On the other hand, $v(\bar{\sigma}_1) = v(\sigma_1)$ is homotopic to γ which implies p_3 is the image of p_1 under the deck transformation (9). Thus,

$$x(p_3) = x(p_1) + L_\gamma, \; y(p_3) = y(p_1).$$

In particular, $p_2 \neq p_3$ and σ must have 2 corners. If we started with 1 corner then we already have the desired contradiction.

We think of p traveling along Σ_2 from p_2 to p_3. As mentioned partially) above, any geodesic is either always transverse to the foliation $\{x = \text{const}\}$ in $\bar{M}$, or always tangent to it. When transverse, it is the graph of a convex function. Thus as p leaves p_2 it may not leave vertically or to the left, if it is to connect with p_3.

So p moves to the right as it leaves p_2. If it leaves above Σ_1 then to reach p_3 it must cross the geodesic determined by Σ_1 which is impossible (e.g., by Gauss-Bonnet formula). So p leaves p_2 moving to the right below Σ_1.

Let l be the line in $\bar{M}$ tangent to Σ_3 at p_3. If p approaches p_3 above l then Σ_2 intersects Σ_3 at 2 points, which is impossible. If p approaches p_3 below l then the angles of $\bar{\sigma}_1$ at p_2 and p_3 from the terminal velocity vector to the initial one at

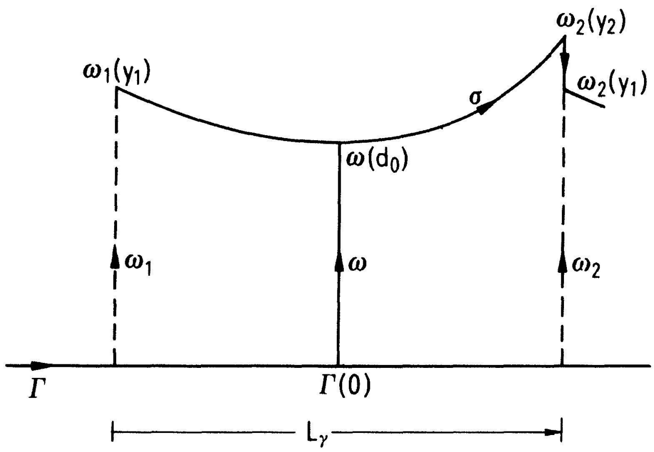

Figure 2

each corner, are of the same sign. (Recall: the discontinuities of the velocity vector field are corners not cusps.)

But the corresponding angles at the corners of σ_1 have opposite sign (Figure 2) – a contradiction, since $\bar{\sigma}_1$ is the isometric image of σ_1 by some element in the deck transformation group.

The proof for case (c) is as in [2, Case #2].

3. The comparison argument

We now restrict ourselves to $\bar{M}$ as in §2, viz., the metric in $\bar{M}$ is lifted from M via v and its Gauss curvature therefore satisfies (2). We apply the apparatus of §2 with v now being the identity map.

Let σ be any geodesic in $\bar{M}$; as mentioned, if σ is transverse to the foliation of $\bar{M}$, $\{x = \text{const}\}$, at one point then it is always transverse to the foliation.

When σ is transverse to the foliation, we can then write σ as the graph of a function $y(x)$. Standard calculation then shows that

$$y''(x) = E_y\left\{\frac{1}{2}+\frac{(y')^2}{E}\right\}+\frac{y'E_x}{2E}. \tag{10}$$

$y'(0) = 0$, so $y''(x) > 0$ in some neighborhood of 0. We wish to show that $y''(x) > 0$ in the entire domain of y. We will restrict our attention to $x > 0$, as the other case follows in a similar manner. Let $\gamma(x)$ be the angle the curve $(x, y(x))$ makes with the line $y \to (x, y)$, i.e., $\tan(\pi/2 - \gamma(x)) = y'(x)$. It suffices to show $\gamma'(x) < 0$. Let R_x be the geodesic quadrilateral bounded above by the graph of $y(x)$, below by the x-axis, on the left by the y-axis, on the right by the line $y \to (x, y)$. Applying the Gauss–Bonnet formula to R_x, we obtain the equation

$$\frac{\pi}{2}-\gamma(x)=-\int_{R_x} K(x,y)\eta(x,y)\,dx\,dy=-\int_0^x\left(\int_0^{y(s)} K(s,t)\eta(s,t)\,dt\right)ds \tag{11}$$

thus,

$$\gamma'(x)=\int_0^{y(x)} K(x,t)\eta(x,t)\,dt<0. \tag{12}$$

Now let M_1 be the hyperbolic plane of constant curvature -1, $\iota_1: TM_1\to TM_1$ the rotation of tangent spaces to M_1 by $\pi/2$ radians, $\gamma_1: R\to M_1, |\gamma_1'|=1$ a geodesic,

$$v_1(x,y)=\exp y\iota_1\gamma_1'(x), \qquad \eta_1=\langle\partial_x v_1, -\iota\,\partial_y v_1\rangle.$$

Then, of course,

$$\eta_1(x,y)=\cosh y,\ \partial_y\eta_1(x,y)=\sinh y.$$

Replace for the moment the inequality (2) by

$$-1<K\leq-\kappa^2<0 \tag{2'}$$

and consider the geodesics σ, τ in $\bar{M}$, M_1 respectively, defined by

$$\sigma(x)=v(x,y)(x)),\qquad \tau(x)=v_1(x,y_1(x))$$

and such that

$$y(0)=y_1(0)=d_0>0,\ y'(0)=y_1(0)=0.$$

We now wish to show that $y(x)\leq y_1(x)$ for all x where $y_1(x)$ is defined. One again only considers the case $x\geq 0$. Let $\gamma_1(x)$ be the analogous angle function for the curve $(x, y_1(x))$, and note that it suffices to show

$$\gamma_1(x)<\gamma'(x)\quad \text{for}\quad x\quad \text{where}\quad y_1(x)\quad \text{is defined.} \tag{13}$$

(13) clearly holds for x in a small neighborhood of 0. Thus if it is to fail we can find some number $x_0>0$, such that $y(x)\leq y_1(x)$ for $x\in[0,x_0]$, $\gamma_1(x)<\gamma'(x)$, $x\in[0,x_0)$ and $\gamma_1(x_0)=\gamma'(x_0)$. Hence

$$\int_0^{y(x_0)} -K(x_0,t)\eta(x_0,t)\,dt=\int_0^{y_1(x_0)}\cosh t\,dt.$$

But (2′) and the inequalities of the lemma show this to be impossible. Thus the domain of $y(x)$ is at least as large as that of $y_1(x)$.

This in turn implies that as in [2],

$$d_0 \geq \operatorname{arccosh}((\tanh(L_\gamma/2))^{-1}). \tag{14}$$

If we are given (2), then for every $\varepsilon > 0$, (2′) is valid for $-1-\varepsilon$ in places of -1. One writes the lower bound for d_0 in this normalization (cf. (13) below), and lets $\varepsilon \downarrow 0$. Then (11) remains valid under the assumption (2). Substituting (11) into (8), we obtain (R).

4. Conclusion

A close look at the estimate for d_0 shows that we only used the fact that the genus of M was ≥ 2 (this hypothesis is used in case (c). cf. [2]), and the assumption $-1 \leq K \leq 0$. We may therefore formulate the estimates as follows.

THEOREM. *Let M be a compact Riemann surface of genus ≥ 2 whose Gauss curvature satisfies*

$$-\delta^2 \leq K \leq 0 \tag{15}$$

for some constant $\delta > 0$. Then for any simple closed geodesic γ of length L_γ, the distance d_0 from γ to its focal cut locus is estimated by

$$d_0 \geq \frac{\operatorname{arccosh}((\tanh(\delta L_\gamma/2)^{-1})}{\delta}; \tag{16}$$

and if we have $\kappa \in [0, \delta]$ such that

$$-\delta^2 \leq K \leq -\kappa^2 \leq 0 \tag{17}$$

on all of M then the area A_γ is estimated by

$$A_\gamma \geq \frac{2L_\gamma}{\kappa} \sinh\left\{\frac{\operatorname{arccosh}((\tanh(\delta L_\gamma/2))^{-1})}{\delta/\kappa}\right\} \tag{18}$$

when $\kappa > 0$, *and*

$$A_\gamma \geq 2L_\gamma \frac{\operatorname{arccosh}\left((\tanh(\delta L_\gamma/2))^{-1}\right)}{\delta}$$

when $\kappa = 0$.

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