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Pseudoconcave homogeneous surfaces

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0. Introduction

The purpose of this note is to list all pseudoconcave, 2-dimensional, homogeneous complex manifolds (*pseudoconcave homogeneous surfaces*). One would expect such manifolds to have compactifications as almost homogeneous surfaces. This turns out to be the case, but one can not proceed directly, because there *are* pseudoconcave surfaces which are not compactifiable [6].

It was noted in [5] that the only pseudoconcave Lie groups are the compact ones. With one type of exception this is also the case for pseudoconcave homogeneous surfaces: *Other than the compact homogeneous surfaces, the only pseudoconcave homogeneous surfaces are the Hirzebruch surfaces with their exceptional divisors removed.*

1. Historical remarks

For our purposes a homogeneous complex manifold is the quotient space of a connected, *complex* Lie group G by a closed subgroup H. Homogeneous algebraic surfaces were of course studied by the Italians (e.g. see [2]). The simply-connected, compact (not necessarily algebraic) homogeneous surfaces were classified by Wang [24]. Later on Tits [23] classified all compact homogeneous surfaces. Using Kodaira's work and going through the cases of transcendence degree of the function field as well as the possibilities for an Albanese fibration, Potters [20] gave a complete list of almost homogeneous compact surfaces. *Almost homogeneous* means that a connected, complex Lie group of automorphisms has an open orbit. In the case of compact Kähler manifolds there is a very good classification by Borel and Remmert [9]: Every compact homogeneous

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Kähler manifold is the product of a homogeneous projective rational manifold with a complex torus.

The purpose of the above remarks is to point out that *quite* a lot is known in the *compact* case. The non-compact case appears to be more complicated. Nevertheless in the "Stein direction" there are a number of interesting results. For example, when G is either reductive or nilpotent, Matsushima ([16], [17]) gave very good descriptions of the Stein quotients G/H. More generally, if G is reductive and G/H is holomorphically separable, then Barth and Otte [7] point out that the homogeneous space G/H is a Zariski open subset of an affine algebraic variety.

It is easy to see (e.g. [10]) that given a homogeneous space G/H there is a complex Lie group J containing H so that the fibration $G/H \to G/J$ (the "separation" map) identifies *exactly* the points which the holomorphic functions identify (i.e. $p, q \in G/H$, $p \sim q \Leftrightarrow f(p) = f(q)\ \forall f \in \mathcal{O}(G/H)$). Hence $\mathcal{O}(G/J)$ separates points on G/J. If for example G is reductive, then the Barth–Otte theorem tells us that G/J is Zariski open in an affine algebraic variety. It would be reasonable to hope that $\mathcal{O}(J/H) \simeq \mathbf{C}$. This is in fact the case when G/J is Stein [10] or when G/H is itself a complex Lie group [18]. However, for example, one can construct homogeneous spaces which are simply $\mathbf{C}^*$-bundle spaces over $\mathbf{C}^2\backslash\{0, 0\}$ such that the holomorphic functions live on the base [7]. Even so, it is still quite reasonable to study homogeneous spaces G/H with $\mathcal{O}(G/H) \simeq \mathbf{C}$ as well as the other extreme where $\mathcal{O}(G/H)$ separates points. In some special cases (e.g. when G is nilpotent) one can classify G/H when $\mathcal{O}(G/H) \simeq \mathbf{C}$(see [11] and [8]). But even for solvable G we do not know what these manifolds are. Thus the stronger curvature assumption of *pseudoconcavity* seems warranted.

DEFINITION. A complex manifold X is called *pseudoconcave* if it contains a relatively compact open subset W with ∂W defined[(3)] by a C^2-function φ whose Levi form restricted to the complex tangent plane of ∂W at any $p \in \partial W$ has at least one negative eigenvalue.

Pseudoconcave manifolds are close relatives of compact manifolds (e.g. $\mathcal{O}(X) \simeq \mathbf{C}$, and certain cohomology groups are finite dimensional). The reader can find the basic properties derived in [1]. Pseudoconcave quotient spaces arise quite naturally. For example, if the isotropy subgroup of a point p in a compact complex space X acts transitively on $X\backslash\{p\}$, then $X\backslash\{p\}$ is a (strongly) pseudoconcave homogeneous space. Andreotti and Grauert [3] observed that the Siegel upper half plane modulo the modular group is pseudoconcave. More generally,

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[3] "Defined" means that W corresponds to $\{\varphi < 0\}$, $\partial W = \{\varphi = 0\}$ and $d\varphi \neq 0$ on W.

any bounded symmetric domain modulo an arithmetic subgroup is pseudoconcave [20]. Applying finiteness and transcendence degree theorems ([1], [4]), these results have numerous interesting ramifications.

2. Statement of the result and a quick sketch of the proof

Our main result is the following:

THEOREM. *Let X be a pseudoconcave, 2-dimensional, complex manifold which admits a transitive complex Lie group of holomorphic automorphisms (i.e. $X = G/H$, where H is a closed subgroup of G). Then X is one of the following:*

(1) *A compact 2-dimensional torus;*
(2) *The product of $\mathbf{P}_1$ with an elliptic curve;*
(3) *A homogeneous Hopf surface;*
(4) *A homogeneous rational manifold (i.e. $\mathbf{P}_1 \times \mathbf{P}_1$ or $\mathbf{P}_2$);*
(5) *A Hirzebruch surface Σ_n $n = 1, 2, \ldots$, with its exceptional divisor[(4)] removed.*

The reader should note that the only non-compact possibilities occur in (5). Cases (1), (2) and (4) are well-known. The homogeneous Hopf surfaces are formed by dividing $\mathbf{C}^2 \backslash \{0, 0\}$ by a discrete finitely generated abelian group of linear transformations Γ. The group Γ contains one copy of $\mathbf{Z}$ which is generated by a dilation $(z_1, z_2) \to (\alpha z_1, \alpha z_2)$, where $|\alpha| > 1$. The torsion part is generated by diagonal matrices whose entries are roots of unity. These facts are routinely derivable from the explicit remarks of Kodaira ([14], p. 694 ff).

Our method of proof is to use the adjoint action of G on a Grassmann manifold. This identifies $G/N_G(H^0)$ with the orbit of G on the point corresponding to the Lie algebra h of H sitting as a subspace of the Lie algebra g of G. This combined with the Albanese fibration was the main tool of Borel and Remmert [9]! Letting $N := N_G(H^0)$, this fibration $G/H \to G/N$ allows us to study G/H in three separate cases depending on the dimension of G/N.

If G/H is 0-dimensional, then X is isomorphic to a complex Lie group S modulo a properly discontinuous subgroup Γ. Since in this case S must be 2-dimensional, it is quite easy to write down all possible such quotients, and, using

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[4] Every Σ_n contains a unique exceptional rational curve T_n with $T_n \cdot T_n = -n$. We note that $\Sigma_1 \backslash T_1$ is the same as $\mathbf{P}_2 \backslash \{\text{point}\}$, so we redefine Σ_1 to be $\mathbf{P}_2$.

the remark of [5], to show that the only possibility for a pseudoconcave S/Γ is a complex torus.

If G/N is 1-dimensional, then the fiber $N/H^0/H/H^0$ is either $\mathbf{C}$, $\mathbf{C}^*$, or an elliptic curve. The base G/N must be $\mathbf{P}_1$, because N contains the radical of G. If in this case G/H is a torus bundle over $\mathbf{P}_1$, then it is either a product or a Hopf surface. The $\mathbf{C}^*$-bundle is eliminated by the pseudoconcavity, and a $\mathbf{C}$-bundle compactifies to a Hirzebruch surface by adding an exceptional(5) rational curve at infinity.

If G/N is 2-dimensional, then G/H is a covering space over the pseudoconcave G/N. By using Andreotti's function field theorem, we compactify G/N to an algebraic variety V in the Grassmann manifold. We then extend the action of G to the minimal desingularization $\tilde{V}$ of V. For pseudoconcavity reasons it is again easy to show that $\tilde{V}$ is a Hirzebruch surface. In particular we see that G/N is simply connected! So $G/H=G/N$ and the classification is finished.

3. Details of the proof

We follow the outline given above, introducing preparatory material as we need it. The basic tool is the action of G on the Grassmann ([23], [9]): Let $\mathrm{ad}: G \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ be the adjoint representation. For a closed subgroup H we consider its Lie algebra $\mathfrak{h}$ to be a point in the Grassmann manifold $G_{k,n}$ of k-planes in n-space, where $k := \dim_{\mathbf{C}} H$ and $n := \dim_{\mathbf{C}} G$. For $g \in G$ it is clear that $\mathrm{ad}(g)$ acts on $G_{k,n}$. We consider the map $g \to \mathrm{ad}(g)(\mathfrak{h})$ which sends G to its orbit on $\mathfrak{h}$. Let $N := N_G(H^0)$ be the normalizer of the connected component H^0 in G, then N is also a closed subgroup of G with G/N being canonically identified with this orbit. We have the fibration

$$G/H \xrightarrow{\alpha} G/N \to G_{k,n},$$

where the fiber $N/H^0/H/H^0$ of α is a complex Lie group modulo a discrete subgroup.

Case 1. Suppose $\dim_{\mathbf{C}} G/N = 0$. Then X can be realized as a simply-connected, complex Lie group S modulo a discrete subgroup Γ. If S is abelian,

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[5] We use the word exceptional for a curve which can be blown down, but the quotient may be singular. Differing slightly from [13], we identify Σ_1 with $\mathbf{P}_2$.

then S/Γ is again an abelian Lie group which, since it is pseudoconcave, must be a torus [5]. If S is non-abelian, then it has $\mathbf{C}^2$ as its underlying manifold with group structure given by

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2, e^{a_1}b_2 + b_1).$$

PROPOSITION. *Let S be a 2-dimensional, complex Lie group and let Γ be a discrete subgroup of S. Then S/Γ can be realized as an abelian Lie group modulo a finite subgroup.*

Proof. This is clear if S itself is abelian. Thus we may assume that S is $\mathbf{C}^2$ with the above non-abelian structure. Let $T := \{(2\pi in, b) \in \Gamma \mid n \in \mathbf{Z}\}$. Then T is a *normal* subgroup of Γ which acts on the left as a group of *translations*. An elementary calculation shows that the derived group $\Gamma^{(1)}$ is contained in $T_2 := \{(0, b) \in \Gamma\}$ which acts discontinuously as a group of translations of the second variable. If there exists a non-trivial $h = (0, c) \in T_2$, then for $g := (a, b) \in \Gamma$ we see that

$$g^{-n}hg^n = (0, e^{-na}c). \tag{*}$$

It is also of use to note that

$$g^n = (na, b(1-e^{na})(1-e^a)^{-1}). \tag{**}$$

If Γ is abelian, then we use a simple affine transformation (see [22]) to facilitate a description of S/Γ: Let $g := (a, b) \in \Gamma$ and suppose that $e^a \neq 1$. Thus we may make the change of variables $(z_1, z_2) \to (z_1, z_2 - b(1-e^a)^{-1})$ so that in this new system (using the old letters) $g(z_1, z_2) = (z_1 + a, e^a z_2)$. If h is an arbitrary element of Γ, then $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$, where $h = (c, d)$ in the old system and $t(h) = d + b(1 - e^c)(1 - e^a)^{-1}$. The statement that g and h commute amounts to $(e^a - 1)t(h) = 0$. Thus $t(h) = 0$ for all $h \in \Gamma$. Hence the general element $h \in \Gamma$ is defined by $h(z_1, z_2) = (z_1 + c, e^c z_2)$. The restriction of Γ to the z_1-axis is a faithful representation. Thus Γ is free abelian having rank at most 2.

Let $q: \mathbf{C}^2 \to S/\Gamma$ be the quotient map. If Γ has rank 2, then $q\{z_2 = 0\}$ is an elliptic curve E. But $h(E)$ is also an elliptic curve for all $h \in S$. If $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$, then $h\{z_2 = 0\} = \{(z_1, t(h)) \mid z_1 \in \mathbf{C}\}$. Now $q(z_1, t(h)) = q(z_1', t(h))$ iff there is an $f \in \Gamma$ such that $f(z_1, t(h)) = (z_1', t(h))$ (i.e. $(z_1 + c, e^c t(h)) = (z_1', t(h))$). In particular, for $h(E) = qh\{z_2 = 0\}$ to be a torus it would be necessary for Γ to contain a rank 2 subgroup with $e^c = 1$ for all h in that group. This is of course impossible. Even if rank $\Gamma = 1$, the same argument shows that Γ would have to contain a rank 1 subgroup $\Gamma^{\#}$ whose elements act on $\mathbf{C}^2$ by

$(z_1, z_2) \to (z_1 + 2\pi in, z_2)$, $n \in \mathbb{Z}$, and such that $\Gamma/\Gamma^{\#}$ is finite. Thus S/Γ is $\mathbf{C}^* \times \mathbf{C}$ modulo a finite abelian group.

In summary we have shown above that if Γ is abelian then either S/Γ is $\mathbf{C}^* \times \mathbf{C}$ modulo a finite group or $e^a = 1$ for all $g = (a, b) \in \Gamma$. In the latter case $\Gamma = T$, Γ acts as a lattice on $\mathbf{C}^2$, and S/Γ is obviously realizable as a complex Lie group,

It remains to consider the non-abelian case. In this situation we know that T_2 contains a non-trivial element $(0, c)$. Thus if there exists some $g = (a, b) \in \Gamma$ with a not a root of unity, then (*) implies that $g^n h g^{-n}$ (or $g^{-n} h g^n$) has a cluster point. Since Γ is a discrete group, this is impossible and consequently for every $g = (a, b) \in \Gamma$ we know that $a = 2\pi iq$ for some $q \in \mathbf{Q}$. In particular this says that Γ/T is a torsion group. Considering two separate cases, we show that Γ/T is finite.

First, assume that rank $T_2 = 1$. So there exists $c \in \mathbf{C}$ such that $T_2 = \{(0, nc) \mid n \in \mathbf{Z}\}$. Take $h = (0, c) \in T_2$ and $g = (2\pi iq, b) \in \Gamma$. Then

$$ghg^{-1}h^{-1} = (0, c(e^{2\pi iq} - 1)) \in T_2.$$

This is only possible if q is an integer or an integer plus $\frac{1}{2}$. Therefore Γ/T is certainly finite in this case.

If rank $T_2 = 2$, then we note that given a class in Γ/T we can pick a representative $g = (2\pi iq, b)$ where $|b| < M$ and M is some a priori bound determined by Γ. Furthermore if $Nq = m \in \mathbf{Z}$, then $g^N = (2\pi im, 0) \in T$. Hence we may additionally pick a representative $h = (2\pi ir, b)$ with $|r| < |m|$. If Γ/T were infinite then we could therefore find infinitely many different elements of Γ in a compact region of $\mathbf{C}^2$. This is contrary to the fact that Γ is discrete. Hence Γ/T is finite and S/Γ is the quotient by a finite group of automorphisms of the space $\mathbf{C}^2/T$ which is obviously realizable as a complex Lie group.

COROLLARY. *Let $X = S/\Gamma$ be a homogeneous space where S is a 2-dimensional complex Lie group and Γ is a discrete subgroup. If X is pseudoconcave, then it is a compact torus.*

Proof. By the above proposition, X has an abelian Lie group G as a *finite* cover. Thus G is also pseudoconcave. But [5] implies that G is a torus, and therefore X is likewise.

Case 2. Suppose $\dim_{\mathbf{C}} G/N = 1$. Then the fact that X has no non-constant holomorphic functions implies that $\mathcal{O}(G/N) \simeq \mathbf{C}$ and consequently G/N is a compact Riemann surface. We now apply a theorem of Borel and Remmert ([9], p. 435): The base G/N is *rational* (i.e. it is $\mathbf{P}_1$) and N is connected. Consequently X is a bundle space over $\mathbf{P}_1$ whose fiber is either $\mathbf{C}$, $\mathbf{C}^*$ or an elliptic curve. The $\mathbf{C}^*$ case can be eliminated rather easily, because such bundles are classified by their Chern numbers. If the Chern number is $n>0$ then the bundle has a non-trivial section s, and if z is a fiber coordinate, then sz^{-n} is a well-defined holomorphic function on X. If $n<0$, then the dual bundle has a section s and sz^n does the job.

The case of an elliptic curve as a fiber is handled classically [23], being a product in the Kähler case and a homogeneous Hopf surface otherwise. It only remains to consider $\mathbf{C}$-fiber bundles over $\mathbf{P}_1$. Call such a bundle space E. We note that E has a natural compactification $\bar{E}$ which comes from adding the section s at infinity. It is clear that the automorphisms of E extend to $\bar{E}$. Thus E is a compact, rational, almost homogeneous surface which is a $\mathbf{P}_1$-bundle over $\mathbf{P}_1$. Hence $\bar{E}$ is a Hirzebruch surface, where we blow down the fixed set when its self intersection number is -1. This finishes *Case* 2.

Case 3. Suppose $\dim_{\mathbf{C}} G/N=2$. Let W be the relatively compact set in G/H which displays its pseudoconcavity, and let $\Omega := \alpha(W)$, where $\alpha: G/H \to G/N$ is the fibration map. Certainly $\partial\Omega \subset \alpha(\partial W)$. Thus Ω is pseudoconcave in the following more general sense: Given $p \in \partial\Omega$, there is an open neighborhood U of p in G/N so that every function holomorphic on $U \cap \Omega$ extends to a function holomorphic on U. One finds such a U by taking $q \in \partial W$ with $\alpha(q)=p$. Then one constructs a neighborhood $\tilde{U}$ of q in G/H which is small enough so that $\alpha \mid \tilde{U}$ is biholomorphic, and such that $\tilde{U}$ has the extendibility property required of U. Finally one defines U as $\alpha(\tilde{U})$. For a normal complex space Y containing a relatively compact open set having this more general pseudoconcave property, Andreotti [1] proved the following:

The field $\mathfrak{K}(Y)$ of meromorphic functions on Y is an algebraic function field having transcendence degree $t(Y)$ at most dim Y *over* $\mathbf{C}$.

It follows that we have Andreotti's transcendence degree theorem for $Y := G/N$. But Y is contained in $G_{k,n}$ which is in turn contained in some $\mathbf{P}_N$. We now follow a standard proof of Chow's Theorem: Let V be the smallest compact algebraic variety which contains Y (i.e. V is the intersection of all such varieties which contain Y). It is clear that V is irreducible. Suppose $k := \dim_{\mathbf{C}} V > 2$. Then $\mathfrak{K}(V)=\mathbf{C}(f_1, \ldots, f_k)[g]$, where $f_1, \ldots, f_k$ are algebraically independent rational functions on V and g is some algebraically dependent function of maximal degree. Since $t(Y)=2$, there exists a polynomial $P \in \mathbf{C}[X_1, \ldots, X_k]$ so that $P(f_1, \ldots, f_k) \equiv 0$ on Y. But $f_1, \ldots, f_k$ are algebraically independent on V. So $V \cap \{P(f_1, \ldots, f_k)=0\}$ is an algebraic subvariety of dimension $k-1$ which contains Y. This contradicts the minimality of V and implies that $\dim_{\mathbf{C}} V=2$. Thus *there is an irreducible, 2-dimensional, compact, algebraic subvariety of $G_{k,n}$ which contains G/N as an open subset.*

Now G acts on the whole Grassmann manifold. So for $g \in G$ it follows that $g(V)$ is an algebraic subvariety of $G_{k,n}$. But $g(V) \cap V$ contains Y as an open subset. Consequently $g(V) = V$, and we have extended the action of G to all of V. We now need the following:

LEMMA. *Let g be a biholomorphic map $g: V \to V$ of a 2-dimensional analytic space. Let $\rho: \tilde{V} \to V$ be the minimal desingularization of V. Then there is a unique biholomorphic map $\tilde{g}: \tilde{V} \to \tilde{V}$ so that*

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{g}} & \tilde{V} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ V & \xrightarrow[g]{} & V \end{array}$$

is commutative.

Proof. The maps $\rho: \tilde{V} \to V$ and $g \circ \rho: \tilde{V} \to V$ are both minimal resolutions of the singularities of V. Thus the lemma follows directly from the uniqueness of a minimal resolution in dimension 2 (e.g. see [15]).

Hence every automorphism g of V is ur iquely liftable to an automorphism $\tilde{g}$ of the minimal desingularization $\tilde{V}$ of V. Thus *G/N can be compactified to a non-singular algebraic surface $\tilde{V}$ where the group G acts and renders it almost homogeneous.*

It is easy to see that an open orbit is the complement of a proper analytic subset (e.g. [21]). Thus $Y = G/N$ is Zariski open in $\tilde{V}$. Let $S := \tilde{V} \setminus Y$ be the fixed set. The only algebraic, almost homogeneous surfaces $\tilde{V}$ such that $\mathcal{O}(\tilde{V} \setminus S) \simeq \mathbf{C}$ are rational surfaces. One can see this by looking at Potter's classification [21] where the only case which needs checking is that of the $\mathbf{P}_1$-bundles over elliptic curves. These are so explicitly described that constructing non-constant holomorphic functions on the complement of the fixed set is a triviality. For example in *Type* II (see p. 252 of [21]), the function $f([w_1 : w_2], z) := \exp(2\pi i w_1 (w_2 b)^{-1})$ is well-defined and holomorphic on the complement of S.

In summary we have shown that G/N compactifies to a Hirzebruch surface Σ_n by adding the fixed rational curve T_n which has self-intersection number $-n$ and (possibly) finitely many more points. The group G, assuming that it is acting effectively, is therefore contained in the stabilizer of the set of the finitely many points which were added.

If $S = \varnothing$, then $V = G/N$ is compact rational and is either $\mathbf{P}_1 \times \mathbf{P}_1$ or $\mathbf{P}_2$. Every compact rational almost homogeneous surface can be blown down to some Σ_n. Thus the minimality of $\tilde{V}$ implies that it is a Hirzebruch surface. The only exceptional curve in Σ_n is T_n. Hence, when it is not empty, S consists of T_n plus isolated points. However, by a very general theorem of Oeljeklaus [19], if S contains an isolated point, then $\tilde{V} = \mathbf{P}_2$.

It is shown in [13] that $\Sigma_{2n} \backslash T_{2n}$ is homeomorphic to $S^2 \times (S^2 \backslash \{p\})$ and $\Sigma_{2n+1} \backslash T_{2n+1}$ is homeomorphic to $\mathbf{P}_2 \backslash \{p\}$. Thus G/N is simply-connected. Hence $G/H = G/N$ and our homogeneous space is in class (4) or (5) of the theorem.

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Added in proof.
The Proposition in Case 1 is false as stated; a corrected version appears in the thesis of J. Snow. The point is to eliminate the parallelizable case S/Γ in the classification. Since S is solvable, this follows immediately from a general theorem of Huckleberry and D. Snow (Three fibrations for pseudoconcave homogeneous manifolds, to appear). However as S is 2-dimensional, one may eliminate this case directly. Since $\mathcal{O}(S/\Gamma)=\mathbf{C}$, there is a proper normal subgroup L of S with closed orbits in S/Γ, yielding the homogeneous fibration $S/\Gamma \to S/L \cdot \Gamma = T$ [8, Satz 3.2]. Clearly T is a torus. One checks quite easily (see later in the paper) that there are no pseudoconcave homogeneous $\mathbf{C}$ or $\mathbf{C}^*$ bundles over an elliptic curve. Thus S/Γ must be compact and a fortiori S is abelian.