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The periodic structure of non-singular Morse–Smale flows

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Morse–Smale flows on smooth compact manifolds are roughly those flows which exhibit only two relatively simple types of recurrent behavior – periodic closed orbits and rest points which are of hyperbolic type. They have been studied by several authors. In particular Smale and Palis [7] showed that they are structurally stable and Asimov [2] showed the existence of non-singular Morse–Smale flows on an essentially arbitrary manifold and in every homotopy class of non-singular vector fields. The Morse–Smale flows are a large class which is open in the C^1 topology, and also a class whose study has led to insights into the structure of flows with more complicated recurrent behavior.

The object of the present article is to investigate the interplay between the dynamics of a non-singular Morse–Smale flow, i.e. the kind of periodic behavior it exhibits, and the topology of the manifold on which the flow occurs.

The local behavior of a hyperbolic periodic closed orbit is completely determined by the dimension of its unstable manifold (this dimension minus one is called the index) and whether or not the orbit is *twisted* (has an unorientable unstable manifold) or *untwisted* (has an orientable unstable manifold). We are concerned with the problem of whether given periodic behavior in the form of a specified number of closed orbits of each index can be realized with a non-singular Morse–Smale flow on a given manifold. An article of Smale [8] deals with precisely this problem and relates the number of closed orbits of each index to the Betti numbers of the manifold through "Morse inequalities." Our aim is to strengthen these necessary conditions until conditions which are both necessary and sufficient are obtained.

THEOREM A. *Suppose M^n admits a non-singular Morse–Smale flow with A_k untwisted closed orbits of index k, then if $R_k = \dim H_k(M; Q)$,*

(a) $A_k \geq R_k - R_{k-1} + \cdots + R_0$ *for all k.*
(b) $A_1 \geq A_0 - 1$, *and* $A_{n-2} \geq A_{n-1} - 1$.
(c) *If* $A_{k-1} = A_{k+1} = 0$ *and* $R_k - R_{k-1} + \cdots \pm R_0 \leq 0$, *then* $A_k = 0$.

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We remark that (a) is essentially the Morse inequalities of Smale [8], the only difference being that A_k is the number of untwisted closed orbits of index k, not of all orbits of index k.

For a large class of manifolds the necessary conditions of (a), (b), and (c) of Theorem A are sufficient as well.

THEOREM B. *Suppose M is a simply connected compact manifold of dimension greater than five. If the Euler characteristic of M vanishes and $H_*(M; Z)$ is torsion free, then to any set of non-negative integers $\{A_k\}$, satisfying* (a), (b) *and* (c) *above, there corresponds a non-singular Morse–Smale flow with A_k untwisted closed orbits of index k and no twisted closed orbits.*

The techniques used will apply to a number of manifolds of dimension ≤ 5. In particular §4 deals with the case $M = S^3$ and complete necessary and sufficient conditions for the existence of flows with prescribed numbers of closed orbits (twisted as well as untwisted) of each index is given.

1. Background and definitions

A closed orbit γ of a flow φ_t on M is said to be *hyperbolic* provided the tangent bundle of M restricted to γ, TM_γ is the sum of three $D\varphi_t$ invariant bundles $E^c \oplus E^u \oplus E^s$ such that

(1) E^c is spanned by the vector field X tangent to the flow.
(2) There are constants $C, \lambda > 0$ such that $\|D\varphi_t(v)\| \geq Ce^{\lambda t} \|v\|$ for $v \in E^u$, $t \geq 0$ and $\|D\varphi_t(v)\| \leq C^{-1}e^{-\lambda t} \|v\|$ for $v \in E^s$, $t \geq 0$, where $\| \ \|$ is some Riemannian metric.

A rest point x for a flow φ_t is called hyperbolic provided $TM_X = E^u \oplus E^s$ and the inequalities of (2) are valid for $v \in E^u$ or E^s.

The stable and unstable manifolds of a hyperbolic closed orbit γ are defined by

$$W^u(\gamma) = \{x \mid d(\varphi_t x, \varphi_t y) \to 0 \quad \text{as} \quad t \to -\infty \quad \text{for some} \quad y \in \gamma\}$$

$$W^s(\gamma) = \{x \mid d(\varphi_t x, \varphi_t y) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for some} \quad y \in \gamma\},$$

where d is a metric on M. These are injectively immersed copies of the vector bundles E^u or E^s, (see [9]). For a rest point x the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ are defined analogously.

(1.1) DEFINITION. The *index* of a hyperbolic closed orbit or rest point is defined to be the fiber dimension of the bundle E^u.

A point x of M is called chain recurrent (see [5]) for the flow φ_t provided that for any T, $\epsilon > 0$ there are points $x_i \in M$ and real numbers $t_i > T$, $i = 0, 1, \ldots, n$, such that $x = x_0 = x_n$ and $d(\varphi_{t_i} x_i, x_{i+1}) < \epsilon$.

The set of all chain recurrent points R is a compact invariant set of the flow, if M is compact.

(1.2) DEFINITION. A smooth flow φ_t on M is called *Morse–Smale* if

(a) The chain-recurrent set R of φ_t consists of a finite number of hyperbolic closed orbits and hyperbolic rest points, and
(b) The unstable manifold of any closed orbit or rest point has transversal intersection with the stable manifold of any closed orbit or rest point.

A flow which is Morse–Smale possesses a *filtration* (see [5] or [7]). That is, there are manifolds with boundary and with the same dimension as M,

$$N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_l = M$$

such that for each i, $N_{i-1} \subset \operatorname{int} N_i$, the vector field tangent to the flow points inward on the boundary of M_i, and $\operatorname{int}(M_i - M_{i-1})$ contains precisely one closed orbit or one rest point. In fact if the closed orbits and rest points $\{\gamma_i\}$ are indexed so that $W^s(\gamma_i) \cap W^u(\gamma_j) = \varnothing$ whenever $i > j$, we may choose the filtration so $\gamma_i \subset N_i - N_{i-1}$, (see [7]).

We shall have occasion to use a somewhat less fine filtration. The transversality of stable and unstable manifolds for Morse–Smale flows ((b) of (1.2)) guarantees that if γ_i and γ_j are closed orbits with index $\gamma_i >$ index γ_j then $W^s(\gamma_i) \cap W^u(\gamma_j) = \varnothing$. Hence we may choose a filtration $\{N_i\}$ with $\gamma_i \subset N_i - N_{i-1}$ such that if index $\gamma_i >$ index γ_j then $i > j$. For a non-singular flow this allows us to make a coarser filtration which will have all closed of the same index in the same level.

(1.3) *Remark.* If φ_t is a non-singular Morse–Smale flow and the N_i are chosen as above then we can define $M_k = \{x \mid x \in N_i \text{ and } \gamma_i \text{ has index} \leq k\}$. It is then clear that

$$M_0 \subset M_1 \subset \cdots \subset M_{n-1} = M$$

has all the properties of a filtration except that $M_i - M_{i-1}$ contains all closed orbits of index i.

2. The spectral sequence associated with a filtration

Let $E^1_{p,q}$ be a spectral sequence over a field, satisfying $E^1_{p,q}=0$ if $q\neq 0, 1$, and such that $\dim E^1_{k,0}=\dim E^1_{k,1}$ for all k. We define a number of integers for subsequent use:

$$a_k = \dim E^1_{k,0} = \dim E^1_{k,1}$$

$$R_k = \dim \bigoplus_{p+q=k} E^\infty_{p,q} = \dim E^\infty_{k,0} + \dim E^\infty_{k-1,1}$$

$$\alpha_k = \operatorname{rank} d^1 : E^1_{k,0} \to E^1_{k-1,0}$$

$$\beta_k = \operatorname{rank} d^1 : E^1_{k,1} \to E^1_{k-1,1}$$

$$\gamma_k = \operatorname{rank} d^2 : E^2_{k,0} \to E^2_{k-2,1}$$

$$c_k = \dim E^3_{k,0} = \dim E^\infty_{k,0}$$

$$\bar{c}_k = \dim E^3_{k-1,1} = \dim E^\infty_{k-1,1}$$

We will be concerned with the spectral sequence associated with the filtration of M as described in §1. In this case R_k will be $\dim H_k(M)$ and a_k will be the dimension of $H_k(M_k, M_{k-1}) = \dim H_{k+1}(M_k, M_{k-1})$.

(2.1) LEMMA. *With the above definitions*

$$a_k = \beta_k + \beta_{k+1} + \gamma_{k+2} + \bar{c}_{k+1} = \alpha_k + \alpha_{k+1} + \gamma_k + c_k.$$

Proof. We start with the fact that $c_k = \dim E^3_{k,0}$ from which it follows that $\dim E^2_{k,0} = \dim E^3_{k,0} + \operatorname{rank}(d^2 : E^2_{k,0} \to E^2_{k-2,1}) = c_k + \gamma_k$. Hence we have

$$\begin{aligned} a_k = \dim E^1_{k,0} &= \dim E^2_{k,0} + \operatorname{rank}(d^1 : E^1_{k,0} \to E^1_{k-1}) \\ &+ \operatorname{rank}(d^1 : E^1_{k+1,0} \to E^1_{k,0}) = c_k + \gamma_k + \alpha_k + \alpha_{k+1} \end{aligned}$$

A similar computation shows that

$$a_k = \dim E^1_{k,1} = \beta_k + \beta_{k+1} + \gamma_{k+1} + \bar{c}_{k+1}.$$

(2.2) LEMMA. *With the above definitions*

$$R_k - R_{k-1} + R_{k-1} - \cdots \pm R_0 = \gamma_{k+2} - \gamma_{k+1} + \beta_{k+1} - \alpha_{k+1} + \bar{c}_{k+1}.$$

Proof. The previous lemma gives two expressions for a_k. Using each of these expressions we evaluate $a_k - a_{k-1} + a_{k-2} - \cdots \pm a_0$ and then equate the results. In this way we obtain

$$\begin{aligned}\beta_{k+1}+(\gamma_{k+2}-\gamma_{k+1}+\gamma_k-\cdots\pm\gamma_0)+(\bar{c}_{k+1}-\bar{c}_k+\cdots\pm\bar{c}_0)\\ =\alpha_{k+1}+(\gamma_k-\gamma_{k-1}+\cdots\pm\gamma_0)+(c_k-c_{k-1}+\cdots\pm c_0).\end{aligned}$$

Hence

$$\bar{c}_{k+1}+[-(\bar{c}_k+c_k)+(\bar{c}_{k-1}+c_{k-1})-\cdots\pm(\bar{c}_0+c_0)]=\alpha_{k+1}-\beta_{k+1}+\gamma_{k+1}-\gamma_{k+2}.$$

Since $R_k=\bar{c}_k+c_k$ we have

$$R_k-R_{k-1}+\cdots\pm R_0=\gamma_{k+2}-\gamma_{k+1}+\beta_{k+1}-\alpha_{k+1}+\bar{c}_{k+1}.$$

(2.3) COROLLARY. $a_k\geq R_k-R_{k-1}+\cdots\pm R_0$.

Proof. By (2.1) and (2.2) we have

$$a_k=\beta_k+\beta_{k+1}+\gamma_{k+2}+\bar{c}_{k+2}$$

and

$$R_k-R_{k-1}+\cdots\pm R_0=\gamma_{k+2}-\gamma_{k+1}+\beta_{k+1}-\alpha_{k+1}-\bar{c}_{k+1}.$$

Therefore $a_k-\beta_k=(R_k-R_{k-1}+\cdots\pm R_0)+\gamma_{k+1}+\alpha_{k+1}$. Since β_k, γ_{k+1} and α_{k+1} are all non-negative the result follows.

(2.4) PROPOSITION. *If R_k and a_k are as previously defined and $R_0=R_n=1$, then*

(a) $a_1\geq a_0-1$ *and* $a_{n-2}\geq a_{n-1}-1$,
(b) *If* $a_{k-1}=a_{k+1}=0$ *and* $R_k-R_{k-1}+\cdots\pm R_0\leq 0$, *then* $a_k=0$.

Proof. By (2.1) $a_0=\alpha_0+\alpha_1+\gamma_0+c_0$; and since $c_0=R_0=1$, and $\alpha_0=\gamma_0=0$, it follows $a_0=\alpha_1+1$. Since by (2.1) again, $a_1\geq\alpha_1$ we have $a_1\geq a_0-1$. A similar computation shows $a_{n-2}\geq a_{n-1}-1$.

To prove (b) we note that from (2.1) it follows that if $a_{k-1}=a_{k+1}=0$ then $\alpha_k=\alpha_{k+1}=\beta_k=\beta_{k+1}=\gamma_{k+1}=0$. But from (2.2) $R_k-R_{k-1}+\cdots\pm R_0=\gamma_{k+2}-\gamma_{k+1}+\beta_{k+1}-\alpha_{k+1}+\bar{c}_{k+1}\leq 0$, so we have $\gamma_{k+2}+\bar{c}_{k+1}\leq 0$ which implies $\gamma_{k+2}=\bar{c}_{k+2}=0$ since γ_{k+2} and $\bar{c}_{k+2}$ are non-negative. Hence

$$a_k=\beta_k+\beta_{k+1}+\gamma_{k+2}+\bar{c}_{k+2}=0.$$

(2.5) PROPOSITION. *Given non-negative integers R_0, R_1, $R_2, \dots, R_n$, such that $R_n - R_{n-1} + \cdots \pm R_0 = 0$, there exists a spectral sequence $E^1_{p,q}$ over any field F such that*

(a) $E^1_{p,q} = 0$ *if* $q \neq 0, 1$ *or if* $p < 0$ *or* $p \geq n$.
(b) $R_k = \dim \oplus_{p+q=k} E^\infty_{p,q} = \dim E^3_{k,0} + \dim E^3_{k-1,1}$.
(c) $\dim E^1_{k,0} = \dim E^1_{k,1} = \max\{0, R_k - R_{k-1} + \cdots \pm R_0\}$.

Proof. We can specify a spectral sequence satisfying (a) by specifying the $\dim (E^3_{p,q} = E^\infty_{p,q})$ and the ranks of the non-zero differentials. That is, we need only define the numbers α_k, β_k, γ_k, c_k and $\bar{c}_k$ defined in the beginning of this section. We will define $\alpha_k = \beta_k = 0$ for all k (so that in fact $E^1_{p,q} = E^2_{p,q}$) and inductively define γ_k, c_k and $\bar{c}_k$ as follows:

Let $c_0 = R_0$, and $c_k = \bar{c}_k = \gamma_k = 0$ if $k < 0$, and define

$$c_k = \max\{0, R_k - c_{k-1} - \gamma_{k-1}\} \tag{1}$$

$$\gamma_k = \max\{0, c_{k-2} + \gamma_{k-2} - R_{k-1}\} \tag{2}$$

$$\bar{c}_k = R_k - c_k = \min\{R_k, c_{k-1} + \gamma_{k-1}\}. \tag{3}$$

With these numbers specified, if we define $E^1_{p,q} = 0$ for $q \neq 0, 1$, we have a well defined spectral sequence and as in (2.1), one can compute

$$\dim E^1_{k,0} = \gamma_k + c_k, \tag{4}$$

and

$$\dim E^1_{k,1} = \gamma_{k+2} + \bar{c}_{k+1}. \tag{5}$$

Notice that by definition $E^1_{p,q} = 0$ if $q \neq 0, 1$, but we will postpone the proof of the remainder of (a), namely that $E_{p,0} = E_{p,1} = 0$ if $p \geq n$.

To show (b), note that $\dim \oplus_{p+q=k} E^\infty_{p,q} = \dim E^3_{k,0} + \dim E^3_{k-1,1}$ and by construction $\dim E^3_{k,0} = c_k$, and $\dim E^3_{k-1,1} = \bar{c}_k$. Thus

$$\dim E^3_{k,0} + \dim E^3_{k-1,1} = c_k + \bar{c}_k = c_k + R_k - c_k = R_k$$

by the definition of $\bar{c}_k$. So (b) is satisfied.

The first equality of (c) is obtained as follows, using (1)–(5) above

$$\begin{aligned}\dim E^1_{k,1} = \gamma_{k+2} + \bar{c}_{k+1} &= \max\{0, c_k + \gamma_k - R_{k+1}\} \\ &\quad + R_{k+1} - \max\{0, R_{k+1} - c_k - \gamma_k\} \\ &= R_{k+1} + (-R_{k+1} + c_k + \gamma_k) = c_k + \gamma_k \\ &= \dim E^1_{k,0}.\end{aligned}$$

We now set $a_k = \dim E^1_{k,0} = \dim E^1_{k,1}$ and note the spectral sequence we are considering satisfies all the hypotheses of (2.2) and hence we can conclude

$$R_k - R_{k-1} + \cdots \pm R_0 = \gamma_{k+2} - \gamma_{k+1} + \bar{c}_{k+1}$$

since $\alpha_{k+1} = \beta_{k+1} = 0$.

Since $a_k = \gamma_{k+2} + \bar{c}_{k+1}$ we have

$$\begin{aligned} R_k - R_{k-1} + \cdots \pm R_0 &= a_k - \gamma_{k+1} = a_k - \max\{0, c_{k-1} + \gamma_{k-1} - R_k\} \\ &= a_k - \max\{0, a_{k-1} - R_k\}. \end{aligned}$$

So

$$a_k = R_k - R_{k-1} + \cdots \pm R_0 + \max\{0, a_{k-1} - R_k\}. \tag{6}$$

We now assume inductively that

$$a_{k-1} = \max\{0, R_{k-1} - R_{k-2} + \cdots \pm R_0\}$$

and note that it follows that

$$\max\{0, a_{k-1} - R_k\} = \begin{cases} 0 & \text{if} \quad R_k \geq a_{k-1} \\ a_{k-1} - R_k & \text{if} \quad R_k < a_{k-1} \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad R_k \geq R_{k-1} - R_{k-2} + \cdots \pm R_0 \\ -R_k + R_k - R_{k-2} + \cdots \pm R_0 & \text{if} \quad R_k < R_{k-1} - R_{k-2} + \cdots \pm R_0. \end{cases}$$

Hence by (6) above

$$a_k = \max\{0, R_k - R_{k-1} + \cdots \pm R_0\}$$

and by induction on k we have completed the proof of (c). Since we assume $R_n - R_{n-1} + \cdots \pm R_0 = 0$ and define $R_k = 0$ for $k > n$, it follows that

$$R_k - R_{k-1} + \cdots \pm R_0 = 0 \quad \text{whenever} \quad k \geq n.$$

Hence $\dim E^1_{k,0} = \dim E^1_{k,1} = a_k = 0$ for all $k \geq n$ and we have completed the proof of (a).

(2.6) *Proof of Theorem A*

We will consider a filtration $\varnothing = M_{-1} \subset M_0 \subset \cdots \subset M_{n-1} = M$ for which all closed orbits of the flow of index k are contained in $M_k - M_{k-1}$ (see (1.3)) and

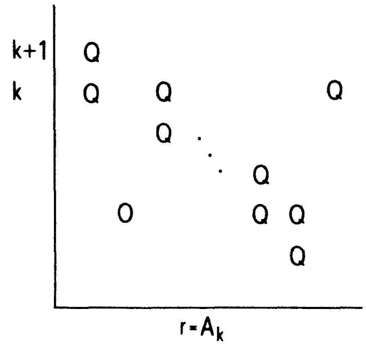

Figure 1

make some calculations concerning $H_*(M_k, M_{k-1}; Q)$. To do this we consider a further filtration $M_k = X_0 \subset X_1 \subset \cdots \subset X_r = M_{k+1}$ such that there is precisely one untwisted closed orbit in $X_p - X_{p-1}$ so we will have $r = A_k$. It is then not a difficult computation to see

$$H_i(X_p, H_{p-1}; Q) = \begin{cases} Q & \text{if} \quad i = k \quad \text{or} \quad k+1 \\ 0 & \text{if} \quad i = k, k+1 \end{cases}$$

This is, for example, a special case of (5.3) of [4]. Notice in particular that the possible existence of twisted closed orbits of index k in $X_p - X_{p-1}$ does not alter this result because we are using rational coefficients.

Consider now the E^1 spectral sequence with $E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; Q)$. It has the form shown in Figure 1 and converges to an associated graded vector space of $H_*(M_k, M_{k-1}; Q)$.

From this two facts follow: If $a_k = \dim H_k(M_k, M_{k-1}; Q)$ then $A_k \geq a_k$ and $A_k \neq 0$ implies that $a_k \neq 0$. Notice also that $H_i(M_k, M_{k-1}; Q) = 0$ if $i \neq k$, $k+1$, so since the Euler characteristic $\chi(M_k, M_{k-1}) = 0$, we must have $a_k = \dim H_{k+1}(M_k, M_{k-1}; Q)$ also.

We can now consider the E^1 spectral sequence associated with the filtration $M_0 \subset M_1 \subset \cdots \subset M_{n-1}$. Thus we have $E^1_{p,q} = H_{p+q}(M_p, M_{p-1}; Q)$ and this spectral sequence converges to an associated graded vector space for $H_*(M; Q)$. So if $R^k = \dim H_k(M; Q)$ we will have $R_k = \dim E^\infty_{p,q}$. Since this spectral sequence satisfies all the conditions for the spectral sequence considered at the beginning of this section it satisfies the hypotheses of (2.3) and (2.4). We thus have $A_k \geq a_k \geq R_k - R_{k-1} + \cdots \pm R_0$ which proves part (a) of the theorem. Part (c) of the theorem follows from (b) of (2.4) since, as remarked above $A_{k-1} = A_{k+1} = 0$ implies $a_{k-1} = a_{k+1} = 0$ and $a_k = 0$ implies $A_k = 0$.

Finally to prove part (b) of the theorem we note A_0 is the number of components of M_0 so $A_0 = a_0$ and dually (considering the inverse flow) $A_{n-1} =$

a_{n-1}. Thus from (a) of (2.4) we have

$$A_1 \geq a_1 \geq a_0 - 1 = A_0 - 1$$

and

$$A_{n-2} \geq a_{n-2} \geq a_{n-1} - 1 = A_{n-1} - 1.$$

3. Constructing Morse–Smale flows

In this section we give the proof of Theorem B by constructing a non-singular Morse–Smale flow with the required number of untwisted closed orbits of each index. The main tool for doing this is the following result of Asimov.

(3.1) THEOREM. (Asimov [1], [2]). *Suppose $f : W \to [a, b]$ is a Morse function with two critical points p and q of index k and $k+1$ respectively, such that $W^u(q) \cap W^s(p) = \varnothing$, if $k \neq 0$, then there is a non-singular Morse–Smale vector field X on W satisfying*

(1) *$X = -\nabla f$ on a neighborhood of ∂W, and*
(2) *the flow of X has exactly one closed orbit and this orbit has index k and is untwisted.*

(3.2) DEFINITION. The Morse function f will be called an associated Morse function for the flow X on M.

The important property of the associated Morse function is that it has two critical points, one of index k and one of index $k+1$; and that the unstable manifold of the index $k+1$ critical point does not intersect the stable manifold of the index k critical point if $k \neq 0$. It is not clear that every Morse–Smale flow on W with a single untwisted closed orbit has an associated Morse function, but all of the Morse–Smale flows we construct will be obtained by applying Theorem (3.1) and then piecing together.

(3.3) LEMMA. *Let $f : W \to [a, b]$ be a Morse function defined on a connected n-dimensional manifold such that all critical points of index 0 are in $f^{-1}(a)$ and all critical points of index n are in $f^{-1}(b)$. Then given c, d such that $a < c < d < b$, and $0 \leq k < n$, there exists a Morse function $g : W \to [a, b]$ such that*

(a) *The functions f and g agree on a neighborhood of ∂W and on a neighborhood of the critical points of f.*

(b) *The function g has two additional critical points: one of index k in $g^{-1}(c)$ and one of index $k+1$ in $g^{-1}(d)$.*

(c) *The unstable manifolds of the additional critical points are disjoint from the stable manifolds of any of the previous critical points except perhaps those of index 0 and this is also true for the dual Morse function $-g$.*

Proof. Because all critical points of f of index 0 and n are in $f^{-1}(a) \cup f^{-1}(b)$ and W is connected there is an integral curve I of $-\nabla f$ running from a point in $f^{-1}(d+\epsilon)$ to a point in $f^{-1}(c-\epsilon)$, for some small $\epsilon > 0$. We may assume I is disjoint from stable and unstable manifolds of all critical points of index $\neq 0$, n; since these manifolds will have dimension $< n$. By standard arguments there is a neighborhood U of I (also disjoint from stable and unstable manifolds) and a chart map $\varphi : U \to \varphi(U) \subset R^n$ such that if $x \in U$ $f(x) = h(\varphi(x))$ where $h : R^n \to R$ is a linear height function. Now by (8.2) of [6] the function $h : \varphi(U) \to R$ can be altered to $\hat{h}$ which has two cancelling critical points of index k and $k+1$ at the desired level. Finally, g is defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin U \\ \hat{h}(\varphi(x)) & \text{if } x \in U. \end{cases}$$

(3.4) PROPOSITION. *Suppose X is a Morse–Smale flow W with a single closed orbit and with an associated Morse function f'. If the closed orbit is untwisted and has index k such that $1 \leq k < n-1$, then there exist non-singular Morse–Smale flows Y_1 and Y_2 on W such that,*

(a) *On a neighborhood of ∂W, $X = Y_1 = Y_2$.*

(b) *The flow of Y_1 has precisely two closed orbits, both untwisted and of index k.*

(c) *The flow Y_2 has precisely two closed orbits, both untwisted, but one of index k and one of index $k+1$. (For this, $k=0$ is permissable).*

We remark that the construction of Y_2 in the case $K=0$ has been done by Asimov in [3].

Proof. Suppose q_k and q_{k+1} are the critical points of f'. Then, since $W^u(q_{k+1}) \cap W^s(q_k) = \varnothing$, by (4.1) of [6] we can change f' to a new Morse function f which is unchanged on a neighborhood of ∂W, and has the same critical points as f', but satisfies $f(q_k) > f(q_{k+1})$.

Now by (3.3) we can form a new Morse function g, agreeing with f on a neighborhood of ∂W but having two additional critical points p_{k+1} and p_k of index $k+1$ and k. It is clear we can do this in such a way that

$$g(p_{k+1}) > g(q_k) = f(q_k) > f(q_{k+1}) = g(q_{k+1}) > g(p_k).$$

If we now consider the flow $-\nabla g$ then by (3.3) we have $W^u(p_{k+1}) \cap W^s(q_k) = \emptyset$ and $W^U(q_{k+1}) \cap W^s(p_k) = \emptyset$.

We are now in a position to apply (3.1) to construct the vector field Y_1. Namely if we choose $c \in R$ such that $g(q_{k+1}) < c < g(q_k)$ and define $W_1 = g^{-1}([c, \infty))$ and $W_2 = g^{-1}((-\infty, c])$ then (3.1) asserts the existence of a flow X_i on W_i agreeing with $-\nabla g$ on a neighborhood of ∂W_i, and with a single closed orbit of index k which is untwisted. Defining Y_1 to be X_i on W_i gives the desired flow.

The construction of Y_2 is similar. For this, we alter to Morse function f' to g', leaving it unchanged on a neighborhood of ∂W but adding two cancelling critical points (using (3.3)) q_{k+2} and q_{k+1} of index $k+2$ and $k+1$. We can do this in such a way that

$$g'(q_{k+2}) > g'(p_{k+1}) = f'(p_{k+1}) > g'(q_{k+1}) > g'(p_k) = f'(p_k).$$

If we now choose c' such that $g'(q_{k+1}) < c' < g'(p_{k+1})$ and define $W_1' = f'^{-1}([a, c])$ and $W_2' = f'^{-1}([c, b])$, then as before we are in a position to apply (3.1). The key point is that by construction the unstable manifold of q_{k+2} (with respect to the flow of $-\nabla g'$) is disjoint from $W^s(p_{k+1})$. Likewise $W^u(q_{k+1}) \cap W^s(p_k) = \emptyset$. Thus by (3.1) we can construct X_1' on W_1' with an untwisted closed orbit of index k and X_2' on W_2' with an untwisted closed orbit of index $k+1$. Defining Y_2 on W to be X_1' on W_1' and X_2' on W_2' gives the desired flow.

(3.5) LEMMA. *Suppose $f: W \to [a, b]$ is a Morse function with no critical points, then there exists a non-singular Morse–Smale vector field X on M such that*

(a) *X agrees with $-\nabla f$ on a neighborhood of ∂W.*
(b) *X has two closed orbits, both untwisted; one of index k and one of index $k+1$.*

Proof. Choose numbers c_i, $i = 1, 2, 3, 4$ such that $a < c_1 < c_2 < c_3 < c_4 < b$. By applications of (3.3) we can construct a Morse function g on W agreeing with f on a neighborhood of ∂W and with critical points $p_i \in g^{-1}(c_i)$ such that p_1 and p_2 form a cancelling pair of index k and $k+1$ and p_2 and p_4 for a cancelling pair of $k+1$ and $k+2$. Moreover by (3.3) we can assume

$$W^u(p_2) \cap W^s(p_1) = \emptyset$$

and

$$W^u(p_4) \cap W^s(p_3) = \emptyset.$$

Now choose $c \in (c_2, c_3)$ and let $W_1 = g^{-1}([a, c])$, $W_2 = g^{-1}([c, b])$. Two applications of (3.1) now produce the desired flow on W.

The following proposition is simply a restatement of Proposition (2.5).

(3.5) PROPOSITION. *Given non-negative integers* $R_0, \ldots, R_n$ *such that* $R_n - R_{n-1} + \cdots \pm R_0 = 0$, *there exist non-negative integers* c_k, $\bar{c}_k$, γ_k, $k = 0, \ldots, n$ *such that*

(a) $\gamma_k + c_k = \gamma_{k+2} + \bar{c}_{k+1} = \max\{0, R_k - R_{k-1} + \cdots \pm R_0\}$.
(b) $R_k = c_k + \bar{c}_k$, $\bar{c}_0 = 0$.

(3.7) *Proof of Theorem B.* The hypothesis that M is simply connected, has dimension greater than 5, and has torsion free homology guarantees that M admits a self-indexing Morse function f whose type numbers are equal to its Betti numbers. (This is (6.3) of [10]).

Now using the techniques of (6.4) of [6] and in particular (6.6) of [6] we can alter this Morse function so that

$$W^u(p_{k+1}) \cap W^s(q_k) = \varnothing. \tag{1}$$

whenever q_k and p_{k+1} are critical points of index k and $k+1$ respectively. Note that since M is simply connected there are no critical points of index 1 or $n-1$.

(3.8) *Remarks.* It is only to obtain this Morse function that we need the hypotheses that $\dim M > 5$. So in fact if M is any simply connected manifold admitting a self-indexing Morse function with type numbers equal to the Betti numbers and satisfying (1) above, then the conclusion of Theorem B holds.

Using (4.1) of [6] we can alter f again so that there are c_k critical points of index k in $f^{-1}(k+\epsilon)$ and $\bar{c}_k$ critical points of index k in $f^{-1}(k-\varepsilon)$.

We are using the fact that $c_k + \bar{c}_k = R_k$ and moving c_k points up slightly and the remaining $\bar{c}_k$ down slightly. This can be done without disturbing the property that $W_u(p_{k+1}) \cap W^s(q_k) = \varnothing$.

We now add γ_k pairs of cancelling pairs of critical points of index k and $k-1$ using (3.3) in such a way that those of index $k-1$ are in $f^{-1}(k-1-\epsilon)$ and those of index k are in $f^{-1}(k+\epsilon)$. We do this for each k.

Now let $M_k = f^{-1}((-\infty, k+1])$ and $W_k = f^{-1}([k, k+1])$. By construction W_k contains $c_k + \gamma_k$ critical points of index k (in $f^{-1}(k+\epsilon)$) and $\bar{c}_{k+1} + \gamma_{k+2}$ critical points of index $k+1$ (in $f^{-1}(k+1-\epsilon)$). Also if $1 \le k < n-1$ the unstable manifold of each index $k+1$ point is disjoint from the stable manifold of each index k point.

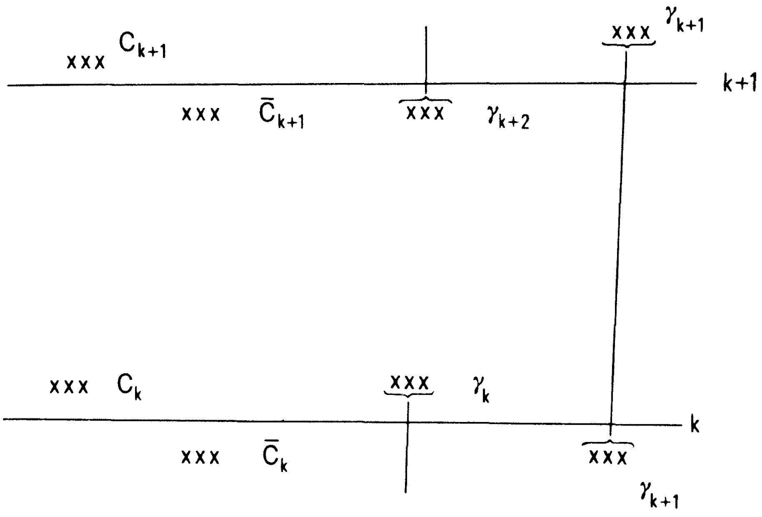

Figure 2

The situation is shown schematically in Figure 2.

Thus using (4.1) of [6] to rearrange the levels on which critical points occur and repeatedly applying (3.1) we can construct a Morse–Smale vector field X_k on W_k such that $X_k = -\nabla f$ on a neighborhood of ∂W_k and such that the flow of X_k has precisely $c_k + \gamma_k = \bar{c}_{k+1} + \gamma_{k+2}$ closed orbits all untwisted and of index k.

We take special note of the cases $k = 0, 1, n-1, n$. By hypothesis $R_0 = R_n = 1$, $R_1 = R_{n-1} = 0$. Thus $c_0 = R_0 = 1$ and hence $\gamma_0 + c_0 = R_0$ implies $\gamma_0 = 0$. Also $R_1 - R_0 = -1$ so $\gamma_1 + c_1 = 0$ and hence both γ_1 and c_1 are zero. The equations $\gamma_2 + \bar{c}_1 = R_0 = 1$ and $c_1 + \bar{c}_1 = R_1 = 0$, then imply $\bar{c}_1 = 0$ and $\gamma_2 = 1$. It follows that M_0 contains exactly two critical points, one of index zero and one of index one. Hence M_0 is diffeomorphic to $S^1 \times D^{n-1}$ and admits a vector field agreeing with $-\nabla f$ on ∂M_0 and having one closed orbit of index zero. A completely dual argument (using the fact that $R_n - R_{n-1} + \cdots \pm R_0 = 0$) shows that $\hat{M} = f^{-1}([n-1, \infty))$ admits a vector field agreeing with $-\nabla f$ on $\partial \hat{M}$ and having a single closed orbit of index $n-1$. Thus piecing together the vector fields on the W_k, M_0 and $\hat{M}$, we obtain a flow which almost proves the theorem in the special case $A_k = \max\{0, R_k - R_{k-1} + \cdots \pm R_0\}$. We have constructed a flow which has a filtration and which satisfies all the properties of Morse–Smale except the transversality of stable and unstable manifolds. But if we take a Kupka–Smale approximation (see [9]) we can achieve this transversality and because the filtration will be a filtration for the approximation, we will not change the periodic behavior or add additional chain recurrent points.

To prove the general case we use Lemmas (3.4) and (3.5) to add closed orbits to this flow. More precisely, given non-negative integers $\{A_k\}$ satisfying (a), (b) and (c) of Theorem A we first use (3.5) to add $A_0 - 1$ cancelling pairs of closed orbits of index 0 and 1 to M_0 and $A_{n-1} - 1$ cancelling pairs of index $n-2$ and

$n-1$ to $\hat{M}$. If $A_1 > A_0 - 1$ we then use (3.4) to add $A_1 - (A_0 - 1)$ closed orbits of index 1 to M_0 so we will have A_0 closed orbits of index 0 and A_1 of index 1. Similarly we can alter the flow so that there are A_{n-2} closed orbits of index $n-2$ and A_{n-1} of index $n-1$.

Now inductively suppose the flow has been altered so we have the desired number of closed orbits through index $k-1$, and we want to alter it further so there are A_k of index k. Suppose first that $R_k - R_{k-1} + \cdots \pm R_0 > 0$ then there is already at least one closed orbit of index k and by repeated application of (3.4) we can alter the flow so there are A_k of index k. Otherwise either $A_{k-1} \neq 0$ and repeated application of (3.4) works, or $A_{k+1} \neq 0$ in which case we use (3.5) to add a cancelling pair of index k and $k+1$ and then apply (3.4) repeatedly to obtain A_k of index k and A_{k+1} of index $k+1$. This last case is done slightly differently if $k = n-3$, for then there are already present $A_{k+1} = A_{n-2}$ orbits of index $n-2$ so we can use the dual of (c) of (3.4) (obtained from applying 3.4 to the inverse flow) to add closed orbits of index $n-3$. We repeat this until we have A_{n-3} such orbits. We again take a Kupka–Smale approximation to get the desired Morse–Smale flow.

4. Morse–Smale flows on S^3

In this section we consider the periodic behavior of a non-singular Morse–Smale flow on S^3. Conversations with D. Asimov were valuable for the preparation of this section.

(4.1) THEOREM. *Necessary and sufficient conditions for the existence of a non-singular Morse–Smale flow on S^3 with A_k untwisted closed orbits of index k are:*

(a) $A_0 \geq 1$, $A_2 \geq 1$.
(b) $A_1 \geq A_0 - 1$, $A_1 \geq A_2 - 1$.

With any specified numbers of untwisted orbits the number of twisted orbits of index 1 is completely arbitrary. There can be no twisted orbits of index 0 or 2.

Proof. We first note that the necessity of (a) and (b) follows from (a) and (b) of Theorem A.

For the sufficiency we consider Figure 3.

We have two round handles (see [1]) of index 0 (i.e. $S^1 \times D^2$) labelled R_0 and R_0' on which we can put a vector field perpendicular to the boundaries and pointing inward, and such that there will be a single closed orbit of index zero in each of R_0 and R_0'. To these we attach a round one handle $R_1 = S^1 \times D^1 \times D^1$.

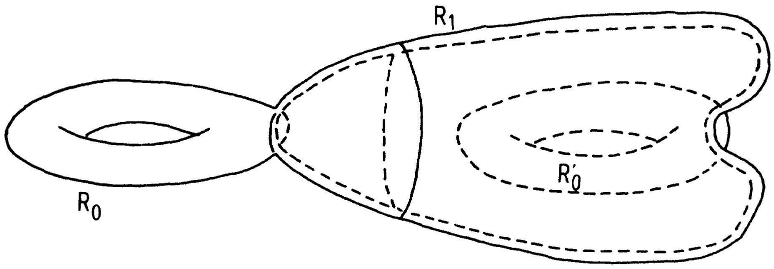

Figure 3

They are attached as shown by identifying $S^1 \times D^1 \times S^0$ with embedded copies of $S^1 \times D^1$ in the boundary of R_0 and R_0'. On R_1 we can construct a vector field with a single untwisted closed orbit of index 1 in such a way that the vector field agrees with the vector field on R_0 and R_0' where identifications are made and is transverse inward on the remainder of R_1. In co-ordinates (θ, x, y) for $R_1 = S^1 \times D^1 \times D^1$ this vector field could be $X = \partial/\partial\theta - x(\partial/\partial x) + y(\partial/\partial y)$.

Noting now that the complement of $R_0 \cup R_0' \cup R_1$ in S^3 is two disjoint copies of $S^1 \times D^2$, we add two such solid tori on each of which an outward pointing vector field with one closed orbit of index 2 has been constructed. These vector fields are constructed to match up with those already defined on the boundary of $R_0 \cup R_0' \cup R_1$. We have thus constructed a Morse–Smale flow on S^3 with $A_0 = 2$, $A_1 = 1$, $A_2 = 2$.

We can however iterate this construction as shown in Figure 4 to create a flow with $A_0 = m$, $A_1 = m - 1$, $A_2 = m$.

For the general case we suppose that A_0, A_1, A_2 satisfying (a) and (b) are given and let $m = \min\{A_0, A_2\}$. We consider the case $A_2 = m$ the other being similar. First, as above we can construct a flow with m closed orbits of index 0, $m-1$ of index 1, and m of index 2. Then using (3.5) we add $A_0 - m$ cancelling pairs of closed orbits of index zero and one. Finally we use (3.4) (c) to add $A_1 - A_0$ untwisted orbits of index one.

It is clear that there can be no twisted orbits of index 0 or 2 since it is not possible to embed an unoriented three manifold in S^3.

To see that the number of twisted orbits of index 1 is arbitrary we consider the following example. It is not difficult to construct an embedding of the disk D^2 in

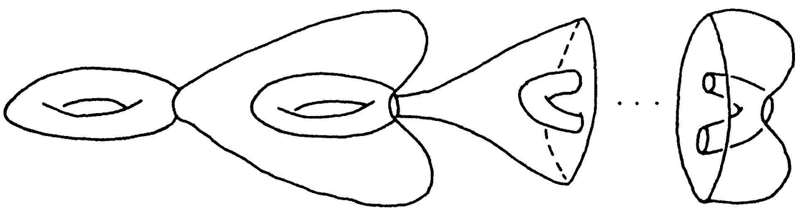

Figure 4

its interior with precisely three hyperbolic periodic points: a sink of period 2 and a saddle whose unstable manifold has its orientation reversed. By taking the suspension (or mapping torus) of this embedding and rounding off corners we obtain a flow on the solid torus pointing inward on the boundary and with one untwisted orbit of index zero and one twisted orbit of index one.

If we now take any non-singular Morse–Smale flow on S^3, cut out a tubular neighborhood of an index zero closed orbit and replace it by the example above we have increased the number of twisted closed orbits by one without changing the numbers of untwisted ones. Repeated application will give any desired number of twisted closed orbits of index one. Finally we take a Kupka–Smale approximation (see [9]) to achieve transversality of stable and unstable manifolds, and this will not change the periodic behavior.

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