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## Level sets of infinite length

GEORGE PIRANIAN AND ALLEN WEITSMAN

Dedicated to Professor Albert Pfluger on his seventieth birthday

The geometry of level sets plays an important role in the analysis of various function-theoretic problems. Often, however, the level sets are so complicated that one must either choose sets associated with special levels (see [3], [4]) or resort to the approximation of level sets by shorter curves (see [1, pp. 550–553]).

In [3] and [4], there are some weak estimates on the length  $\ell(r, R)$  of the sets  $\{z : |f(z)| = R, |z| < r\}$  associated with a function  $f$  meromorphic in the plane. For such an  $f$  we do not know whether the quantity  $\ell(r, R)$  can be bounded in terms of Nevanlinna's characteristic function  $T$  without reference to exceptional levels. But the following is implicit in the results in [3, pp. 121–123] and [4, p. 44]: *If  $f$  is meromorphic in  $|z| \leq 2r$  and  $f(0) = 1$ , then each subinterval  $[\alpha, \beta]$  of  $(0, \infty)$  contains a set  $I$  of measure  $(\beta - \alpha)/2$  such that the inequality*

$$\ell(r, R) \leq 2\pi r \sqrt{\frac{\beta T(2r, f)}{(\beta - \alpha) \log 2}}$$

*holds for each  $R$  in  $I$ .*

For functions univalent in the disk  $D = \{|z| < 1\}$ , it follows from the basic inequalities of length and area [2, p. 18] that almost all level sets have finite length; but we do not know of any univalent function in  $D$  having even one level set of infinite length. Neither do we know whether there exists a bounded holomorphic function  $f$  in  $D$  such that  $\ell(1, R) = \infty$  for all  $R$  in some interval  $(\alpha, \beta)$ , or for all  $R$  in some set of positive measure. In this paper, we show that bounded holomorphic functions can have level sets of infinite length.

**THEOREM.** *If  $M$  is a finite or countable set in the interval  $(0, 1)$ , then there exists a holomorphic function  $f$  in  $D$  such that  $|f(z)| < 1$  for all  $z$  in  $D$  and  $\ell(1, R) = \infty$  for each  $R$  in  $M$ .*

Because for  $0 \leq r < 1$  and  $0 < R < \infty$  the quantity  $\ell(r, R)$  is a continuous

function of  $r$  and  $R$  (for each function  $f$  holomorphic in  $D$ ), the set of values  $R$  for which  $\ell(1, R) = \infty$  is of type  $G_\delta$ . It follows immediately that if the set  $M$  in the theorem is dense in  $(0, 1)$ , then the set of values  $R$  for which  $\ell(1, R) = \infty$  is residual in  $(0, 1)$ .

To prove the theorem, we begin with the case where the set  $M$  consists of a single value  $R$ , and we construct a nondecreasing continuous function  $\mu$  such that the level set  $\ell(1, R)$  of the function

$$f(z) = \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t) \quad (1)$$

has infinite length.

We write  $z = \rho e^{i\theta}$ , and we observe that

$$|f(\rho e^{i\theta})| = \exp \frac{-1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \theta, t) d\mu(t),$$

where  $P$  denotes the Poisson kernel

$$P(\rho, \theta, t) = \frac{1 - \rho^2}{1 - 2\rho \cos(t - \theta) + \rho^2}.$$

If  $\mu'(t)$  exists at the point  $t$ , then the radial limit of  $|f(z)|$  at  $e^{it}$  is  $\exp(-\mu'(t))$ . We shall now construct a sequence  $\{\mu_n\}$  such that on some set of positive measure the sequence  $\{\mu'_n(t) + \log R\}$  converges to 0 with alternating signs.

On some subinterval  $[t_0, t_1]$  of  $(-\pi, \pi)$ , let  $\mu_0$  be linear, and let  $\mu'_0(t)$  be slightly less than  $-\log R$ . On  $[-\pi, t_0]$  and on  $[t_1, \pi]$ , let  $\mu_0$  be constant. Then the function

$$|f_0(\rho e^{i\theta})| = \exp \frac{-1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \theta, t) d\mu_0(t)$$

is slightly greater than  $R$ , on some arc  $\gamma_0$  joining  $e^{it_0}$  to  $e^{it_1}$ .

To construct  $\mu_1$ , we divide the interval  $(t_0, t_1)$  into subintervals in such a way that for each  $\varepsilon > 0$  both  $[t_0, t_0 + \varepsilon]$  and  $[t_1 - \varepsilon, t_1]$  contain infinitely many of the subintervals. On each of the subintervals, we define  $\mu_1$  so that

- (i)  $\mu_1(t) = \mu_0(t)$  at the endpoints,
- (ii)  $\mu'_1$  is constant and slightly greater than  $-\log R$ , on a portion beginning at the left end of the subinterval,
- (iii)  $\mu'_1(t) = 0$  on the remainder of the subinterval (see Figure 1).

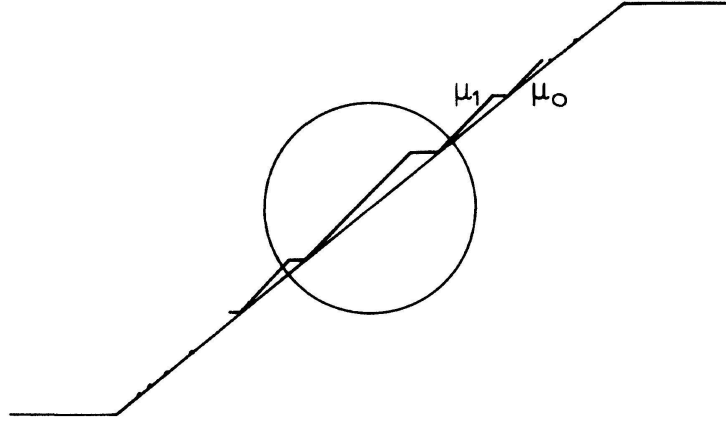


Figure 1

If the subdivision of  $(t_0, t_1)$  is fine enough, then the function  $f_1$  determined by  $\mu_1$  has modulus greater than  $R$  on the previously established arc  $\gamma_0$ . But on a family  $\{\gamma_{1k}\}$  of other arcs (corresponding to the intervals on which  $\mu'_1 > -\log R$ ) separated from the origin by the arc  $\gamma_0$ , the value of  $|f_1(z)|$  is less than  $R$ . Moreover, if the positive values of  $\mu'_0$  and  $\mu'_1$  are sufficiently close to  $-\log R$ , the sum of the diameters of the arcs  $\gamma_{1k}$  is greater than  $(t_1 - t_0)/2$ .

On each segment on which  $\mu'_1(t) = 0$ , let  $\mu_2(t) = \mu_1(t)$ . Each of the remaining segments of  $(t_0, t_1)$  we divide into infinitely many subsegments. On each of these subsegments we define  $\mu_2$  so that  $\mu'_2(t)$  is constant and slightly less than  $-\log R$  in a large part of the segment, while  $\mu'_2(t) = -2 \log R$  in the remainder (see Figure 2).

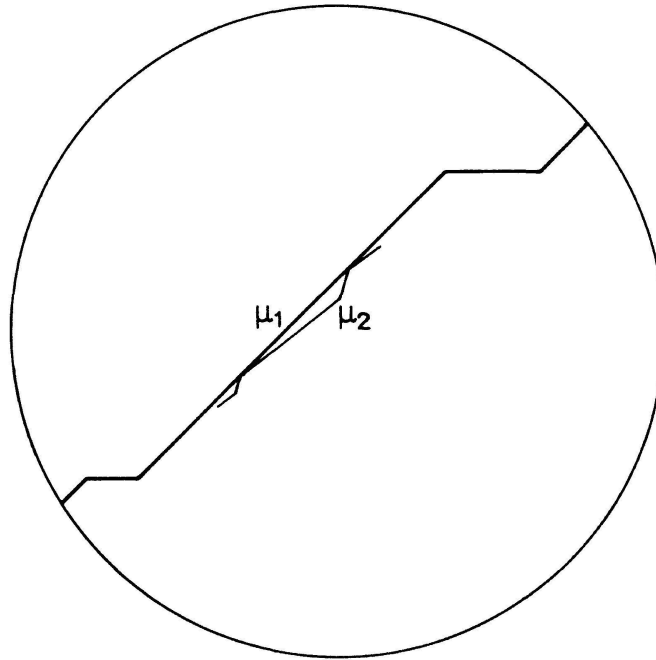


Figure 2

The sequence  $\{\mu_n\}$  constructed by the obvious continuation of our process converges to a function  $\mu$ , and the function (1) has the property that on each of

certain arcs  $\gamma_{nk}$  its modulus is greater or less than  $R$  according as the index  $n$  is even or odd. If the values  $\mu'_n(t)$  converge to  $-\log R$  rapidly enough in the portions where they are neither 0 nor  $-2 \log R$ , then for each  $n$  the sum of the diameters of the arcs  $\gamma_{nk}$  is greater than  $(t_1 - t_0)/2$ . If  $\gamma_{nh}$  separates the origin from  $\gamma_{n+1,k}$  in  $D$ , then by continuity of  $f$  an arc of the level set  $\{z : |f(z)| = R\}$  separates  $\gamma_{nh}$  from  $\gamma_{n+1,k}$ . It follows that the level set has infinite length.

We have proved the theorem for the special case where the set  $M$  is a singleton. In the general case, let  $M = \{R_j\}$ . We divide the interval  $(-\pi, \pi)$  into infinitely many disjoint subintervals  $T_j$ , and we construct the sequence  $\{\mu_n\}$  as in the special case, except that now the limit of  $\mu'_n(t)$  is  $-\log R_j$  in a substantial part of  $T_j$ . A detailed description would require the development of tedious notation; but the underlying principle is simple: if in the passage from  $\mu_{n-1}$  to  $\mu_n$  we make the new system of subdivisions fine enough, then on the arcs of the families  $\{\gamma_{mk}\}$  ( $m = 1, 2, \dots, n-1$ ) the appropriate differences  $|f_n| - R_j$  and  $|f_{n-1}| - R_j$  have the same sign.

Because the limit  $\mu$  of the sequence  $\{\mu_n\}$  is absolutely continuous, our function (1) is an outer function. We do not know whether a nonconstant inner function can have a level set of infinite length. Since the radial limit of an inner function has modulus 1 almost everywhere, the construction of an example may well require delicate computations.

This paper developed from conversations held in Lexington during the function-theoretic conference of May 19th to 22nd, 1976, in honor of Professor S. M. Shah on the occasion of his retirement. The authors acknowledge with gratitude the warm hospitality of the Mathematics Department of the University of Kentucky.

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