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## Pseudoconcave homogeneous surfaces

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### 0. Introduction

The purpose of this note is to list all pseudoconcave, 2-dimensional, homogeneous complex manifolds (*pseudoconcave homogeneous surfaces*). One would expect such manifolds to have compactifications as almost homogeneous surfaces. This turns out to be the case, but one can not proceed directly, because there *are* pseudoconcave surfaces which are not compactifiable [6].

It was noted in [5] that the only pseudoconcave Lie groups are the compact ones. With one type of exception this is also the case for pseudoconcave homogeneous surfaces: *Other than the compact homogeneous surfaces, the only pseudoconcave homogeneous surfaces are the Hirzebruch surfaces with their exceptional divisors removed.*

### 1. Historical remarks

For our purposes a homogeneous complex manifold is the quotient space of a connected, *complex* Lie group  $G$  by a closed subgroup  $H$ . Homogeneous algebraic surfaces were of course studied by the Italians (e.g. see [2]). The simply-connected, compact (not necessarily algebraic) homogeneous surfaces were classified by Wang [24]. Later on Tits [23] classified all compact homogeneous surfaces. Using Kodaira's work and going through the cases of transcendence degree of the function field as well as the possibilities for an Albanese fibration, Potters [20] gave a complete list of almost homogeneous compact surfaces. *Almost homogeneous* means that a connected, complex Lie group of automorphisms has an open orbit. In the case of compact Kähler manifolds there is a very good classification by Borel and Remmert [9]: Every compact homogeneous

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Kähler manifold is the product of a homogeneous projective rational manifold with a complex torus.

The purpose of the above remarks is to point out that *quite* a lot is known in the *compact* case. The non-compact case appears to be more complicated. Nevertheless in the “Stein direction” there are a number of interesting results. For example, when  $G$  is either reductive or nilpotent, Matsushima ([16], [17]) gave very good descriptions of the Stein quotients  $G/H$ . More generally, if  $G$  is reductive and  $G/H$  is holomorphically separable, then Barth and Otte [7] point out that the homogeneous space  $G/H$  is a Zariski open subset of an affine algebraic variety.

It is easy to see (e.g. [10]) that given a homogeneous space  $G/H$  there is a complex Lie group  $J$  containing  $H$  so that the fibration  $G/H \rightarrow G/J$  (the “separation” map) identifies *exactly* the points which the holomorphic functions identify (i.e.  $p, q \in G/H, p \sim q \Leftrightarrow f(p) = f(q) \forall f \in \mathcal{O}(G/H)$ ). Hence  $\mathcal{O}(G/J)$  separates points on  $G/J$ . If for example  $G$  is reductive, then the Barth–Otte theorem tells us that  $G/J$  is Zariski open in an affine algebraic variety. It would be reasonable to hope that  $\mathcal{O}(G/J) \simeq \mathbf{C}$ . This is in fact the case when  $G/J$  is Stein [10] or when  $G/H$  is itself a complex Lie group [18]. However, for example, one can construct homogeneous spaces which are simply  $\mathbf{C}^*$ -bundle spaces over  $\mathbf{C}^2 \setminus \{0, 0\}$  such that the holomorphic functions live on the base [7]. Even so, it is still quite reasonable to study homogeneous spaces  $G/H$  with  $\mathcal{O}(G/H) \simeq \mathbf{C}$  as well as the other extreme where  $\mathcal{O}(G/H)$  separates points. In some special cases (e.g. when  $G$  is nilpotent) one can classify  $G/H$  when  $\mathcal{O}(G/H) \simeq \mathbf{C}$  (see [11] and [8]). But even for solvable  $G$  we do not know what these manifolds are. Thus the stronger curvature assumption of *pseudoconcavity* seems warranted.

**DEFINITION.** A complex manifold  $X$  is called *pseudoconcave* if it contains a relatively compact open subset  $W$  with  $\partial W$  defined<sup>(3)</sup> by a  $C^2$ -function  $\varphi$  whose Levi form restricted to the complex tangent plane of  $\partial W$  at any  $p \in \partial W$  has at least one negative eigenvalue.

Pseudoconcave manifolds are close relatives of compact manifolds (e.g.  $\mathcal{O}(X) \simeq \mathbf{C}$ , and certain cohomology groups are finite dimensional). The reader can find the basic properties derived in [1]. Pseudoconcave quotient spaces arise quite naturally. For example, if the isotropy subgroup of a point  $p$  in a compact complex space  $X$  acts transitively on  $X \setminus \{p\}$ , then  $X \setminus \{p\}$  is a (strongly) pseudoconcave homogeneous space. Andreotti and Grauert [3] observed that the Siegel upper half plane modulo the modular group is pseudoconcave. More generally,

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<sup>3</sup> “Defined” means that  $W$  corresponds to  $\{\varphi < 0\}$ ,  $\partial W = \{\varphi = 0\}$  and  $d\varphi \neq 0$  on  $W$ .

any bounded symmetric domain modulo an arithmetic subgroup is pseudoconcave [20]. Applying finiteness and transcendence degree theorems ([1], [4]), these results have numerous interesting ramifications.

## 2. Statement of the result and a quick sketch of the proof

Our main result is the following:

**THEOREM.** *Let  $X$  be a pseudoconcave, 2-dimensional, complex manifold which admits a transitive complex Lie group of holomorphic automorphisms (i.e.  $X = G/H$ , where  $H$  is a closed subgroup of  $G$ ). Then  $X$  is one of the following:*

- (1) *A compact 2-dimensional torus;*
- (2) *The product of  $\mathbf{P}_1$  with an elliptic curve;*
- (3) *A homogeneous Hopf surface;*
- (4) *A homogeneous rational manifold (i.e.  $\mathbf{P}_1 \times \mathbf{P}_1$  or  $\mathbf{P}_2$ );*
- (5) *A Hirzebruch surface  $\Sigma_n$   $n = 1, 2, \dots$ , with its exceptional divisor<sup>(4)</sup> removed.*

The reader should note that the only non-compact possibilities occur in (5). Cases (1), (2) and (4) are well-known. The homogeneous Hopf surfaces are formed by dividing  $\mathbf{C}^2 \setminus \{0, 0\}$  by a discrete finitely generated abelian group of linear transformations  $\Gamma$ . The group  $\Gamma$  contains one copy of  $\mathbf{Z}$  which is generated by a dilation  $(z_1, z_2) \rightarrow (\alpha z_1, \alpha z_2)$ , where  $|\alpha| > 1$ . The torsion part is generated by diagonal matrices whose entries are roots of unity. These facts are routinely derivable from the explicit remarks of Kodaira ([14], p. 694 ff).

Our method of proof is to use the adjoint action of  $G$  on a Grassmann manifold. This identifies  $G/N_G(H^0)$  with the orbit of  $G$  on the point corresponding to the Lie algebra  $\mathfrak{h}$  of  $H$  sitting as a subspace of the Lie algebra  $\mathfrak{g}$  of  $G$ . This combined with the Albanese fibration was the main tool of Borel and Remmert [9]! Letting  $N := N_G(H^0)$ , this fibration  $G/H \rightarrow G/N$  allows us to study  $G/H$  in three separate cases depending on the dimension of  $G/N$ .

If  $G/H$  is 0-dimensional, then  $X$  is isomorphic to a complex Lie group  $S$  modulo a properly discontinuous subgroup  $\Gamma$ . Since in this case  $S$  must be 2-dimensional, it is quite easy to write down all possible such quotients, and, using

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<sup>4</sup>Every  $\Sigma_n$  contains a unique exceptional rational curve  $T_n$  with  $T_n \cdot T_n = -n$ . We note that  $\Sigma_1 \setminus T_1$  is the same as  $\mathbf{P}_2 \setminus \{\text{point}\}$ , so we redefine  $\Sigma_1$  to be  $\mathbf{P}_2$ .

the remark of [5], to show that the only possibility for a pseudoconcave  $S/\Gamma$  is a complex torus.

If  $G/N$  is 1-dimensional, then the fiber  $N/H^0/H/H^0$  is either  $\mathbf{C}$ ,  $\mathbf{C}^*$ , or an elliptic curve. The base  $G/N$  must be  $\mathbf{P}_1$ , because  $N$  contains the radical of  $G$ . If in this case  $G/H$  is a torus bundle over  $\mathbf{P}_1$ , then it is either a product or a Hopf surface. The  $\mathbf{C}^*$ -bundle is eliminated by the pseudoconcavity, and a  $\mathbf{C}$ -bundle compactifies to a Hirzebruch surface by adding an exceptional<sup>(5)</sup> rational curve at infinity.

If  $G/N$  is 2-dimensional, then  $G/H$  is a covering space over the pseudoconcave  $G/N$ . By using Andreotti's function field theorem, we compactify  $G/N$  to an algebraic variety  $V$  in the Grassmann manifold. We then extend the action of  $G$  to the minimal desingularization  $\tilde{V}$  of  $V$ . For pseudoconcavity reasons it is again easy to show that  $\tilde{V}$  is a Hirzebruch surface. In particular we see that  $G/N$  is simply connected! So  $G/H = G/N$  and the classification is finished.

### 3. Details of the proof

We follow the outline given above, introducing preparatory material as we need it. The basic tool is the action of  $G$  on the Grassmann ([23], [9]): Let  $\text{ad}: G \rightarrow \text{Hom}(\mathfrak{g}, \mathfrak{g})$  be the adjoint representation. For a closed subgroup  $H$  we consider its Lie algebra  $\mathfrak{h}$  to be a point in the Grassmann manifold  $G_{k,n}$  of  $k$ -planes in  $n$ -space, where  $k := \dim_{\mathbf{C}} H$  and  $n := \dim_{\mathbf{C}} G$ . For  $g \in G$  it is clear that  $\text{ad}(g)$  acts on  $G_{k,n}$ . We consider the map  $g \rightarrow \text{ad}(g)(\mathfrak{h})$  which sends  $G$  to its orbit on  $\mathfrak{h}$ . Let  $N := N_G(H^0)$  be the normalizer of the connected component  $H^0$  in  $G$ , then  $N$  is also a closed subgroup of  $G$  with  $G/N$  being canonically identified with this orbit. We have the fibration

$$G/H \xrightarrow{\alpha} G/N \rightarrow G_{k,n},$$

where the fiber  $N/H^0/H/H^0$  of  $\alpha$  is a complex Lie group modulo a discrete subgroup.

*Case 1.* Suppose  $\dim_{\mathbf{C}} G/N = 0$ . Then  $X$  can be realized as a simply-connected, complex Lie group  $S$  modulo a discrete subgroup  $\Gamma$ . If  $S$  is abelian,

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<sup>5</sup> We use the word exceptional for a curve which can be blown down, but the quotient may be singular. Differing slightly from [13], we identify  $\Sigma_1$  with  $\mathbf{P}_2$ .

then  $S/\Gamma$  is again an abelian Lie group which, since it is pseudoconcave, must be a torus [5]. If  $S$  is non-abelian, then it has  $\mathbf{C}^2$  as its underlying manifold with group structure given by

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2, e^{a_1}b_2 + b_1).$$

**PROPOSITION.** *Let  $S$  be a 2-dimensional, complex Lie group and let  $\Gamma$  be a discrete subgroup of  $S$ . Then  $S/\Gamma$  can be realized as an abelian Lie group modulo a finite subgroup.*

*Proof.* This is clear if  $S$  itself is abelian. Thus we may assume that  $S$  is  $\mathbf{C}^2$  with the above non-abelian structure. Let  $T := \{(2\pi in, b) \in \Gamma \mid n \in \mathbf{Z}\}$ . Then  $T$  is a normal subgroup of  $\Gamma$  which acts on the left as a group of translations. An elementary calculation shows that the derived group  $\Gamma^{(1)}$  is contained in  $T_2 := \{(0, b) \in \Gamma\}$  which acts discontinuously as a group of translations of the second variable. If there exists a non-trivial  $h = (0, c) \in T_2$ , then for  $g := (a, b) \in \Gamma$  we see that

$$g^{-n}hg^n = (0, e^{-na}c). \quad (*)$$

It is also of use to note that

$$g^n = (na, b(1 - e^{na})(1 - e^a)^{-1}). \quad (**)$$

If  $\Gamma$  is abelian, then we use a simple affine transformation (see [22]) to facilitate a description of  $S/\Gamma$ : Let  $g := (a, b) \in \Gamma$  and suppose that  $e^a \neq 1$ . Thus we may make the change of variables  $(z_1, z_2) \rightarrow (z_1, z_2 - b(1 - e^a)^{-1})$  so that in this new system (using the old letters)  $g(z_1, z_2) = (z_1 + a, e^a z_2)$ . If  $h$  is an arbitrary element of  $\Gamma$ , then  $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$ , where  $h = (c, d)$  in the old system and  $t(h) = d + b(1 - e^c)(1 - e^a)^{-1}$ . The statement that  $g$  and  $h$  commute amounts to  $(e^a - 1)t(h) = 0$ . Thus  $t(h) = 0$  for all  $h \in \Gamma$ . Hence the general element  $h \in \Gamma$  is defined by  $h(z_1, z_2) = (z_1 + c, e^c z_2)$ . The restriction of  $\Gamma$  to the  $z_1$ -axis is a faithful representation. Thus  $\Gamma$  is free abelian having rank at most 2.

Let  $q: \mathbf{C}^2 \rightarrow S/\Gamma$  be the quotient map. If  $\Gamma$  has rank 2, then  $q\{z_2 = 0\}$  is an elliptic curve  $E$ . But  $h(E)$  is also an elliptic curve for all  $h \in S$ . If  $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$ , then  $h\{z_2 = 0\} = \{(z_1, t(h)) \mid z_1 \in \mathbf{C}\}$ . Now  $q(z_1, t(h)) = q(z'_1, t(h))$  iff there is an  $f \in \Gamma$  such that  $f(z_1, t(h)) = (z'_1, t(h))$  (i.e.  $(z_1 + c, e^c t(h)) = (z'_1, t(h))$ ). In particular, for  $h(E) = qh\{z_2 = 0\}$  to be a torus it would be necessary for  $\Gamma$  to contain a rank 2 subgroup with  $e^c = 1$  for all  $h$  in that group. This is of course impossible. Even if  $\text{rank } \Gamma = 1$ , the same argument shows that  $\Gamma$  would have to contain a rank 1 subgroup  $\Gamma^\#$  whose elements act on  $\mathbf{C}^2$  by

$(z_1, z_2) \rightarrow (z_1 + 2\pi in, z_2)$ ,  $n \in \mathbb{Z}$ , and such that  $\Gamma/\Gamma^*$  is finite. Thus  $S/\Gamma$  is  $\mathbb{C}^* \times \mathbb{C}$  modulo a finite abelian group.

In summary we have shown above that if  $\Gamma$  is abelian then either  $S/\Gamma$  is  $\mathbb{C}^* \times \mathbb{C}$  modulo a finite group or  $e^a = 1$  for all  $g = (a, b) \in \Gamma$ . In the latter case  $\Gamma = T$ ,  $\Gamma$  acts as a lattice on  $\mathbb{C}^2$ , and  $S/\Gamma$  is obviously realizable as a complex Lie group,

It remains to consider the non-abelian case. In this situation we know that  $T_2$  contains a non-trivial element  $(0, c)$ . Thus if there exists some  $g = (a, b) \in \Gamma$  with  $a$  not a root of unity, then (\*) implies that  $g^n h g^{-n}$  (or  $g^{-n} h g^n$ ) has a cluster point. Since  $\Gamma$  is a discrete group, this is impossible and consequently for every  $g = (a, b) \in \Gamma$  we know that  $a = 2\pi iq$  for some  $q \in \mathbb{Q}$ . In particular this says that  $\Gamma/T$  is a torsion group. Considering two separate cases, we show that  $\Gamma/T$  is finite.

First, assume that  $\text{rank } T_2 = 1$ . So there exists  $c \in \mathbb{C}$  such that  $T_2 = \{(0, nc) \mid n \in \mathbb{Z}\}$ . Take  $h = (0, c) \in T_2$  and  $g = (2\pi iq, b) \in \Gamma$ . Then

$$ghg^{-1}h^{-1} = (0, c(e^{2\pi iq} - 1)) \in T_2.$$

This is only possible if  $q$  is an integer or an integer plus  $\frac{1}{2}$ . Therefore  $\Gamma/T$  is certainly finite in this case.

If  $\text{rank } T_2 = 2$ , then we note that given a class in  $\Gamma/T$  we can pick a representative  $g = (2\pi iq, b)$  where  $|b| < M$  and  $M$  is some a priori bound determined by  $\Gamma$ . Furthermore if  $Nq = m \in \mathbb{Z}$ , then  $g^N = (2\pi im, 0) \in T$ . Hence we may additionally pick a representative  $h = (2\pi ir, b)$  with  $|r| < |m|$ . If  $\Gamma/T$  were infinite then we could therefore find infinitely many different elements of  $\Gamma$  in a compact region of  $\mathbb{C}^2$ . This is contrary to the fact that  $\Gamma$  is discrete. Hence  $\Gamma/T$  is finite and  $S/\Gamma$  is the quotient by a finite group of automorphisms of the space  $\mathbb{C}^2/T$  which is obviously realizable as a complex Lie group.

**COROLLARY.** *Let  $X = S/\Gamma$  be a homogeneous space where  $S$  is a 2-dimensional complex Lie group and  $\Gamma$  is a discrete subgroup. If  $X$  is pseudoconcave, then it is a compact torus.*

*Proof.* By the above proposition,  $X$  has an abelian Lie group  $G$  as a finite cover. Thus  $G$  is also pseudoconcave. But [5] implies that  $G$  is a torus, and therefore  $X$  is likewise.

*Case 2.* Suppose  $\dim_{\mathbb{C}} G/N = 1$ . Then the fact that  $X$  has no non-constant holomorphic functions implies that  $\mathcal{O}(G/N) \cong \mathbb{C}$  and consequently  $G/N$  is a compact Riemann surface. We now apply a theorem of Borel and Remmert ([9], p. 435): The base  $G/N$  is rational (i.e. it is  $\mathbf{P}_1$ ) and  $N$  is connected. Consequently  $X$  is a bundle space over  $\mathbf{P}_1$  whose fiber is either  $\mathbb{C}$ ,  $\mathbb{C}^*$  or an elliptic curve. The  $\mathbb{C}^*$  case can be eliminated rather easily, because such bundles are classified by

their Chern numbers. If the Chern number is  $n > 0$  then the bundle has a non-trivial section  $s$ , and if  $z$  is a fiber coordinate, then  $sz^{-n}$  is a well-defined holomorphic function on  $X$ . If  $n < 0$ , then the dual bundle has a section  $s$  and  $sz^n$  does the job.

The case of an elliptic curve as a fiber is handled classically [23], being a product in the Kähler case and a homogeneous Hopf surface otherwise. It only remains to consider  $\mathbf{C}$ -fiber bundles over  $\mathbf{P}_1$ . Call such a bundle space  $E$ . We note that  $E$  has a natural compactification  $\bar{E}$  which comes from adding the section  $s$  at infinity. It is clear that the automorphisms of  $E$  extend to  $\bar{E}$ . Thus  $E$  is a compact, rational, almost homogeneous surface which is a  $\mathbf{P}_1$ -bundle over  $\mathbf{P}_1$ . Hence  $\bar{E}$  is a Hirzebruch surface, where we blow down the fixed set when its self intersection number is  $-1$ . This finishes *Case 2*.

*Case 3.* Suppose  $\dim_{\mathbf{C}} G/N = 2$ . Let  $W$  be the relatively compact set in  $G/H$  which displays its pseudoconcavity, and let  $\Omega := \alpha(W)$ , where  $\alpha : G/H \rightarrow G/N$  is the fibration map. Certainly  $\partial\Omega \subset \alpha(\partial W)$ . Thus  $\Omega$  is pseudoconcave in the following more general sense: Given  $p \in \partial\Omega$ , there is an open neighborhood  $U$  of  $p$  in  $G/N$  so that every function holomorphic on  $U \cap \Omega$  extends to a function holomorphic on  $U$ . One finds such a  $U$  by taking  $q \in \partial W$  with  $\alpha(q) = p$ . Then one constructs a neighborhood  $\tilde{U}$  of  $q$  in  $G/H$  which is small enough so that  $\alpha|_{\tilde{U}}$  is biholomorphic, and such that  $\tilde{U}$  has the extendibility property required of  $U$ . Finally one defines  $U$  as  $\alpha(\tilde{U})$ . For a normal complex space  $Y$  containing a relatively compact open set having this more general pseudoconcave property, Andreotti [1] proved the following:

*The field  $\mathfrak{R}(Y)$  of meromorphic functions on  $Y$  is an algebraic function field having transcendence degree  $t(Y)$  at most  $\dim Y$  over  $\mathbf{C}$ .*

It follows that we have Andreotti's transcendence degree theorem for  $Y := G/N$ . But  $Y$  is contained in  $G_{k,n}$  which is in turn contained in some  $\mathbf{P}_N$ . We now follow a standard proof of Chow's Theorem: Let  $V$  be the smallest compact algebraic variety which contains  $Y$  (i.e.  $V$  is the intersection of all such varieties which contain  $Y$ ). It is clear that  $V$  is irreducible. Suppose  $k := \dim_{\mathbf{C}} V > 2$ . Then  $\mathfrak{R}(V) = \mathbf{C}(f_1, \dots, f_k)[g]$ , where  $f_1, \dots, f_k$  are algebraically independent rational functions on  $V$  and  $g$  is some algebraically dependent function of maximal degree. Since  $t(Y) = 2$ , there exists a polynomial  $P \in \mathbf{C}[X_1, \dots, X_k]$  so that  $P(f_1, \dots, f_k) \equiv 0$  on  $Y$ . But  $f_1, \dots, f_k$  are algebraically independent on  $V$ . So  $V \cap \{P(f_1, \dots, f_k) = 0\}$  is an algebraic subvariety of dimension  $k - 1$  which contains  $Y$ . This contradicts the minimality of  $V$  and implies that  $\dim_{\mathbf{C}} V = 2$ . Thus *there is an irreducible, 2-dimensional, compact, algebraic subvariety of  $G_{k,n}$  which contains  $G/N$  as an open subset.*

Now  $G$  acts on the whole Grassmann manifold. So for  $g \in G$  it follows that  $g(V)$  is an algebraic subvariety of  $G_{k,n}$ . But  $g(V) \cap V$  contains  $Y$  as an open subset. Consequently  $g(V) = V$ , and we have extended the action of  $G$  to all of  $V$ . We now need the following:

LEMMA. *Let  $g$  be a biholomorphic map  $g: V \rightarrow V$  of a 2-dimensional analytic space. Let  $\rho: \tilde{V} \rightarrow V$  be the minimal desingularization of  $V$ . Then there is a unique biholomorphic map  $\tilde{g}: \tilde{V} \rightarrow \tilde{V}$  so that*

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{g}} & \tilde{V} \\ \rho \downarrow & & \downarrow \rho \\ V & \xrightarrow{g} & V \end{array}$$

*is commutative.*

*Proof.* The maps  $\rho: \tilde{V} \rightarrow V$  and  $g \circ \rho: \tilde{V} \rightarrow V$  are both minimal resolutions of the singularities of  $V$ . Thus the lemma follows directly from the uniqueness of a minimal resolution in dimension 2 (e.g. see [15]).

Hence every automorphism  $g$  of  $V$  is uniquely liftable to an automorphism  $\tilde{g}$  of the minimal desingularization  $\tilde{V}$  of  $V$ . Thus  $G/N$  can be compactified to a non-singular algebraic surface  $\tilde{V}$  where the group  $G$  acts and renders it almost homogeneous.

It is easy to see that an open orbit is the complement of a proper analytic subset (e.g. [21]). Thus  $Y = G/N$  is Zariski open in  $\tilde{V}$ . Let  $S := \tilde{V} \setminus Y$  be the fixed set. The only algebraic, almost homogeneous surfaces  $\tilde{V}$  such that  $\mathcal{O}(\tilde{V} \setminus S) \simeq \mathbb{C}$  are rational surfaces. One can see this by looking at Potter's classification [21] where the only case which needs checking is that of the  $\mathbf{P}_1$ -bundles over elliptic curves. These are so explicitly described that constructing non-constant holomorphic functions on the complement of the fixed set is a triviality. For example in Type II (see p. 252 of [21]), the function  $f([w_1:w_2], z) := \exp(2\pi i w_1 (w_2 b)^{-1})$  is well-defined and holomorphic on the complement of  $S$ .

In summary we have shown that  $G/N$  compactifies to a Hirzebruch surface  $\Sigma_n$  by adding the fixed rational curve  $T_n$  which has self-intersection number  $-n$  and (possibly) finitely many more points. The group  $G$ , assuming that it is acting effectively, is therefore contained in the stabilizer of the set of the finitely many points which were added.

If  $S = \emptyset$ , then  $V = G/N$  is compact rational and is either  $\mathbf{P}_1 \times \mathbf{P}_1$  or  $\mathbf{P}_2$ . Every compact rational almost homogeneous surface can be blown down to some  $\Sigma_n$ . Thus the minimality of  $\tilde{V}$  implies that it is a Hirzebruch surface. The only

exceptional curve in  $\Sigma_n$  is  $T_n$ . Hence, when it is not empty,  $S$  consists of  $T_n$  plus isolated points. However, by a very general theorem of Oeljeklaus [19], if  $S$  contains an isolated point, then  $\tilde{V} = \mathbf{P}_2$ .

It is shown in [13] that  $\Sigma_{2n} \setminus T_{2n}$  is homeomorphic to  $S^2 \times (S^2 \setminus \{p\})$  and  $\Sigma_{2n+1} \setminus T_{2n+1}$  is homeomorphic to  $\mathbf{P}_2 \setminus \{p\}$ . Thus  $G/N$  is simply-connected. Hence  $G/H = G/N$  and our homogeneous space is in class (4) or (5) of the theorem.

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**Added in proof.**

The Proposition in Case 1 is false as stated; a corrected version appears in the thesis of J. Snow. The point is to eliminate the parallelizable case  $S/\Gamma$  in the classification. Since  $S$  is solvable, this follows immediately from a general theorem of Huckleberry and D. Snow (Three fibrations for pseudoconcave homogeneous manifolds, to appear). However as  $S$  is 2-dimensional, one may eliminate this case directly. Since  $\mathcal{O}(S/\Gamma) = \mathbf{C}$ , there is a proper normal subgroup  $L$  of  $S$  with closed orbits in  $S/\Gamma$ , yielding the homogeneous fibration  $S/\Gamma \rightarrow S/L \cdot \Gamma = T$  [8, Satz 3.2]. Clearly  $T$  is a torus. One checks quite easily (see later in the paper) that there are no pseudoconcave homogeneous  $\mathbf{C}$  or  $\mathbf{C}^*$  bundles over an elliptic curve. Thus  $S/\Gamma$  must be compact and a fortiori  $S$  is abelian.