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Cohomology theories and infinite CW-complexes

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Dedicated to Prof. B. Eckmann on the occasion of his sixtieth birthday

Introduction

Let h^* be a representable cohomology theory on $\prod CW$, the pointed homotopy category of CW-complexes. Suppose that $X \in \prod CW$ is an infinite complex and that $\{X_\alpha\}$ denotes the system of all finite subcomplexes, directed by inclusion. Then we consider the canonical map

$$\theta : h^n(X) \to \varprojlim_\alpha h^n(X_\alpha);$$

by [1, Theorem 1.8] θ is surjective. We are interested in the *kernel* of θ. For h^* of *finite type* it was proved in [5] that $\ker \theta \cong \varprojlim{}^1 h^{n-1}(X_\alpha)$; in other words there exists a (generalized) Milnor exact sequence

$$0 \longrightarrow \varprojlim_\alpha{}^1 h^{n-1}(X_\alpha) \longrightarrow h^n(X) \xrightarrow{\theta} \varprojlim_\alpha h^n(X_\alpha) \longrightarrow 0.$$

The main goal of this paper is to show that this sequence remains valid for important classes of cohomology theories which are not of finite type. Instead we assume that there exists a homology theory F_* of finite type (defined by a CW-spectrum F) and an abelian group G fitting into a natural short exact sequence

$$0 \to \operatorname{Ext}(F_{n-1}(X), G) \to h^n(X) \to \operatorname{Hom}(F_n(X), G) \to 0.$$

Under these conditions (Theorem 1.1)

(1) $\ker \theta \cong \varprojlim{}^1_\alpha h^{n-1}(X_\alpha)$;
(2) $\ker \theta \cong \operatorname{Pext}(F_{n-1}(X), G)$;
(3) $\varprojlim{}^i_\alpha h^n(X_\alpha) = 0$ *for all* $i \geq 2$.

The second statement extends Theorem 3 of [19], while (3) says that the Bousfield–Kan homotopy spectral sequence

$$E_2^{p,q} = \varprojlim_\alpha{}^p h^q(X_\alpha) \overset{p}{\Longrightarrow} h^{p+q}(X)$$

collapses, which in fact implies (1). Our proof however does not rely on this spectral sequence. It is rather based on several algebraic propositions which partially seem to be new and which may also be useful in other contexts. Among other results the following are established (Corollary 1.5):

Let $\{A_\alpha\}$ be a direct system of finitely generated abelian groups, and G any abelian group. Then

(i) $\varprojlim{}^1 \operatorname{Hom}(A_\alpha, G) \cong \operatorname{Pext}(\varinjlim A_\alpha, G)$;
(ii) $\varprojlim{}^i \operatorname{Hom}(A_\alpha, G) = 0$ for all $i \geq 2$;
(iii) $\varprojlim{}^i \operatorname{Ext}(A_\alpha, G) = 0$ for all $i \geq 1$.

In part II we use Theorem 1.1 (2) to investigate the structure of $\ker \theta$ (Proposition 2.2). In particular we give sufficient conditions for the vanishing of $\ker \theta$ (Proposition 2.1).

In the third part we assume, for any abelian group G, the existence of a universal coefficient sequence

$$0 \to \operatorname{Ext}(F_{n-1}(X), G) \to EG^n(X) \to \operatorname{Hom}(F_n(X), G) \to 0,$$

where EG^* is defined by a spectrum E with coefficients in G [2]. We study the question whether such a sequence *splits*. The case $E = KO$ (the spectrum of real K-theory) shows that this is not always true. But we can give sufficient conditions for such a universal coefficient sequence to split (Theorem 3.7). In particular we show that *in complex K-theory the sequence*

$$0 \to \operatorname{Ext}(K_{n-1}(X), G) \to KG^n(X) \to \operatorname{Hom}(K_n(X), G) \to 0$$

is split exact under rather general conditions; e.g., *if the torsion subgroup of G is divisible or bounded* (Corollary 3.8). To prove this corollary we use essentially that $G \mapsto KG^*(X)$ is a *functor* (Proposition 3.3).

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I. The kernel of θ and inverse systems $\{\operatorname{Hom}(A_\alpha, G)\}$

I.1 We consider a representable cohomology theory h^* on $\prod CW$ which satisfies the following condition:

(C) There is a homology theory F_* of finite type (defined by a CW-spectrum F)

and an abelian group G fitting into natural short exact sequences

$$0 \to \operatorname{Ext}(F_{*-1}(X), G) \to h^*(X) \to \operatorname{Hom}(F_*(X), G) \to 0.$$

THEOREM 1.1. *Let h^* be a representable cohomology theory on $\prod CW$ satisfying* (C). *If, for $X \in \prod CW$, $\{X_\alpha\}$ denotes any directed system of finite subcomplexes of X with $\bigcup_\alpha X_\alpha = X$, then, for every $n \in \mathbf{Z}$,*

(1) *there is a short exact sequence*

$$0 \longrightarrow \varprojlim_\alpha{}^1\, h^{n-1}(X_\alpha) \longrightarrow h^n(X) \xrightarrow{\theta} \varprojlim_\alpha h^n(X_\alpha) \longrightarrow 0;$$

(2) $\ker \theta \cong \operatorname{Pext}(F_{n-1}(X), G)$;
(3) $\varprojlim_\alpha{}^i\, h^n(X_\alpha) = 0$ *for all $i \geq 2$.*

For abelian groups A and G, $\operatorname{Pext}(A, G)$ denotes the subgroup of $\operatorname{Ext}(A, G)$ whose elements correspond to the classes of pure extensions (see e.g. [9]). Recall that an extension $0 \to G \xrightarrow{\mu} E \to A \to 0$ is said to be *pure* provided $\mu(G)$ is a pure subgroup of E, i.e., for every $n \in \mathbf{N}$, $nE \cap \mu(G) = n\mu(G)$.

The assumptions of Theorem 1.1 are satisfied for every cohomology theory of finite type [19]. Furthermore they hold for the theories HG^*, KG^*, KOG^* (ordinary cohomology, complex and real K-theory with coefficients in G, respectively), as was shown by Anderson [3] and Yosimura [19]; these theories are not of finite type if the group G is not. The corresponding homologies are H_*, K_* and KSp_* (ordinary homology, complex K-homology and symplectic K-homology, respectively). Moreover, if E is a CW-spectrum with $\pi_*(E)$ of finite type, and G is either a direct sum or a direct product of finitely generated abelian groups, then condition (C) holds for $h^* = EG^*$ (which is defined by the spectrum $E \wedge SG$, where SG denotes the Moore spectrum of type G [2]). This follows immediately from [19] (Theorem 2 and the proof of Lemma 7).

I.2 For the first step in the proof of Theorem 1.1 we consider the diagram

$$\text{(D)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(F_{n-1}(X), G) & \longrightarrow & h^n(X) & \longrightarrow & \operatorname{Hom}(F_n(X), G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & \varprojlim_\alpha \operatorname{Ext}(F_{n-1}(X_\alpha), G) & \longrightarrow & \varprojlim_\alpha h^n(X_\alpha) & \longrightarrow & \varprojlim_\alpha \operatorname{Hom}(F_n(X_\alpha), G), & & \end{array}$$

which is commutative and has exact rows. Since F_* is given by a spectrum, we have $\varinjlim_\alpha F_n(X_\alpha) \cong F_n(X)$ [17, Corollary 8.35], hence ψ is an isomorphism. So the ker-coker-sequence of (D) yields isomorphims

$$\ker \varphi \cong \ker \theta \quad \text{and} \quad \operatorname{coker} \varphi \cong \operatorname{coker} \theta.$$

We therefore investigate the canonical map

$$\varphi : \operatorname{Ext}(\varinjlim A_\alpha, G) \to \varprojlim \operatorname{Ext}(A_\alpha, G),$$

where $\{A_\alpha\}$ is an arbitrary direct system of abelian groups.

PROPOSITION 1.2. *Let $\{A_\alpha\}$ be a direct system of abelian groups over a directed index set, and let G be any abelian group. Then there exist a 4-term exact sequence*

$$0 \longrightarrow \varprojlim{}^1 \operatorname{Hom}(A_\alpha, G) \longrightarrow \operatorname{Ext}(\varinjlim A_\alpha, G) \xrightarrow{\varphi} \varprojlim \operatorname{Ext}(A_\alpha, G) \longrightarrow \varprojlim{}^2 \operatorname{Hom}(A_\alpha, G) \longrightarrow 0$$

and isomorphisms

$$\varprojlim{}^i \operatorname{Ext}(A_\alpha, G) \xrightarrow{\sim} \varprojlim{}^{i+2} \operatorname{Hom}(A_\alpha, G) \quad \textit{for} \quad i \geq 1.$$

This proposition is contained in [18]. It can be proved by a spectral sequence argument of Roos. We give here an elementary proof which is based on the following special case of a theorem of Nöbeling [15].

LEMMA 1.3. *Let D be an injective abelian group, and let $\{A_\alpha \mid \alpha \in I\}$ be a direct system over a directed set I. Then*

$$\varprojlim{}^i \operatorname{Hom}(A_\alpha, D) = 0 \quad \textit{for all} \quad i \geq 1.$$

Proof of Proposition 1.2. Suppose that $0 \to G \to D \to S \to 0$ is an injective presentation of G. Then there exist two commutative diagrams (E1) and (E2) with exact rows:

$$\text{(E1)} \quad \begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(A, G) & \longrightarrow & \operatorname{Hom}(A, D) & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\kappa} & & \\
0 & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, G) & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, D) & \xrightarrow{\delta} & \varprojlim K_\alpha & \longrightarrow & \varprojlim{}^1 \operatorname{Hom}(A_\alpha, G) \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \operatorname{Hom}(A, S) & \longrightarrow & \operatorname{Ext}(A, G) & \longrightarrow & 0 \\
 & & \downarrow{\kappa} & & \downarrow{\cong} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & \varprojlim K_\alpha & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, S) & \xrightarrow{\omega} & \varprojlim \operatorname{Ext}(A_\alpha, G) & \longrightarrow & \varprojlim{}^1 K_\alpha \longrightarrow \cdots
\end{array}$$

Thereby $A = \varinjlim A_\alpha$, K denotes the kernel of $\operatorname{Hom}(A, S) \to \operatorname{Ext}(A, G)$ and K_α the kernel of $\operatorname{Hom}(A_\alpha, S) \to \operatorname{Ext}(A_\alpha, G)$ respectively. Since the bottom sequence of (E1) is exact, by Lemma 1.3 there exist isomorphisms

(i) $\varprojlim^i K_\alpha \cong \varprojlim^{i+1} \operatorname{Hom}(A_\alpha, G)$ for $i \geq 1$

and $\operatorname{coker} \delta \cong \varprojlim^1 \operatorname{Hom}(A_\alpha, G)$. But $\operatorname{im} \kappa = \operatorname{im} \delta$, hence $\operatorname{coker} \kappa \cong \varprojlim^1 \operatorname{Hom}(A_\alpha, G)$. From this isomorphism and from the ker-coker-sequence of the diagram (E2) it follows that

$$\ker \varphi \cong \varprojlim{}^1 \operatorname{Hom}(A_\alpha, G).$$

Now by Lemma 1.3 $\varprojlim^i \operatorname{Hom}(A_\alpha, S) = 0$ for all $i \geq 1$ since S is injective. Hence by exactness of the bottom sequence of (E2)

(ii) $\varprojlim^i \operatorname{Ext}(A_\alpha, G) \cong \varprojlim^{i+1} K_\alpha$ for $i \geq 1$

and $\operatorname{coker} \omega \cong \varprojlim^1 K_\alpha$. The latter isomorphism together with (i) implies that $\operatorname{coker} \omega \cong \varprojlim^2 \operatorname{Hom}(A_\alpha, G)$. Thus, since $\operatorname{im} \varphi = \operatorname{im} \omega$, there exists an isomorphism

$$\operatorname{coker} \varphi \cong \varprojlim{}^2 \operatorname{Hom}(A_\alpha, G).$$

Finally, it follows from (i) and (ii) that

$$\varprojlim{}^i \operatorname{Ext}(A_\alpha, G) \cong \varprojlim{}^{i+2} \operatorname{Hom}(A_\alpha, G) \quad \text{for} \quad i \geq 1.$$

With regard to our main theorem 1.1 the above proposition gives us the following partial result:

(1) $\ker \theta \cong \varprojlim_\alpha^1 \operatorname{Hom}(F_{n-1}(X_\alpha), G)$;

(2) $\operatorname{coker} \theta \cong \varprojlim_\alpha^2 \operatorname{Hom}(F_{n-1}(X_\alpha), G)$.

1.3 Recall that F_* is assumed of *finite type*. As is well-known this implies that the groups $F_*(X_\alpha)$ are finitely generated. To take advantage of this fact we need another proposition.

PROPOSITION 1.4. *Let $\{A_\alpha\}$ be a direct system of abelian groups over a directed index set, and let G be any abelian group. Then there exist a 4-term exact* *sequence*

$$0 \to \varprojlim{}^1 \operatorname{Hom}(A_\alpha, G) \to \operatorname{Pext}(\varinjlim A_\alpha, G) \to \varprojlim \operatorname{Pext}(A_\alpha, G) \to \varprojlim{}^2 \operatorname{Hom}(A_\alpha, G) \to 0$$

and isomorphisms

$$\varprojlim{}^i \operatorname{Pext}(A_\alpha, G) \xrightarrow{\simeq} \varprojlim{}^{i+2} \operatorname{Hom}(A_\alpha, G) \quad \textit{for} \quad i \geqslant 1.$$

This will be proved in I.4. We shall make use of the following consequence.

COROLLARY 1.5. *Let $\{A_\alpha \mid \alpha \in I\}$ be a direct system of finitely generated abelian groups over a directed set I, and let G be any abelian group. Then*

(i) $\varprojlim^1 \operatorname{Hom}(A_\alpha, G) \cong \operatorname{Pext}(\varinjlim A_\alpha, G)$;
(ii) $\varprojlim^i \operatorname{Hom}(A_\alpha, G) = 0$ *for all* $i \geqslant 2$;
(iii) $\varprojlim^i \operatorname{Ext}(A_\alpha, G) = 0$ *for all* $i \geqslant 1$.

Proof. Since every finitely generated abelian group is *pure-projective*, i.e. projective relative to pure exact sequences [9, Theorem 30.2], we have $\operatorname{Pext}(A_\alpha, G) = 0$ for every $\alpha \in I$. Now the corollary follows immediately from Propositions 1.2 and 1.4.

Remark. The statements (i) and (ii) of Corollary 1.5 are implicit in [12, p. 37].

Proof of Theorem 1.1. The exact sequence of inverse systems

$$0 \to \{\operatorname{Ext}(F_{*-1}(X_\alpha), G)\} \to \{h^*(X_\alpha)\} \to \{\operatorname{Hom}(F_*(X_\alpha), G)\} \to 0$$

induces a long exact sequence

$$0 \to \varprojlim_\alpha \operatorname{Ext}(F_{*-1}(X_\alpha), G) \to \varprojlim_\alpha h^*(X_\alpha) \to \varprojlim_\alpha \operatorname{Hom}(F_*(X_\alpha), G) \to$$
$$\varprojlim_\alpha{}^1 \operatorname{Ext}(F_{*-1}(X_\alpha), G) \to \varprojlim_\alpha{}^1 h^*(X_\alpha) \to \varprojlim_\alpha{}^1 \operatorname{Hom}(F_*(X_\alpha), G) \to \cdots$$

Now, since the systems $\{F_*(X_\alpha)\}$ consist of finitely generated abelian groups, by Corollary 1.5 we have $\varprojlim^i_\alpha \operatorname{Hom}(F_n(X_\alpha), G) = 0$ for all $i \geqslant 2$, $\varprojlim^i_\alpha \operatorname{Ext}(F_{n-1}(X_\alpha), G) = 0$ for all $i \geqslant 1$, as well as

$$\varprojlim_\alpha{}^1 \operatorname{Hom}(F_{n-1}(X_\alpha), G) \cong \operatorname{Pext}(F_{n-1}(X), G).$$

Hence, using the result of I.2, we obtain

(1) $\ker \theta \cong \varprojlim_{\alpha}{}^1 \operatorname{Hom}(F_{n-1}(X_\alpha), G) \cong \varprojlim_{\alpha}{}^1 h^{n-1}(X_\alpha)$;

(2) $\ker \theta \cong \operatorname{Pext}(F_{n-1}(X), G)$;

(3) $\varprojlim_{\alpha}{}^i h^n(X_\alpha) = 0$ for all $i \geq 2$.

This completes the proof of Theorem 1.1.

Remark. Furthermore, we see that

$$\operatorname{coker} \theta \cong \varprojlim_{\alpha}{}^2 \operatorname{Hom}(F_{n-1}(X_\alpha), G) = 0;$$

thus our method provides us in addition with an alternative proof, in this case, for θ being surjective.

I.4 An abelian group Q is called *pure-injective* or *algebraically compact* if, for every pure exact sequence $0 \to A' \xrightarrow{\mu} A \to A'' \to 0$ and for every $\alpha : A' \to Q$, there exists a $\beta : A \to Q$ making

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "\mu"] \arrow[d, "\alpha"'] & A \arrow[dl, "\beta"] \\ & Q & \end{tikzcd}$$

commute. For the structure of algebraically compact groups see [9, Ch. VII].

In the proof of Proposition 1.4 we shall make use of the following generalization of Lemma 1.3.

LEMMA 1.6. *Let Q be algebraically compact. Then*

$$\varprojlim{}^i \operatorname{Hom}(A_\alpha, Q) = 0$$

for every direct system $\{A_\alpha \mid \alpha \in I\}$ of abelian groups over a directed set I, and for all $i \geq 1$.

Proof. The groups $\bigoplus_{\beta \leq \alpha} A_\beta$ together with the natural injections form a direct system over I. Define

$$\zeta_\alpha : \bigoplus_{\beta \leq \alpha} A_\beta \to A_\alpha$$

by the maps $A_\beta \to A_\alpha$ of the given system. Thus one obtains a short exact sequence

$$0 \to \{K_\alpha\} \to \left\{\bigoplus_{\beta \leq \alpha} A_\beta\right\} \to \{A_\alpha\} \to 0$$

of direct systems where K_α is the kernel of ζ_α. For every $\alpha \in I$ the sequence

$$0 \to K_\alpha \to \bigoplus_{\beta \leq \alpha} A_\beta \to A_\alpha \to 0$$

is split exact. Thus by [9, Theorem 29.4] the induced sequence

$$0 \to \varinjlim K_\alpha \to \bigoplus_{\alpha \in I} A_\alpha \to \varinjlim A_\alpha \to 0$$

is pure exact. This sequence gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(\varinjlim A_\alpha, Q) \arrow[r] \arrow[d, "\cong"] & \prod_{\alpha \in I} \operatorname{Hom}(A_\alpha, Q) \arrow[r, "\chi"] \arrow[d, "\cong"] & \operatorname{Hom}(\varinjlim K_\alpha, Q) \arrow[d, "\cong"] & \\
0 \arrow[r] & \varprojlim \operatorname{Hom}(A_\alpha, Q) \arrow[r] & \varprojlim_\alpha \prod_{\beta \leq \alpha} \operatorname{Hom}(A_\beta, Q) \arrow[r, "\psi"] & \varprojlim \operatorname{Hom}(K_\alpha, Q) \arrow[r] & \\
& & & \varprojlim^1 \operatorname{Hom}(A_\alpha, Q) \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows. Since Q is algebraically compact, χ and thus ψ are epimorphic. On the other hand by [12, Théorème 1.8]

$$\varprojlim_\alpha{}^i \prod_{\beta \leq \alpha} \operatorname{Hom}(A_\beta, Q) = 0 \quad \text{for all} \quad i \geq 1.$$

Hence

$$\varprojlim{}^1 \operatorname{Hom}(A_\alpha, Q) = 0$$

and

$$\varprojlim{}^i \operatorname{Hom}(K_\alpha, Q) \cong \varprojlim{}^{i+1} \operatorname{Hom}(A_\alpha, Q) \quad \text{for all} \quad i \geq 1.$$

Since we have proved this for an arbitrary system $\{A_\alpha\}$, the lemma follows by induction.

Remark. Lemma 1.6 is sharp in the following sense: If G is an abelian group such that, for every direct system $\{A_\alpha\}$ and for all $i \geq 1$, $\varprojlim^i \operatorname{Hom}(A_\alpha, G) = 0$, then G is algebraically compact. This follows easily from Proposition 1.4.

Proof of Proposition 1.4. By [9, Corollary 38.4] there exists a pure exact sequence $0 \to G \to Q \to H \to 0$ with Q algebraically compact. Note that in this situation H is algebraically compact as well. By [9, Theorem 53.7] this sequence induces commutative diagrams

$$\text{(P1)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(A, G) & \longrightarrow & \operatorname{Hom}(A, Q) & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, G) & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, Q) & \longrightarrow & \varprojlim K_\alpha & \longrightarrow & \\ & & & & & & \varprojlim^1 \operatorname{Hom}(A_\alpha, G) & \longrightarrow & \cdots \end{array}$$

and

$$\text{(P2)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \operatorname{Hom}(A, H) & \longrightarrow & \operatorname{Pext}(A, G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim K_\alpha & \longrightarrow & \varprojlim \operatorname{Hom}(A_\alpha, H) & \longrightarrow & \varprojlim \operatorname{Pext}(A_\alpha, G) & \longrightarrow & \varprojlim^1 K_\alpha \longrightarrow \cdots \end{array}$$

with exact rows. Thereby $A = \varinjlim A_\alpha$, K denotes the kernel of $\operatorname{Hom}(A, H) \to \operatorname{Pext}(A, G)$, and K_α the kernel of $\operatorname{Hom}(A_\alpha, H) \to \operatorname{Pext}(A_\alpha, G)$, respectively. We may proceed now just in the same way as in the proof of Proposition 1.2, using Lemma 1.6 instead of Lemma 1.3.

II. The structure of ker θ.

II.1. The functors h^n are assumed to be representable; i.e., for every $n \in \mathbf{Z}$ there exists a space B_n such that $h^n(X) = [X, B_n]$. The elements of the kernel of $\theta: h^n(X) \to \varprojlim_\alpha h^n(X_\alpha)$ may therefore be interpreted as the classes of maps from X into B_n whose restriction to any finite subcomplex X_α is nullhomotopic. If such a map on X is not nullhomotopic it is called a *phantom map*.

As an application of Theorem 1.1 we can give sufficient conditions for the non-existence of phantom maps from X into B_n.

PROPOSITION 2.1. *Let h^* be a cohomology satisfying the assumptions of Theorem 1.1 for some group G. If G is algebraically compact then*

$$h^*(X) \cong \varprojlim_\alpha h^*(X_\alpha).$$

Proof. The natural map $\theta: h^*(X) \to \varprojlim_\alpha h^*(X_\alpha)$ is epimorphic and, by Theorem 1.1, $\ker\theta \cong \operatorname{Pext}(F_{n-1}(X), G)$. If G is algebraically compact then $\operatorname{Pext}(F_{n-1}(X), G) = 0$. Hence θ is an isomorphism.

In particular there exist no phantom maps if G is compact or a vector space over an arbitrary field. Thus the above Proposition is a generalization of a statement in [8, Ch VIII]. By the way, for $h^n = EG^n$ and $G = \mathbf{Q}$ or $\hat{\mathbf{Z}}$ the group $EG^n(X)$ coincides with $[X, (B_n)_{(0)}]$ or $[X, (B_n)^\wedge]$ respectively, where $(B_n)_{(0)}$ is the rationalization of the classifying space B_n of E^n and $(B_n)^\wedge$ is the profinite completion of B_n in the sense of [16]. Furthermore Proposition 2.1 holds if G is a direct product of finite groups. It fails however for direct sums of finite groups (see example below).

For a representable cohomology theory h^* of finite type the following holds [18]: The kernel of $\theta: h^n(X) \to \varprojlim_\alpha h^n(X_\alpha)$ is the maximal divisible subgroup of $h^n(X)$. This is not true in general for theories h^* which satisfy the assumptions of Theorem 1.1.

Example. Take $X = M(\mathbf{Z}(p^\infty), 2)$, the Moore space of type $\mathbf{Z}(p^\infty)$ in dimension 2, $G = \bigoplus_{k=1}^\infty \mathbf{Z}/p^k\mathbf{Z}$ and $h^* = H^*(\ ,G)$. Then

$$\begin{aligned}\ker(\theta: H^3(X, G) \to \varprojlim_\alpha H^3(X_\alpha, G)) &\cong \operatorname{Pext}(H_2X, G) = \operatorname{Pext}(\mathbf{Z}(p^\infty), G)\\ &\cong \varprojlim_n{}^1 \operatorname{Hom}(\mathbf{Z}/p^n\mathbf{Z}, G) \neq 0\end{aligned}$$

(cf. [7, Ch. VI]), and $\ker\theta$ is *reduced* (by [9, Lemma 55.3]), i.e. it doesn't contain divisible elements.

However we can extend the above result of [18] as follows.

PROPOSITION 2.2. *Let h^* be a cohomology theory satisfying the assumptions of Theorem 1.1 for some group G. If* $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$, *then the kernel of $\theta: h^n(X) \to \varprojlim_\alpha h^n(X_\alpha)$ is divisible for every $X \in \prod CW$ and every $n \in \mathbf{Z}$. Suppose in addition that G is reduced. Then* $\ker\theta$ *is the maximal divisible subgroup of $h^n(X)$.*

Remark. The assumption $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$ is not very restrictive. It is satisfied e.g. if the p-torsion subgroup of G is finite for every prime p or if the torsion subgroup tG is algebraically compact. In particular $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$ if tG is divisible or bounded.

II.2 For the proof of Proposition 2.2 we need two lemmata.

LEMMA 2.3. *Let A, G be arbitrary abelian groups. Then* $\operatorname{Pext}(A, G)$ *is* *divisible if and only if* $\operatorname{Pext}(tA, G)=0$ (*where* tA *denotes the torsion subgroup of* A).

Proof. Since $0 \to tA \to A \to A/tA \to 0$ is pure exact, it gives rise to an exact sequence

$$\operatorname{Pext}(A/tA, G) \to \operatorname{Pext}(A, G) \to \operatorname{Pext}(tA, G) \to 0.$$

Now $\operatorname{Pext}(A/tA, G) = \operatorname{Ext}(A/tA, G)$ is divisible. Hence $\operatorname{Pext}(A, G)$ is divisible if and only if $\operatorname{Pext}(tA, G)$ is divisible. But by [9, Lemma 55.3] $\operatorname{Pext}(T, G)$ is reduced for every torsion group T. Thus $\operatorname{Pext}(A, G)$ is divisible exactly if $\operatorname{Pext}(tA, G) = 0$.

LEMMA 2.4. *A group* G *has the property that* $\operatorname{Pext}(A, G)$ *is divisible for every* A *if and only if* $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$.

Proof. Let first G be a group with the property that $\operatorname{Pext}(A, G)$ is divisible for every A. Then in particular $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G)$ is divisible. Hence by Lemma 2.3 $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$.

Now assume that G satisfies $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$. By Lemma 2.3 it is sufficient to prove that this implies $\operatorname{Pext}(T, G) = 0$ for every torsion group T. Let first p be a prime and P a p-group. Then there exists a pure subgroup B of P with the properties that B is a direct sum of cyclic p-groups and P/B is a divisible p-group, i.e. a direct sum of groups $\mathbf{Z}(p^\infty)$ [9, Theorem 32.3]. The inclusion $B \subseteq P$ gives rise to an exact sequence

$$\operatorname{Pext}(P/B, G) \to \operatorname{Pext}(P, G) \to \operatorname{Pext}(B, G) = 0.$$

Since $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$, one has $\operatorname{Pext}(P/B, G) = 0$ and thus $\operatorname{Pext}(P, G) = 0$. Suppose now that T is an arbitrary torsion group. Then $T = \bigoplus_p T_p$ where T_p is the p-primary component of T. Hence

$$\operatorname{Pext}(T, G) \cong \prod_p \operatorname{Pext}(T_p, G) = 0.$$

Proof of Proposition 2.2. The first part is an immediate consequence of Lemma 2.4 and Theorem 1.1. Suppose in addition that G is reduced. Then

$$\operatorname{Hom}(\mathbf{Q}, \operatorname{Hom}(F_n(X_\alpha), G)) \cong \operatorname{Hom}(\mathbf{Q} \otimes F_n(X_\alpha), G) = 0,$$

hence $\operatorname{Hom}(F_n(X_\alpha), G)$ is reduced. Since $F_{n-1}(X_\alpha)$ is finitely generated,

$\operatorname{Ext}(F_{n-1}(X_\alpha), G)$ is isomorphic to $\operatorname{Ext}(t(F_{n-1}X_\alpha), G)$ and thus reduced [9, Lemma 55.3]. As an extension of reduced groups $h^n(X_\alpha)$ is reduced again, and therefore $\varprojlim_\alpha h^n(X_\alpha)$ is reduced, too. But then, by the first part, $\ker\theta$ is the maximal divisible subgroup of $h^n(X)$.

III. On the splitting of universal coefficient sequences

III.1 We now assume that the cohomology theory E^* and the homology theory F_* are related, for any abelian group G, by a universal coefficient sequence

$$0 \to \operatorname{Ext}(F_{n-1}(X), G) \to EG^n(X) \to \operatorname{Hom}(F_n(X), G) \to 0, \tag{3.1}$$

as in the case of H^*, K^* and KO^*. The natural question arises whether such a sequence is *split exact* as in the case of ordinary cohomology. Mislin [14] as well as Hilton and Deleanu [11] have discussed the corresponding question for Künneth theorems and universal coefficient sequences of the form

$$0 \to h^n(X) \otimes G \to h^n(X, G) \to \operatorname{Tor}(h^{n+1}(X), G) \to 0 \tag{3.2}$$

where $h^*(X, G)$ denotes the cohomology h^* with coefficients in G in the definition given by Hilton [10]. For infinite complexes X and $h = H$ Hilton's definition doesn't agree in general with the ordinary cohomology $H^*(X, G)$ [10] and therefore doesn't coincide with $HG^*(X)$.

We proceed as follows. First we discuss pure exactness of (3.1) as it is done in [11] for the sequence (3.2). In III.2 we give then conditions for the groups A, B and G in order that every pure exact sequence of the form

$$0 \to \operatorname{Ext}(A, G) \to H \to \operatorname{Hom}(B, G) \to 0$$

splits.

PROPOSITION 3.1. [11] *Let $0 \to R \to S \to T \to 0$ be a short exact sequence of additive functors: $C \to Ab$, where C is either the category Ab of all abelian groups or the category Ab_2 of 2-torsion-free abelian groups. If T is left exact, the sequence $0 \to R(A) \to S(A) \to T(A) \to 0$ is pure exact for every $A \in C$.*

To apply the criterion to the sequence (3.1) we have to know whether putting coefficients into E^* (in the sense of [2]) is a functorial process.

PROPOSITION 3.2. *Let $X \in \prod CW$ and E an arbitrary spectrum. Then $G \mapsto EG^n(X)$ is a functor $Ab_2 \to Ab$.*

Proof. Obviously it is enough to show that $G \mapsto SG$ is a functor from Ab_2 into the homotopy category of spectra. Since $H_0(SG) = G$, this is done by proving that the map

$$\Phi : [SG_1, SG_2] \to \mathrm{Hom}\,(H_0(SG_1), H_0(SG_2))$$

is an isomorphism if G_1 is 2-torsion-free.

The map Φ corresponds to

$$\Phi' : [M(G_1, 3), M(G_2, 3)] \to \mathrm{Hom}\,(H_3(M(G_1, 3), H_3(M(G_2, 3)))$$

where $M(G, 3)$ is the Moore space of type G in dimension 3. Now the universal coefficient theorem for homotopy groups gives an exact sequence

$$0 \to \mathrm{Ext}\,(G_1, \pi_4(M(G_2, 3)) \to \pi_3(G_1, M(G_2, 3)) \xrightarrow{\Phi'} \mathrm{Hom}\,(G_1, \pi_3(M(G_2, 3))) \longrightarrow 0.$$

To determine the kernel of Φ' we compute $\pi_4(M(G_2, 3))$:

$$\pi_4(M(G_2, 3)) = \pi_4^s(M(G_2, 3)) = (\pi G_2^s)_1(S^0)$$

(the first stable homotopy group of S^0 with coefficients in G_2). From the universal coefficient theorem for stable homotopy [2] we have

$$(\pi G_2^s)_1(S^0) = \pi_1^s(S^0) \otimes G_2 = \mathbf{Z}/2\mathbf{Z} \otimes \mathrm{G}_2.$$

This implies $\ker \Phi' = \mathrm{Ext}\,(G_1, \mathbf{Z}/2\mathbf{Z} \otimes G_2) = 0$, since $G_1 \in Ab_2$.

Proposition 3.2 is analogous to a result of [10]. However for "good theories" E^*, i.e. cohomology theories which don't detect the Hopf map $\eta : S^3 \to S^2$, we are not able to prove that $G \mapsto EG^n(X)$ is a functor on Ab, corresponding to [11, Appendix]. But we can show this in an important special case.

PROPOSITION 3.3. *Let $X \in \prod CW$. Then $G \mapsto KG^n(X)$ is a functor $Ab \to Ab$, where K^* is complex K-theory.*

Proof. Let $G_1, G_2 \in Ab$. We claim that the map

$$K \wedge - : [SG_1, SG_2] \to [K \wedge SG_1, K \wedge SG_2],$$

defined in the obvious way, factors through Φ:

$$\begin{tikzcd} {[SG_1, SG_2]} \arrow[r, "\Phi"] \arrow[d, "K \wedge -"] & \mathrm{Hom}\,(G_1, G_2) \arrow[dl, dotted, "\tau"] \\ {[K \wedge SG_1, K \wedge SG_2]} & \end{tikzcd}$$

From the existence of τ it follows easily that $G \mapsto KG^n(X)$ is a functor.

We prove the claim by showing that

$$\ker \Phi \subseteq \ker (K \wedge -).$$

Let $f: SG_1 \to SG_2$ be a map whose homotopy class is in the kernel of Φ. Then there is a factorization of f (up to homotopy)

$$\begin{tikzcd} SG_1 \arrow[rr, "f"] \arrow[dr] & & SG_2 \\ & \sum SR \arrow[ur] & \end{tikzcd}$$

where $0 \to R \to F \to G_1 \to 0$ is a free presentation of G_1. From the induced diagram

$$\begin{tikzcd} \pi_*(K \wedge SG_1) \arrow[rr, "f_*"] \arrow[dr] & & \pi_*(K \wedge SG_2) \\ & \pi_*(K \wedge \sum SR) \arrow[ur] & \end{tikzcd}$$

we see that f_* is trivial. By the Künneth theorem for complex K-theory we have the diagram

$$\begin{tikzcd} K_m(X) \otimes K_n(\sum SR) \arrow[r, "\simeq"] \arrow[d] & K_{m+n}(X \wedge \sum SR) \arrow[d] \\ K_m(X) \otimes K_n(SG_2) \arrow[r] & K_{m+n}(X \wedge SG_2) \end{tikzcd}$$

Hence $f_*:(KG_1)_*(X)\to(KG_2)_*(X)$ is trivial for any $X\in\prod CW$ (or spectrum X). If V is any finite spectrum and V^* its S-dual, $KG^{-n}(V)=KG_n(V^*)$ and $(K\wedge f)_*:KG_1^*(V)\to KG_2^*(V)$ is the zero map. In particular $(K\wedge f)_*$: $[V_\alpha, K\wedge SG_1]\to[V_\alpha, K\wedge SG_2]$ is trivial for every finite subspectrum V_α of $K\wedge SG_1$. Therefore, to prove that $K\wedge f$ is nullhomotopic we only need to show that there exist no phantom maps (in the category of spectra) from $K\wedge SG_1$ into $K\wedge SG_2$. For this purpose we extend our result (Theorem 1.1)

$$\ker\theta\cong\varprojlim_\alpha{}^1 KG^{n-1}(X_\alpha)\cong\operatorname{Pext}(K_{n-1}(X), G)$$

to arbitrary spectra X. This is possible since the universal coefficient sequence in K-theory is valid for arbitrary spectra. Moreover the relation

$$E_*(X)\cong\varinjlim_\alpha E_*(X_\alpha)$$

subsists for spectra E and X, where $\{X_\alpha\}$ runs over all finite subspectra [17, p. 331, Remark 1]. By definition $[K\wedge SG_1, K\wedge SG_2]=(KG_2)^0(K\wedge SG_1)$. Therefore, the phantom maps correspond to $\operatorname{Pext}(K_1(K\wedge SG_1), G_2)$. But $K_1(K\wedge SG_1)=(KG)_1(K)=0$, since $K_1(K)=0$ and $K_0(K)$ is torsion-free [17, p. 423]. Hence $K\wedge f$ is nullhomotopic.

COROLLARY 3.4

(a) *Let E^* be a cohomology theory with a universal coefficient sequence*

$$0\to\operatorname{Ext}(F_{n-1}(X), G)\to EG^n(X)\to\operatorname{Hom}(F_n(X), G)\to 0.$$

Then this sequence is pure exact for every 2-torsion-free group G.

(b) *The sequence*

$$0\to\operatorname{Ext}(K_{n-1}(X), G)\to KG^n(X)\to\operatorname{Hom}(K_n(X), G)\to 0$$

is pure exact for every group G.

For $E=KO$ the condition on G is necessary.

Example. The sequence

$$0\to\operatorname{Ext}(KO_1(S^0), \mathbf{Z}/2\mathbf{Z})\to KO^{-2}(S^0, \mathbf{Z}/2\mathbf{Z})\to\operatorname{Hom}(KO_2(S^0), \mathbf{Z}/2\mathbf{Z})\to 0$$

doesn't split because $KO^{-2}(S^0, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/4\mathbf{Z}$ [4]. Moreover it can't be pure exact since Hom $(KO_2(S^0), \mathbf{Z}/2\mathbf{Z})$ is pure-projective by [9, Theorem 30.2].

III.2 It remains to give conditions for the groups A, B and G in order that every pure extension of Hom (B, G) by Ext (A, G) splits or, equivalently, in order that Pext (Hom (B, G), Ext $(A, G)) = 0$.

PROPOSITION 3.5. *Let* A, B *and* G *be abelian groups. Then* Pext (Hom (B, G), Ext $(A, G)) = 0$, *if* (i) *or* (ii) *is satisfied:*

(i) Pext $(tA, G) = 0$;
(ii) t(Hom (B, G)) *is a direct sum of cyclic groups.*

Remark. Condition (i) holds if tA is a direct sum of cyclic groups or, by Lemmata 2.3 and 2.4, if Pext $(\mathbf{Q}/\mathbf{Z}, G) = 0$. Condition (ii) is satisfied e.g. if B is a bounded torsion group.

For the proof of Proposition 3.5 we need the following lemma:

LEMMA 3.6. *For arbitrary abelian groups* A, C *and* G

$$\text{Pext}\,(C, \text{Ext}\,(A, G)) \cong \text{Pext}\,(tC, \text{Ext}\,(tA, G)).$$

Proof. From

$$\text{Ext}\,(C, \text{Ext}\,(A, G)) \cong \text{Ext}\,(\text{Tor}\,(C, A), G)$$

we deduce that

$$\text{Pext}\,(C, \text{Ext}\,(A, G)) \cong \text{Pext}\,(\text{Tor}\,(C, A), G).$$

Now the lemma follows from the fact that Tor $(C, A) \cong$ Tor (tC, tA).

Proof of Proposition 3.5. By Lemma 3.6 Pext (Hom (B, G), Ext $(A, G)) = 0$, if Ext (tA, G) is algebraically compact or if t(Hom (B, G)) is a direct sum of cyclic groups. Thus it remains to prove that Ext (tA, G) is algebraically compact suppose Pext $(tA, G) = 0$. But this follows from [9, Proposition 54.2], since Ext (tA, G) is reduced and

$$\text{Ext}\,(\mathbf{Q}, \text{Ext}\,(tA, G)) \cong \text{Ext}\,(\text{Tor}\,(\mathbf{Q}, tA), G) = 0.$$

III.3 We are now ready to state our main result about the splitting of universal coefficient sequences.

THEOREM 3.7. *Let E^* be a cohomology theory with a universal coefficient sequence*

$$0 \to \mathrm{Ext}\,(F_{n-1}(X), G) \to EG^n(X) \to \mathrm{Hom}\,(F_n(X), G) \to 0.$$

Suppose that G is 2-torsion-free or that $G \mapsto EG^n(X)$ is a functor $Ab \to Ab$. Then each of the following conditions implies the splitting of this sequence:

(a) $\mathrm{Pext}\,(\mathbf{Q}/\mathbf{Z}, G) = 0$ (cf. II.1);

(b) *$t(F_{n-1}X)$ is a direct sum of cyclic groups;*

(c) *$F_n(X)$ is a bounded torsion group;*

(d) *E^* is of finite type and there exist no phantom maps from X into the representing space of EG^n.*

Proof. Suppose that G is 2-torsion-free or that $G \mapsto EG^n(X)$ is a functor on Ab. Then by Corollary 3.4 the sequence

$$0 \to \mathrm{Ext}\,(F_{n-1}(X), G) \to EG^n(X) \to \mathrm{Hom}\,(F_n(X), G) \to 0$$

is pure exact. Hence it is sufficient to prove that

$$\mathrm{Pext}\,(\mathrm{Hom}\,(F_n(X), G), \mathrm{Ext}\,(F_{n-1}(X), G)) = 0,$$

if any of the conditions (a)–(d) is satisfied.

If (a), (b) or (c) holds, then

$$\mathrm{Pext}\,(\mathrm{Hom}\,(F_n(X), G), \mathrm{Ext}\,(F_{n-1}(X), G)) = 0$$

by Proposition 3.5 and the remark following it. Now suppose that condition (d) is satisfied. Then by [6, Lemma 2.6] F_* is of finite type. Thus it follows from Theorem 1.1 that

$$\mathrm{Pext}\,(F_{n-1}(X), G) \cong \ker\,(\theta : EG^n(X) \to \varprojlim_{\alpha} EG^n(X_\alpha)),$$

hence $\mathrm{Pext}\,(F_{n-1}(X), G) = 0$. Then by exactness of

$$\mathrm{Pext}\,(F_{n-1}X, G) \to \mathrm{Pext}\,(t(F_{n-1}X), G) \to 0$$

one has $\mathrm{Pext}\,(t(F_{n-1}X), G) = 0$ and thus, by Proposition 3.5,

$$\mathrm{Pext}\,(\mathrm{Hom}\,(F_n(X), G), \mathrm{Ext}\,(F_{n-1}(X), G)) = 0.$$

As a consequence the universal coefficient sequence for complex K-theory splits under rather general conditions.

COROLLARY 3.8. *Suppose that G has divisible or bounded torsion. Then the sequence*

$$0 \to \operatorname{Ext}(K_{n-1}(X), G) \to KG^n(X) \to \operatorname{Hom}(K_n(X), G) \to 0$$

(cf. [19]) *is split exact for every $X \in \prod CW$.*

Proof. Suppose that tG is divisible or bounded. Then $\operatorname{Pext}(\mathbf{Q}/\mathbf{Z}, G) = 0$ by the remark following Proposition 2.2. The assertion follows now from Theorem 3.7 since $G \mapsto KG^n(X)$ is a functor on Ab (Proposition 3.3).

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