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Quadruple points of 3-manifolds in S^4

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A folk theorem (see Banchoff [B]) says that the number of normally triple points of a closed surface normally immersed in 3-space is congruent modulo two to its Euler characteristic. In general, a normal immersion of a compact n -manifold in an $n + 1$ -manifold will have a finite number, θ , of $(n + 1)$ -tuple points. θ , taken mod 2, is well defined under bordism of both the immersion and ambient manifold. An attractive place to try to evaluate θ is on the abelian group, “(oriented bordism of immersed n -manifolds in S^{n+1} , connected sum)” = B_n , since B_n is naturally isomorphic to the stable homotopy group π_n . Counting $(n + 1)$ -tuple points determines a homomorphism, $\theta_n : \pi_n \rightarrow Z_2$. The figure eight immersion of a circle shows that θ_1 is an isomorphism; Banchoff’s proof shows that θ_2 is the zero map; the main result of this paper is that θ_3 is the unique epimorphism $\pi_3 \simeq Z_{24} \rightarrow Z_2$. Thus, we show that a (actually any) oriented 3-manifold may be generically immersed in S^4 with an odd number of quadruple points. Like Smale’s inversion of S^2 , our proof is abstract and does not yield an example.

A pleasing conjecture is that θ_n is the stable Hopf invariant for all n .

§1. B_n is the n^{th} Stable Stem

All terminology will be smooth; the spheres, S^i , are given a standard orientation. Let X be a compact oriented $n + 1$ -manifold with boundary components divided into ∂^-X and ∂^+X . $(X; \partial^-X, \partial^+X) \xrightarrow{f} (S^{n+1} \times [-1, +1]; S^{n+1} \times -1, S^{n+1} \times 1)$ is called an immersed bordism between f/∂^-X and f/∂^+X if f is a relative immersion. Let B_n be the set of immersions, g , of compact oriented n -manifolds, M , modulo the equivalence relation of immersed bordism. B_n is a group under connected sum of ambient spheres away from the immersions.

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Since $\nu M \xrightarrow{g} S^{n+1}$ is trivialized by the orientations, g determines a trivialization of $\tau(X) \oplus \varepsilon^1$. According to Smale-Hirsh theory immersions exist (and are unique up to regular homotopy) which induce arbitrary trivializations of $\tau(M) \oplus \varepsilon^1$ and $\tau(X) \oplus \varepsilon^1$. Consequently $B_n \cong \{\text{trivializations of } \tau(M) \oplus \varepsilon^1\} / \{\text{trivializations which extend to trivializations of } \tau(X) \oplus \varepsilon^1, \text{ where } \partial X = M\}$. The Pontryagin-Thom construction determines a homomorphism $i_n : B_n \rightarrow \pi_n$.

Since $\pi_i(S0, S0(n+1)) \cong 0 \ i \leq n$, a stable trivialization of ν_M determines a trivialization of $\tau(M) \oplus \varepsilon^1$; so i_n is epic. Since $\pi_i(S0, S0(n+2)) \cong 0 \ i \leq n+1$, a stable trivialization of ν_X determines a trivialization of $\tau(X) \oplus \varepsilon^1$; so i is monic.

THEOREM 1. $B_n \stackrel{i_n}{\cong} \pi_n$

§2. Generic immersions

Let $G : M \rightarrow S^{n+1}$ be an immersion of a compact manifold. g determines maps $g_i : \underbrace{(Mx \cdots xM)}_{i\text{-copies}} \text{--big diagonal} \rightarrow (S^{n+1}x \cdots xS^{n+1})$. g_i^{-1} (small diagonal) = M_i is the i -tuple set of g^{-1} . It is easy to see that the M_i are compact. An argument using the Thom-transversality theorem shows that g may be C^∞ approximated by an immersion \bar{g} with \bar{g}_i transverse to the small diagonal for all i ; such immersions will be called *normal*. $M_i = \bar{g}_i^{-1}$ (small diagonal) is an orientable submanifold of $\underbrace{Mx \cdots xM}_{i\text{-copies}}$ but does not have a preferred orientation since either $Mx \cdots xM$ or $\underbrace{S^{n+1}x \cdots xS^{n+1}}_{i\text{-copies}}$ will not inherit an orientation from its factors. Since an immersion is locally 1-1 the symmetric group $S(i)$ acts freely on M_i ; let N_i be the quotient manifold. When $i = n+1$ these considerations applied to $f : X \rightarrow S^{n+1}x[-1, 1]$ show that the number of $n+1$ -tuple points of g determine a well defined homomorphism $\theta_n : B_n \rightarrow Z_2$.

The condition that g is a normal immersion has this equivalent form: every point in S^{n+1} should have a chart which intersects $g(M)$ in the l hyperplanes $x_{j_1} = 0, x_{j_2} = 0, \dots, x_{j_l} = 0, 1 \leq j_1, < \dots, < j_l \leq n+1$. (For an open dense set of points l will be zero.)

§3. The computation of θ_3

Here is the program for computing θ_3 . Starting with a generic immersion of an oriented 3-manifold, $g : M \rightarrow S^4$ we find N_2 naturally immersed in S^4 with a

normal bundle having twisted (if N_2 is nonorientable) Euler class zero. Lemma 2 shows that the Hopf invariant of $[g] \in B_3 \cong \pi_3$, $H[g]$, is congruent to the Euler characteristic $\chi(N_2)$. In lemma 4 we replace N_2 by a surface \bar{N}_2 with the same Euler characteristic (mod 2) and also immersed in S^4 with twisted Euler class zero. When g has an even number of quadruple points, we show that the above immersion is regularly homotopic to a generic immersion with an even number of double points. It follows from a theorem of Whitney's [W] that a generically immersed surface in S^4 with an even number of double points and with twisted Euler class zero must have even Euler characteristic. So when $\theta_3[g]=0$, \bar{N}_2 admits an immersion with the above properties. Hence $\chi(N_2) \equiv \chi(\bar{N}_2) \equiv 0 \pmod{2}$. Now by Lemma 2 $\theta_3[g]=0$ implies $H[g]=0$, i.e. $\ker(H) \supset \ker(\theta_3)$. Since $H: \pi_3 \rightarrow Z_2$ is an epimorphism, so is $\theta_3: \pi_3 \rightarrow Z_2$. Knowing $\pi_3 \cong Z_{24}$ now completely determines θ_3 .

Let $\pi: M \times \cdots \times M \rightarrow M$ be the projection from the i -fold product of an oriented n -manifold to the first factor. The following commutative diagram shows that the restriction of π to M_i is an immersion.

$$\begin{array}{ccccc}
 0 & \longrightarrow & \tau(M_i) & \longrightarrow & \tau(\Delta) \\
 & & \downarrow (\pi/M_i)_* & & \downarrow \alpha_* \\
 0 & \longrightarrow & \tau(M) & \xrightarrow{g} & \tau(S^{n+1})
 \end{array}$$

Δ is the small diagonal of $(S^{n+1})^i$. g_i is an immersion so $(g_i|_*) : \tau(M_i) \rightarrow (\Delta)$ is an injection. α is the restriction of projection to the first factor; α_* is an isomorphism and therefore $(\pi/M_i)_*$ is an injection as desired.

Let h be the map making the diagram:

$$\begin{array}{ccc}
 M_i & \xrightarrow{g \circ \pi} & S^{n+1} \\
 \text{proj.} \searrow & & \nearrow h \\
 & & N_i
 \end{array}$$

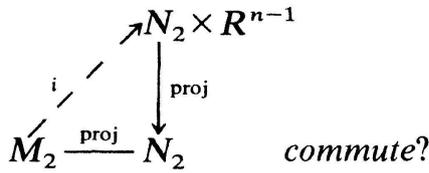
commute. $g \circ \pi$ is an immersion, so h is an immersion.

LEMMA 1. *The normal 2-plane bundle $\nu_{N_2} \xrightarrow{h_2} S^{n+1} = \nu_{h_2}$ has a section.*

Proof. The normal bundle $\nu_{N_2} \xrightarrow{h_2} M$ is trivialized by (say) the normal vector, v . $g_*(v)$ determines a linearly independent pair of vectors v_1 and v_2 in ν_{h_2} . $v_1 + v_2$ defines the desired section.

COROLLARY 1. *If $n = 3$ then $\chi(\nu_{h_2}) = 0 \in H^2(N_2; Z_{\text{twisted}})$ where the coefficients are twisted by $w_1(\tau(N_2))$ when N_2 is nonorientable.*

We need to ask the question: When is there an imbedding $i: M_2 \rightarrow N_2 \times R^{n-1}$ making the diagram

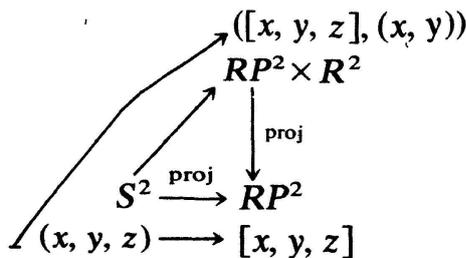


If ζ is the line bundle associated to $(M_2 \xrightarrow{\text{proj}} N_2)$, i will exist if ζ^{-1} has geometric dimension $\leq n - 2$. Since $\dim(N_2) = n - 1$ this will happen if the Stiefel-Whitney class $w_{n-1}(\zeta^{-1}) = 0$

From now on we consider the case $n = 3$. Here $M_2 \xrightarrow{\text{proj}} N_2$ is a two fold covering of a possibly non-orientable surface by an orientable surface. If $w_1(\tau N_2) \neq 0$, $M_2 \xrightarrow{\text{proj}} N_2$ is the orientation covering so $w_1(\zeta) = w_1(\tau N_2)$. In this case $\zeta \oplus \tau N_2$ is trivial since $w_1(\zeta \oplus \tau N_2) = w_1(\zeta) + w_1(\tau N_2) = 0$ and $w_2(\zeta + \tau N_2) = w_1(\zeta) \cdot w_1(\tau N_2) + w_2(\tau N_2) = w_1(\zeta) \cdot w_1(\tau N_2) + (w_1(\tau N_2))^2 = 0$. As a result $\zeta^{-1} = \tau N_2$. If $w_1(\tau N_2) = 0$, $w_1(\zeta \oplus \zeta \oplus \tau N_2) = w_1(\zeta) + w_1(\zeta) = 0$, $w_2(\zeta \oplus \zeta \oplus \tau N_2) = w_1(\zeta)^2 + w_2(\tau N_2) = w_1(\zeta)^2 + w_1(\tau N_2)^2 = 0 + 0 = 0$. So $\zeta^{-1} = \zeta + \tau N_2$. In both cases $w_2(\zeta^{-1}) = w_2(\tau N_2)$, but $w_2(\tau N_2)[N_2]$ is congruent modulo 2 to the Euler characteristic $\chi(N_2)$ so $w_2(\zeta^{-1})[N_2] \equiv \chi(N_2) \pmod{2}$. We now prove:

CLAIM. *If i' is a generic immersion making the preceding diagram commute, then $\#(\text{double points } (i')) \equiv \chi(N_2) \pmod{2}$.*

Proof. If the Euler characteristic of every component of N_2 is even then $w_{n-1}(\zeta^{-1}) = 0$ and, as stated above, i' may be chosen to be an imbedding. Any two choices for i' are regularly homotopic so $\#(\text{double points } (i')) \equiv 0 \pmod{2}$ for any generic i' . For the general case we must consider the following example:



Note that $([0, 0, 1], (0, 0))$ is the only multiple value for i' and that i' is normal.

To remove a generic double point of an arbitrary i' one forms the connected sum $N_2 \# RP^2$ at $[0, 0, 1] \in RP^2$ and (the projection of the double point of i') $\in N_2$. Thus a generic double point of i' over a component of N_2 may be removed at the expense of lowering the Euler characteristic of that component by 1. This reduces the claim to the case first considered.

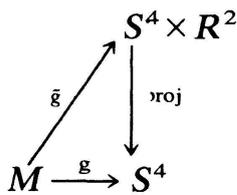
LEMMA 2. *The Hopf invariant $H[g] \equiv \chi(N_2) \pmod{2}$.*

Proof. We use the following definition of the Hopf invariant of $\alpha \in \pi_n$. By the Freudenthal suspension theorem there is an $\alpha' \in \pi_{2n+1}(S^{n+1})$ which stabilizes to α . Let $a : S^{2n+1} \rightarrow S^{n+1}$ represent α' and be transverse to $*$ $\in S^{n+1}$. $a^{-1}(*)$ is a framed submanifold of dimension n in S^{2n+1} . Any frame vector determines a self-linking number $L(a^{-1}(*), a^{-1}(*))$ which, modulo 2, is the Hopf invariant.

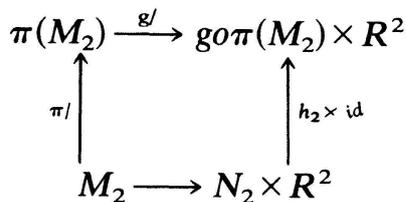
The composition $g' : M \xrightarrow{g} S^{n+1} \hookrightarrow S^{n+1} \times R^{n-1}$ is a framed immersion.
 $s \longmapsto s \times 0$

The number of double points of a generic immersion, \tilde{g} , approximating g' is easily seen to be congruent modulo 2 to the self-linking number of a generic framed imbedding approximating $g'' : M \xrightarrow{g} S^{n+1} \hookrightarrow S^{n+1} \times R^n$. By our definition this self-linking number modulo 2 is $H[g]$. We will show $\#(\text{double points } \tilde{g}) \equiv \chi(N_2) \pmod{2}$.

\tilde{g} can be chosen so that the diagram



commutes. The double points of \tilde{g} are the double points of $\tilde{g} / : \pi(M_2) \rightarrow g \circ \pi(M_2) \times R^2$. There is a generic immersion $j : M_2 \rightarrow N_2 \times R^2$ making



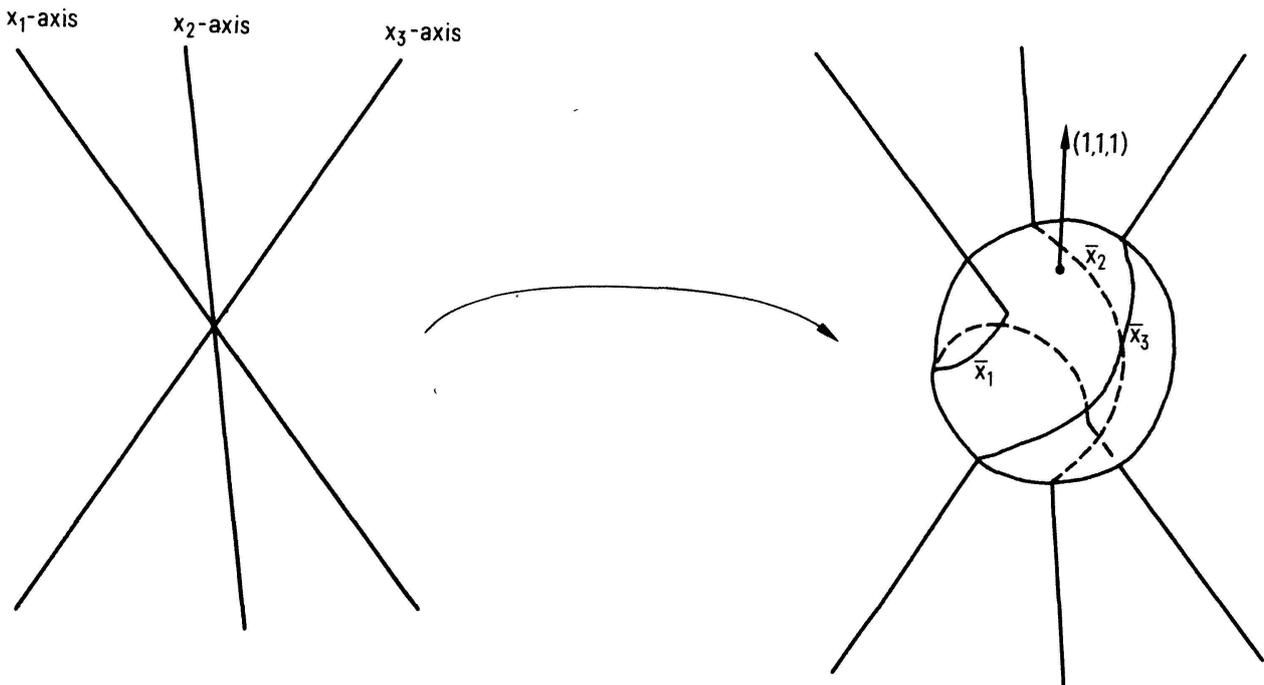
commute. Our characterization of g being generic implies that h_2 only identifies 0 and 1-simplexes of N_2 . So the number of double points of j is equal to the number of double points of \tilde{g} . Lemma 2 now follows by setting $j = i'$ in the discussion immediately preceding its statement. ■

If $g : M \looparrowright S^4$ is a generic immersion of an oriented 3-manifold, $h_2 : N_2 \looparrowright S^4$ though not usually generic does have singularities of a special kind. As an analogy it is helpful to imagine the singularities of the double point set of a generically immersed surface in 3-space. The next lemma considers the case: q has no quadruple points. We analyse the singularities of h_2 to show that h_2 is regularly homotopic to a normal immersion with an even number of double points.

LEMMA 3. *If g has no quadruple points then $h_2 : N_2 \looparrowright S^4$ is regularly homotopic to a generic immersion with an even number of double points.*

Proof. Let T be the subset of S^4 in the image of three distinct points under g . T is a finite family of circles. $h_2|_N : N - h_2^{-1}(1) \rightarrow S^4$ is an imbedding since $g \circ \pi|_M : M_2 \rightarrow M$ is 2-1 on $M_2 \cap (g \times g)^{-1}(T \times T)$. From our characterization of generic maps, we see that some normal open 3-disk ($= d^3$) to T in S^4 may be parametrized to meet $h_2(N_2)$ in a $\{x_1\text{-axis} \cup x_3\text{-axis}\} \subset R^3$. Consider the distortion depicted below as a standard model for separating the sheets of $h_2(N_2)$ in a neighborhood of a point on T . h_2 is moved slightly in the normal directions to T .

Specifically if the x_1, x_2 and x_3 -axes are generated by the vectors $x_1 = (1, 0, 0)$,



$x_2 = (0, 1, 0)$ and $x_3 = (0, 0, 1)$ the curves in diagram 1 are geodesic arcs $\bar{x}_1, \bar{x}_2, \bar{x}_3$, on the unit sphere determined by the condition that their midpoints be $(0, -\sqrt{2}/2, \sqrt{2}/2), (\sqrt{2}/2, 0, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2, 0)$ respectively. Let θ be the 3×3 matrix with these vectors as its rows.

If the model on the left for $h_2(N_2) \cap d^3$ is transported around a circle, c , of T the resulting monodromy of the axes may be represented by a 3×3 -orthogonal matrix, M , with the property that two entries in each row are zero and the remaining entry is ± 1 . The i -th row indicates to which axis (and with which orientation) the i -th axis is transported. (We note that $\nu_{T \rightarrow S^4}$ is orientable so $\text{Det}(M) = +1$). If the model on the right is invariant under the linear transformation (also denoted by M) defined by right multiplication by M , then our model may be used to separate the sheets of $h_2(N_2)$ along all of C . In general, though, separating these sheets along C will result in a finite number of generic double points; our present purpose is to calculate this number in terms of M . Put $\overline{x_i M} = \bar{x}_1, \bar{x}_2$, or \bar{x}_3 as $x_i M = \pm x_1, \pm x_2$, or $\pm x_3$. The model on the right is invariant under M iff $\overline{x_i M} = \overline{x_i M}$ for $i = 1, 2$, and 3 ; if the above equality fails to hold we will see that $D(M) = \sum_{i=1}^3 (1 - (x_i \theta M) \cdot (x_i M \theta)) \pmod{2}$ (\cdot denotes vector dot product) measures the failure. Note that $x_i \theta \perp x_i$ and $x_i M \theta \perp x_i M$. Since M is orthogonal $x_i \theta M \perp x_i M$, as a result $x_i \theta M$ and $x_i M \theta$ both lie in the plane P_i perpendicular to $x_i M$ and must have one of four possible coordinates (restricting our coordinate system to this plane) in that plane: $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$. The number, $(1 - (x_i \theta M) \cdot (x_i M \theta))$, is equal $\pmod{2}$ to the number of times a transverse arc, γ_i , in P_i from $x_i \theta M$ to $x_i M \theta$ must cross the coordinate axes. The arc γ_i determines a homotopy from $\overline{x_i M}$ to $\overline{x_i M}$ through geodesic arcs. Using the model on the right for most of C and then "splicing in" this homotopy at the end we may separate the sheets of $h_2(N_2)$ along all of C with generic double points resulting from transverse crossings of the coordinates axes by γ_i . It follows that h_2 is regularly homotopic to a general immersion with $\sum D(M)$ double points, where the sum is taken over each circle component to T .

We complete the proof of Lemma 3 by showing that for every admissible $M, D(M) \equiv 0 \pmod{2}$. $D(M) \equiv 1 - \sum_{i=1}^3 (x_i \theta M) \cdot (x_i M \theta) \equiv 1 - \sum_{i,j=1}^3 (\theta M)_{ij} \pmod{2}$. Put $(\bar{M})_{ij} = |(M)_{ij}|$. All the non-zero terms in the last sum are $\pm 1/2$, replacing M by \bar{M} reverses an even number of these signs so we have $D(M) \equiv 1 - \sum_{i,j=1}^3 (\theta \bar{M})_{ij} (\bar{M} \theta)_{ij} \pmod{2}$. If \bar{M} is a simple transposition $\theta \bar{M} = (\theta \bar{M})^T = \bar{M}^T \theta^T = -\bar{M} \theta$ so $D(M) \equiv 1 + \sum_{i,j=1}^3 (\theta \bar{M})_{ij}^2 = 1 + \sum_{i,j=1}^3 (\theta)_{ij}^2 = 1 + 3 \equiv 0$. If \bar{M} is a cycle of order 3, one checks that $\theta \bar{M} = \bar{M} \theta$ so again $D(M) \equiv 0 \pmod{2}$. The lemma follows. ■

When g has an even $\neq 0$ number of quadruple points, we perform some oriented 0-surgeries to enlarge our ambient manifold S^4 to $\#(S^1 \times S^3)$. We note

k -copies

that if one chose to, this freedom could be built in from the start; our bordism group, B_n , is isomorphic to “bordism of immersions of oriented 3-manifolds in stably framed 4-manifolds”. An oriented 0-surgery is the operation of removing an imbedded $S^0 \times D^n$ from an oriented n -manifold and gluing back $D^1 \times S^{n-1}$ in a standard manner so as to obtain a new oriented manifold. The notion is often generalized to an operation on a pair, (oriented n -manifold, oriented $(n-1)$ dimensional submanifold). Below we will perform oriented 0-surgery with $S^0 \times 0$ imbedded on a pair of generic quadruple points of a immersed 3-manifold in S^4 ; for this an additional but obvious extension of the notion is required. Rather than give an abstract definition, we have written out the results of our 0-surgery on $(S^4, g(M))$.

Let q, q', \dots, q_k, q'_k be the quadruple points of g arbitrarily paired. For each pair (q_i, q'_i) we perform an oriented 0-surgery on S^4 and a corresponding modification of g . In terms of the image of g the result of a single surgery is: $(S^4, \text{image}(g) - (S^0 \times D^4, S^0 \times (\cup_{\text{hyperplanes}} x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0))$

$$\cup (D^1 \times S^3, D^1 \times (S^3 \cap \bigcup_{\text{hyperplane}} x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0))$$

Call the new immersion $\bar{g}: \bar{M} \rightarrow \#_k S^1 \times S^3$.

If within each chart, D^4 , about a quadruple point of g the positive direction along the 4 axes is consistently determined by the difference of the orientations on S^4 and M , the new manifold \bar{M} will be oriented, and in fact diffeomorphic to $M \#_{j=1}^{4k} (S^1 \times S^2)_j$. Let \bar{M}_2 and \bar{N}_2 correspond to M_2 and N_2 . As proved for N_2 , \bar{N}_2 is immersed (by \bar{h}_2) in $\#_k (S^1 \times S^3)$ with $\chi(\nu_{\bar{h}_2}) = 0$. \bar{N}_2 abstractly is the result of $\binom{4}{2} k = 6k$ 0-surgeries on N_2 . Since a 0-surgery does not change the Euler characteristic modulo 2, $\chi(\bar{N}_2) \equiv \chi(N_2) \pmod{2}$. We are ready to prove:

LEMMA 4. *If g has an even number of quadruple points, there is a surface \bar{N}_2 satisfying:*

- 1) $\chi(\bar{N}_2) \equiv \chi(N_2) \pmod{2}$
- 2) \bar{N}_2 is generically immersed in S^4 with an even number of double points; call its normal bundle ν .
- 3) $\chi(\nu) = 0 \in H^2(\bar{N}_2; Z_{\text{twisted}})$.

Proof. The \bar{N}_2 constructed above is immersed in $\nabla_k (S^1 \times S^3)$ with the above normal bundle condition. The proof of Lemma 3 shows how to regularly homotop this immersion to satisfy condition 2. $\bar{N}_2 \rightarrow \nabla_k S^1 \times S^3$. Framed surgery on k

circles in $(\nabla_k(S^1 \times S^3) - \text{image}(\bar{N}_2))$ returns the ambient manifold to S^4 without affecting the normal bundle of \bar{N}_2 . ■

A theorem of Whitney's [W] says that if a compact surface, Q , is imbedded in S^4 with normal bundle ν and $\chi(\nu) = m \cdot \text{generator} \in H^2(Q; Z_{\text{twisted}})$ then $m \equiv 2\chi(Q) \pmod{4}$. The introduction of a double point changes the twisted Euler class $\chi(\nu)$ by $\pm 2 \cdot \text{generator}$. As a result, Whitney's theorem stated for immersions of Q in S^4 becomes: $m \equiv 2\chi(Q) \pm 2(\# \text{double points of } Q) \pmod{4}$. If g has an even number of quadruple points Whitney's theorem for immersions and Lemma 4 show that $\chi(\bar{N}_2)$ and therefore $\chi(N_2)$ is even. Lemma 2 now says that $H[g] = 0$. Thus we have $\theta_3[g] = 0$ implies $H[g] = 0$, i.e. $\ker(H) \supset \ker(\theta_3)$. Since $H: \pi_3 \rightarrow Z_2$ is well known to be an epimorphism, $\theta_3: \pi_3 \rightarrow Z_2$ is also epic. Since $\pi_3 \cong Z_{24}$, θ_3 is completely determined, we have proved:

THEOREM. $\theta_3: \pi_3 \rightarrow Z_2$ is the unique epimorphism.

§4. Remarks and problems

Remark 1. Since the J_3 -homomorphism: $\pi_3(S^0) \rightarrow \pi_3$ is onto, every element of B_3 is realized by an immersed 3-sphere. In particular there is a generic immersion of S^3 in S^4 with an odd number of quadruple points.

Remark 2. There is no local argument for converting quadruple points of $M \looparrowright S^4$ to double points of $M \looparrowright S^4 \times R^2$ as inspection of the immersion $4(T^3) \looparrowright T^4$ obtained by omitting successive circle factors will show. It seems to be necessary to work down through the strata to prove our theorem, so analogous computations for $n > 3$ are likely to be more difficult.

Remark 3. In this paper we have gone to great trouble to express the Hopf invariant in terms of the lowest dimensional strata of a generic immersion $g: M^3 \rightarrow S^4$, and our arguments have been special to the dimensions involved. There is, however, a simple way in every dimension of reading off the Hopf invariant from the highest dimensional strata, the double point set. If ξ is the line bundle associated to $M_2 \xrightarrow{\text{Proj}} N_2$, $H(g) = 0$ iff $w_{n-1}(\varepsilon^{-1}) = 0$ on all but an even number of path components of N_2 . This is easily seen by comparing our definition of Hopf invariant with our solution to the "question" preceding corollary 2.

PROBLEM 1. Is there a generic immersion of S^3 in S^4 with a single quadruple point?

PROBLEM 2. Explicitly construct a generic immersion of S^3 in S^4 with an odd number of quadruple points.

PROBLEM 3. Compute θ_n for $n > 3$.

Conjecture. θ_n is the stable Hopf invariant.

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