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Periodic maps on Poincaré duality spaces

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0. Introduction

Let T be a homeomorphism of odd prime period p on an oriented integral Poincaré space X^{2n}. The main theorem of this paper is an analogue in this context of the Atiyah–Singer–Segal G-Signature Theorem. It computes the Witt class of the $\mathbf{Z}_p$ quadratic form arising from the fixed set F in terms of the orthogonal (or symplectic, if n is odd) representation of T^* in $H^n(X^{2n})/\mathrm{Tor}$. We shall also prove a similar theorem for circle actions. Our proofs owe much to the paper of Bredon [5]. As Bredon has noted, these theorems yield new information even for differentiable actions on certain non-closed manifolds.

Before precisely stating our principal results, we need some assumptions and definitions.

Blanket Assumptions. All spaces are assumed to be Hausdorff and to have finite covering dimension. In order that Čech and singular cohomology will agree, all integral Poincaré spaces, integral Poincaré pairs, and the doubles of these pairs are assumed to be HLC [6]. ■

Subject to the above assumptions, the terms *integral Poincaré space* or *pair* will be used in the sense of Browder (pp. 14–15 of [7]). That is, there is a homology orientation class which induces an isomorphism between appropriate integral singular cohomology and homology groups via cap product.

If K is a field, then a connected space Y is a (Čech cohomology) *K Poincaré space* if there is an integer m, called the formal dimension over K, such that the cup product pairing

$$\check{H}^*(Y;K)\times\check{H}^{m-*}(Y;K)\rightarrow\check{H}^m(Y;K)$$

is non-singular. Here we do not assume HLC. A K-orientation for Y is a K-vector space isomorphism $[Y]:\check{H}^m(Y;K)\rightarrow K$. If $m\equiv 0 \pmod 4$, then the

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pairing

$$\check{H}^{m/2}(Y;K)\times \check{H}^{m/2}(Y;K)\to K$$

given by $(a,b)\mapsto\langle a\cup b,[Y]\rangle$ defines a class in the Witt group $W_0(K)$ which we denote $w_K(Y)$. For $K=\mathbf{Z}_p$, write $w_p(Y)=w_{\mathbf{Z}_p}(Y)$.

Returning now to the periodic map T of the first paragraph, we know the following from [5]

THEOREM (Bredon). *Each component $\check{F}$ of the fixed set F of T is a $\mathbf{Z}_p$ Poincaré space of even formal codimension in X.*

The cohomology map T^* generates a right C_p-module structure on $H^*(X^{2n};\mathbf{Z})$, where C_p is the cyclic group of order p. There is a non-singular Z_p valued symmetric form on the group cohomology group $H^n(C_p, H^n(X^{2n};\mathbf{Z})/\text{Tor})$ which is induced by the cup products on X and the classifying space BC_p and the canonical isomorphism $H^{2n}(C_p;\mathbf{Z})\approx\mathbf{Z}_p$. Denote the Witt class of this form in $W_0(\mathbf{Z}_p)$ by $q(T,X)$. When n is even, this is equivalent to the invariant introduced by Conner and Raymond [10]. We can now state our main theorem.

THEOREM 1. *If p is an odd prime, then for each component $\check{F}$ of F there exists a $\mathbf{Z}_p$ orientation determined by the orientation of X and the action of C_p so that*

$$q(T,X)=w_p(F)$$

where the latter denotes the sum $\sum_F w_p(\check{F})$.

An essentially equivalent result was proved for differentiable actions on closed $4k$ manifolds by ourselves and James Vick ([3] and [4]). For *PL* actions on closed $4k$ manifolds, Lowell Jones ([12] and [13]) proved a slightly weaker result which can now be extended to Poincaré spaces as follows. There is a unique homomorphism $A: W(\mathbf{Z}_p)\to\mathbf{Z}_8$ which takes the rank 1 form βx^2 to $3-2(\beta/p)-p$ (mod 8) where $(\beta/p)=\pm1$ is the Legendre symbol. Just as in [4] or [9], one can conclude

COROLLARY 2. *If the formal dimension of X is divisible by 4, then*

$$\operatorname{sgn}(X)-p\operatorname{sgn}(X/T)\equiv A(w_p(F))\qquad (\text{mod } 8)$$

where sgn *denotes signature.*

Theorem 1 depends in part on the following proposition and its extension to pairs. The proof of this proposition is quite easy and will be deferred until the last section.

PROPOSITION 3. *If a cycle group C_k acts freely on an integral Poincaré space X, preserving orientation, then X/C_k is also an integral Poincaré space of the same formal dimension and the quotient map $X \to X/C_k$ has degree k.*

Let t be the periodic map on the unit sphere $S^{2s-1} \subset \mathbf{C}^s$ given by $t(z_1, z_2, \ldots, z_s) = (\lambda z_1, \lambda z_2, \ldots, \lambda z_s)$ where $\lambda = \exp(2\pi i/p)$, $s \gg n$, and $s \equiv n \pmod 2$. By Proposition 3, $X^{2n} \times S^{2s-1}/T \times t$ has a torsion linking form whose Witt class in $W_0(\mathbf{Z}_p)$ will be denoted by $Lk(X \times S/T \times t)$. The first step in the proof of Theorem 1 is to describe the orientation of the components of F and to show that $w_p(F) = Lk(X \times S/T \times t)$.

Following an algebraic lemma concerning the graded Witt ring $W_*(\mathbf{Z}, C_p)$ of orthogonal and symplectic representations of C_p, we conclude the proof of Theorem 1 by showing that $q(T, X) = Lk(X \times S/T \times t)$.

We then briefly indicate how to use the techniques of Bredon and Chern–Hirzebruch–Serre to get a similar theorem for circle actions. Recall from Bredon [5] that each component of F in the following is a rational Poincaré space.

THEOREM 4. *If S^1 acts on an oriented rational Poincaré space X, then there is a rational orientation of each component of the fixed set F so that $w_{\mathbf{Q}}(X) = w_{\mathbf{Q}}(F)$.*

Thus we have the following extension of a theorem of Kawakubo and Raymond [14].

COROLLARY 5. *With the above rational orientations,* $\operatorname{sgn}(X) = \operatorname{sgn}(F)$.

1. The linking form and the orientation of F

For any s, recall that $t: S^{2s-1} \to S^{2s-1}$ is given by $t(z_1, \ldots, z_s) = (\lambda z_1, \ldots, \lambda z_s)$. Let t also denote the induced map on $S^\infty = \bigcup_s S^{2s-1}$. We now select s much larger than the covering dimension of X/T and such that $s + n$ is even. There is a unique integral orientation on $X^{2n} \times S^{2s-1}/T \times t$ so that the quotient map $X \times S \to X \times S/T \times t$ has degree p. Reduction mod p gives a $\mathbf{Z}_p$ orientation which we regard as an isomorphism

$$[X \times S/T \times t]: \check{H}^{2(n+s)-1}(X \times S/T \times t) \xrightarrow{\sim} \mathbf{Z}_p.$$

From the transfer homomorphism $H^*(X \times S, \mathbf{Z}) \to H^*(X \times S/T \times t, \mathbf{Z})$, it is seen that $H^{n+s}(X \times S/T \times t, \mathbf{Z})$ is a group of exponent p; hence the torsion linking form $Lk[X\times S/T\times t]\in W(Q/\mathbf{Z})$ actually is an element of $W_0(\mathbf{Z}_p)\subset \oplus_{\text{prime } q} W_0(\mathbf{Z}_q)=W(Q/\mathbf{Z})$ [3]. Furthermore, $Lk[X\times S/T\times t]$ may be computed as the following form on the image of the Bockstein $\beta:\check{H}^{n+s-1}(X\times S/T\times t;\mathbf{Z}_p)\to \check{H}^{n+s}(X\times S/T\times t;\mathbf{Z}_p)$.

$$(x,y)=\langle x\cup y',[X\times S/T\times t]\rangle \quad \text{where} \quad y=\beta(y').$$

For $X_{C_p}=X\times S^\infty/T\times t$ and $F_{C_p}=F\times(S^\infty/t)=F\times BC_p$, the subspace homomorphisms

$$\check{H}^*(F_{C_p};\mathbf{Z}_p)\xleftarrow{j^*}\check{H}^*(X_{C_p};\mathbf{Z}_p)\xrightarrow{i^*}\check{H}^*(X\times S^{2s-1}/T\times t;\mathbf{Z}_pP)$$

are isomorphisms in certain ranges which include dimensions $n+s$ and $n+s-1$. The isomorphic range for j^* also includes dimension $2(n+s)-1$. Abusing notation, we shall denote the composition

$$\check{H}^{2(n+s)-1}(F_{C_p};\mathbf{Z}_p)\xrightarrow{i^*\cdot(j^*)^{-1}}\check{H}^{2(n+s)-1}(X\times S^{2s-1}/T\times t;\mathbf{Z}_p)\xrightarrow{\approx}\mathbf{Z}_p$$

by $[F_{C_p}]$. $Lk[X\times S/T\times t]$ is then isometric to the form on the image of

$$\beta:\check{H}^{n+s-1}(F_{C_p};\mathbf{Z}_p)\to\check{H}^{n+s-1}(F_{C_p};\mathbf{Z}_p)$$

given by

$$(x,y)=\langle x\cup y',[F_{C_p}]\rangle \quad \text{where} \quad y'\in\beta^{-1}(y).$$

We can now compute this latter form, denoted simply as Lk, more explicitly in terms of F as follows.

DEFINITION. Let $\check{F}^{2f}$ be a component of F having formal dimension $2f$ over $\mathbf{Z}_p$. If $t\in\check{H}^1(BC_p;\mathbf{Z}_p)$ denotes the characteristic class of the universal C_p bundle $S^\infty\to S^\infty/t=BC_p$, then the $\mathbf{Z}_p$ orientation $[\check{F}^{2f}]$ is defined to be the composition

$$\check{H}^{2f}(\check{F};\mathbf{Z}_p)\xrightarrow{\cup\, t(\beta(t))^{n+s-(f+1)}}\check{H}^{2(n+s)-1}(\check{F}_{C_p};\mathbf{Z}_p)$$

$$\xrightarrow{\subset}\check{H}^{2(n+s)-1}(F_{C_p};\mathbf{Z}_p)\xrightarrow{[F_C]}\mathbf{Z}_p$$

This is exactly the same orientation as described in [5], where it is shown that $[\check{F}^{2f}]$ is an isomorphism.

The form Lk splits into orthogonal summands

$$\operatorname{im}\{\beta : \check{H}^{n+s-1}(\check{F}_{C_p};\mathbf{Z}_p) \to \check{H}^{n+s}(\check{F}_{C_p};\mathbf{Z}_p)\}$$

for each component $\check{F}$. If we can show that the form on this summand is Witt equivalent to $w_p(\check{F})$, we shall have

PROPOSITION 6. *$Lk[X \times S/T \times t] = w_p(F)$ in $W_0(\mathbf{Z}_p)$.*

Proof. Under the Kunneth isomorphism $\check{H}^*(\check{F}^{2f}_{C_p};\mathbf{Z}_p) \approx \check{H}^*(\check{F}^{2f};\mathbf{Z}_p) \otimes \check{H}^*(BC_p,\mathbf{Z}_p)$, the Bockstein acts by $\beta(u \otimes v) = \beta(u) \otimes v \pm u \otimes \beta(v)$. Recalling that $\beta^2 = 0$ and $\beta : \check{H}^{2i-1}(BC_p;\mathbf{Z}_p) \to \check{H}^{2i}(BC_p;\mathbf{Z}_p)$ is an isomorphism, we see that β maps $\oplus_j \check{H}^{2j}(\check{F}^{2f};\mathbf{Z}_p) \otimes \check{H}^{n+s-2j-1}(BC_p;\mathbf{Z}_p)$ isomorphically onto

$$\operatorname{im}\{\beta : \check{H}^{n+s-1}(\check{F}_{C_p},\mathbf{Z}_p) \to \check{H}^{n+s}(\check{F}_{C_p};\mathbf{Z}_p)\}.$$

If $x \in \check{H}^{2i}(\check{F}^{2f};\mathbf{Z}_p)$ and $y \in \check{H}^{2j}(\check{F}^{2f};\mathbf{Z}_p)$, then

$$\begin{aligned}
&\langle \beta(x \otimes (t \cup \beta(t)^{1/2(n+s)-i-1})) \cup (y \otimes (t \cup \beta(t)^{1/2(n+s)-j-1})), [F_{C_p}]\rangle \\
&\quad = \langle (x \cup y) \otimes (t \cup \beta(t)^{n+s-(i+j)-1}), [F_{C_p}]\rangle \\
&\quad = \begin{cases} 0, & \text{if } i+j > f \\ \langle x \cup y, [\check{F}^{2f}]\rangle, & \text{if } i+j = f. \end{cases}
\end{aligned}$$

Thus the linear subspace $V = \oplus_{j \ge f/2} \check{H}^{2j}(\check{F},\mathbf{Z}_p) \otimes \check{H}^{n+s-2j-1}(BC_p,\mathbf{Z}_p)$ is contained in the annihilator $U^\perp$ of $U = \oplus_{j > f/2} \check{H}^{2j}(\check{F},\mathbf{Z}_p) \otimes \check{H}^{n+s-2j-1}(BC_p;\mathbf{Z}_p)$. But, because V and $U^\perp$ have the same rank, they must be equal. Our form is then Witt equivalent to $U^\perp/U = V/U$, which is $\check{H}^f(\check{F}^{2f};\mathbf{Z}_p) \otimes \check{H}^{n+s-(f+1)}(BC_p;\mathbf{Z}_p)$ if f is even. The above computation for $i+j=f$ shows that the form restricted to this subgroup is just $w_p(\check{F})$.

The orientation $[\check{F}]$ that has been defined is the one that will be used for Theorem 1 and Corollary 2. We should remark that $[\check{F}]$ is not, in general, the reduction of an integral orientation, even for differentiable actions on closed manifolds. In fact, Corollary 2 would be false for $p \equiv 1 \pmod 4$ for any scheme of orienting the fixed set by reductions of integral orientations (example below). Consequently, one must view with care the statement in [12] that the local invariant $w_p(\check{F}^{2f})$ is given by the signature mod 2 and the determinant mod p of the form on $H^f(\check{F}^{2f},\mathbf{Z})/\mathrm{Tor}$.

EXAMPLE. Consider the following example of Conner (page 51 of [15]).

$$R=\{[z_0, z_1, z_2]\in CP(2) \mid z_0^p+z_1^p+z_2^p=0\}$$

is a Riemann surface of genus $(p-1)(p-2)/2$, and $\tau: R\to R$ defined by $\tau[z_0, z_1, z_2]=[z_0, z_1, \lambda z_2]$ is a map of period p with p fixed points. The local representation in the tangent space at each fixed point is given by multiplication by λ. Our example is the map $T=\tau^2\times\tau$ on $X=R\times R$ when $p=5$. Certainly $\operatorname{sgn}(X)=0$ and from the G-signature theorem it can be verified that $\operatorname{sgn}(X/T)=0$. However, for any choice of integral orientations for the fixed points one obtains $A(w_p(F))=4 \pmod 8$. Hence for integral orientations we do not have the equation of Corollary 2.

2. An algebraic remark

THEOREM 7. *If p is a prime and U, V are finitely generated free abelian C_p-modules, then the cup product induces an isomorphism of $\mathbf{Z}_2$-graded groups*

$$H^*(C_p; U)\otimes H^*(C_p; V)\to H^*(C_p; U\otimes_{\mathbf{Z}} V).$$

Proof. Let $\mathbf{Z}(\lambda)$ denote the subring of the complex numbers obtained by adjoining $\lambda=\exp(2\pi i/p)$ to the integers. $\mathbf{Z}(\lambda)$ is made into a C_p-module by letting a specified generator of C_p act by multiplication by λ. A straightforward computation shows a C_p-module isomorphism $\mathbf{Z}(\lambda)\otimes_{\mathbf{Z}}\mathbf{Z}(\lambda)\approx\mathbf{Z}[C_p]^{p-2}\oplus\mathbf{Z}$ where $\gamma=\sum_{0\le i<j\le p-1}\lambda^i\otimes\lambda^j$ can be chosen as the generator for the $\mathbf{Z}$ summand. The product

$$H^1(C_p;\mathbf{Z}(\lambda))\otimes H^1(C_p;\mathbf{Z}(\lambda))\to H^2(C_p;\mathbf{Z}(\lambda)\otimes\mathbf{Z}(\lambda))$$

sends $[1]\otimes[1]$ to $[\gamma]$. Now a simple argument using exact sequences proves

$$H^1(C_p; P)\otimes H^1(C_p; Q)\to H^2(C_p; P\otimes Q)$$

is an isomorphism for any ideals $P, Q\subset\mathbf{Z}(\lambda)$, since P is isomorphic as a $\mathbf{Z}(\lambda)$-module to another $P'\subset\mathbf{Z}(\lambda)$ such that the quotient $\mathbf{Z}(\lambda)/P'$ has order prime to p. The Reiner decomposition theorem says $U\approx U_0\oplus U_1\oplus U_2$ where U_0 is a trivial C_p-module, U_1 is a projective $\mathbf{Z}(\lambda)$-module, and U_2 is a projective C_p-module [11]. We have $H^*(C_p; W\otimes U_2)=0$ for any free abelian C_p-module W. Similarly

$V \approx V_0 \oplus V_1 \oplus V_2$, hence

$$\begin{aligned} H^2(C_p; U \otimes V) &\approx H^2(C_p; U_0 \otimes V_0) \oplus H^2(C_p; U_1 \otimes V_1) \\ &\approx H^2(C_p; U_0) \otimes H^2(C_p; V_0) \oplus H^1(C_p; U_1) \otimes H^1(C_p; V_1) \\ &\approx H^2(C_p; U) \otimes H^2(C_p, V) \oplus H^1(C_p, U) \otimes H^1(C_p, V). \end{aligned}$$

The same works for $H^1(C_p; U \otimes V)$.

COROLLARY 8. *For p an odd prime the map from* $W_*(\mathbf{Z}, C_p)$ *to* $W_0(\mathbf{Z}_p)$ *defined by sending* $[V]$ *(a* $(-1)^j$ *symmetric form,* $j = 1, 2$*) to* $[H^j(C_p; V)]$ *is a ring homomorphism.*

Proof. The form on $H^j(C_p; V)$ is given by $H^j(C_p; V) \times H^j(C_p; V) \to H^{2j}(C_p; V \otimes V) \to H^{2j}(C_p; \mathbf{Z}) \approx \mathbf{Z}_p$ where the first map is cup product and the second is the coefficient pairing defined by the inner product on V. Consider the situation where $[U], [V] \in W_0(\mathbf{Z}, C_p)$. Then $j = 2$ in both cases and $H^2(C_p; U \otimes V) \approx H^2(C_p; U) \otimes H^2(C_p; V) \otimes H^1(C_p; U) \otimes H^1(C_p, V)$. It is easy to see that cup product and the pairings $U \otimes U \to \mathbf{Z}$, and $V \otimes V \to \mathbf{Z}$ define skew forms on $H^1(C_p; U)$ and $H^1(C_p; V)$. Therefore, because p is odd, $H^1(C_p; U) \otimes H^1(C_p; V)$ is a hyperbolic form. Clearly $[H^2(C_p; U \otimes V)] = [H^2(C_p; U)\, [H^2(C_p; V)]$. All other cases work similarly.

3. Proofs of Theorems 1 and 4

We will now show that the global invariant $q(T, X)$ is also equal to $Lk[X^{2n} \times S^{2s-1}/T \times t]$.

Recall the example (τ, R) defined at the end of section 1. The action of $\tau \times \cdots \times \tau = (\tau)^s$ on $R \times \cdot \cdot \times R = (R)^s$ has p^s isolated fixed points. Let $\{D_i \mid 1 \leq i \leq p^s\}$ be a collection of equivariant tubular neighborhoods of these points. Set $V = (R)^s - \bigcup_i D_i$, $\partial V = \bigcup_i (-S_i^{2s-1})$. Then

$$Lk[X^{2n} \times \partial V/T \times (\tau)^s] = -p^s Lk[X^{2n} \times S^{2s-1}/T \times t].$$

Also,

$$H^{n+s}(X \times (R)^s; \mathbf{Z}) \xrightarrow{\sim} H^{n+s}(X \times V; \mathbf{Z}) \xleftarrow{\sim} H^{n+s}(X \times (V, \partial V); \mathbf{Z}).$$

By [1], $q(\tau, R) = p\langle 1 \rangle_p \in W_0(\mathbf{Z}_p)$ where $\langle 1 \rangle_p$ is the rank one form x^2 over $\mathbf{Z}_p$.

Corollary 8 implies

$$p^s q(T, X) = q(T\times(\tau)^s, X\times(R)^s).$$

If I denotes the fixed vectors in $H^{n+s}(X\times(R)^s; Q)$, then using the equivalent Conner–Raymond definition of q [10], we get

$$\begin{aligned} p^s q(T, X) &= q(T\times(\tau)^s, X\times(R)^s) \\ &= \langle p\rangle \otimes I - \operatorname{sgn}(I)\cdot\langle 1\rangle \quad \text{in} \quad \ker\{\operatorname{sgn}: W(Q)\to \mathbf{Z}\}\approx W(Q/\mathbf{Z}) \\ &= w_Q(X\times V/T\times(\tau)^s) - \operatorname{sgn}(X\times V/T\times(\tau)^s)\cdot\langle 1\rangle \\ &= -Lk[X\times\partial V/T\times(\tau)^s] \quad \text{by Coro. 14 in this paper and [3; Thm. 2.3]} \\ &= p^s Lk[X^{2n}\times S^{2s-1}/T\times t]. \end{aligned}$$

Since $W_0(\mathbf{Z}_p)$ is a 2-group and p is odd, this proves

PROPOSITION 9. *$q(T, X^{2n}) = Lk[X^{2n}\times S^{2s-1}/T\times t]$ in $W_0(\mathbf{Z}_p)$.*

This together with Proposition 6 completes the proof of Theorem 1. To illustrate how the theorem may be used, we give an elementary

COROLLARY 10. *Let X^{4k} be an integral Poincaré space and p be an odd prime. Suppose the rank of $H^{2k}(X, \mathbf{Z})/\mathrm{Tor}$ is less than p. If*

(1) $\operatorname{sgn}(X)$ *is odd, or*

(2) $p \equiv -1 \pmod 4$ *and* $4 \nmid \operatorname{sgn}(X)$,

then for any map of period p on X there must be a component of the fixed set whose formal dimension over $\mathbf{Z}_p$ is divisible by 4.

Proof. Let $V = H^{2m}(X^{2m}, \mathbf{Z})/\mathrm{Tor}$. We are assuming $rk(V) < p$. If C_p acts non-trivially on V, then $H^1(C_p; V)\approx \mathbf{Z}_p$ bears a non-singular skew-symmetric form, contradicting p odd. Hence C_p acts trivially on V and $w_p(F) = q(T, X) = \operatorname{sgn}(X)\cdot\langle 1\rangle \neq 0$. ■

Now we turn to S^1 actions and Theorem 4. Suppose X^{2n} is a rational Poincaré space with an orientation $[X^{2n}]$. Consider $(X^{2n}\times S^{2s+1})/S^1$ where S^1 acts canonically on S^{2s+1} so that $S^{2s+1}/S^1 = CP(s)$ and $s+n$ is even.

LEMMA 11. *$(X^{2n}\times S^{2s+1})/S^1$ is a rational Poincaré space of formal dimension $2(n+s)$.*

Proof. Consider the Leray spectral sequence for the constant rational sheaf applied to the fibration

$$X \to (X \times S^{2s+1})/S^1 \to CP(s).$$

The pairing $E_2^{u,v} \times E_2^{2s-u,2n-v} \to E_2^{2s,2n} \approx Q$ is non-singular. Since the differentials are derivations, we get that

$$E_\infty^{u,v} \times E_\infty^{2s-u,2n-v} \to E_\infty^{2s,2n} \approx Q$$

is non-singular. This shows that $(X \times S^{2s+1})/S^1$ is a rational Poincaré space.

The isomorphism $\check{H}^{2(n+s)}((X \times S^{2s+1})/S^1, Q) \approx E_\infty^{2s,2n} \approx E_2^{2s,2n} = \check{H}^{2s}(CP(s), \check{H}^{2n}(X, Q))$ induces a rational orientation $[(X \times S^{2s+1})/S^1]$ from the orientations $[X]$ and $[CP(s)]$. A simple variation of the Chern–Hirzebruch–Serre argument [8] shows

LEMMA 12. *With the above orientation*

$$w_Q((X \times S^{2s+1})/S^1) = w_Q(X).$$

Proof of Theorem 4. The arguments in section 1 can be modified to define an orientation on F and show that

$$w_Q((X \times S^{2s+1})/S^1) = \sum_F w_Q(\check{F}) = w_Q(F).$$

4. Free actions on integral Poincaré spaces

We shall prove Proposition 3. It is sufficient to consider a free action of a cyclic group C_p of prime order p on the integral Poincaré space X^n.

LEMMA 13. *X^n/C_p is a $\mathbf{Z}_p$ Poincaré space of formal dimension n.*

Proof. Choose an integer $2s-1>n$. Crossing X with a free action of C_p on S^{2s-1}, apply the spectral sequence for $\mathbf{Z}_p$ cohomology to the fibration

(1) $X \to (X \times S)/C_p \to S/C_p$

(2) $S \to (X \times S)/C_p \to X/C_p$.

The E_2 term for (1) satisfies Poincaré duality by [5; Lemma 2.1]. As in Lemma

10, it follows that $(X\times S)/C_p$ is a $\mathbf{Z}_p$ Poincaré space. Noting that in the spectral sequence for (2) the filtration of $H^j((X\times S)/C_p;\mathbf{Z}_p)$ yields at most one non-trivial grading, we conclude that E_∞ satisfies Poincaré duality. So does E_2, since the sequence collapses. Hence $H^i(X/C_p;\mathbf{Z}_p)\approx E_2^{i,0}$ is dually paired with $H^{n-i}(X/C_p;\mathbf{Z}_p)\approx E_2^{n-i,2s-1}$ in $H^n(X/C_p;\mathbf{Z}_p)\approx E_2^{n,2s-1}$.

Proof of Proposition 3. The Serre spectral sequence for (1) and (2) in Lemma 13 show that X^n/C_p has finitely generated integral homology. If K is a field of characteristic other than p, $H^*(X/C_p, K)$ is isomorphic to the fixed vectors in $H^*(X, K)$ and thus X/C_p is a K Poincaré space of formal dimension n. Then $H_n(X/C_p, \mathbf{Z})\approx\mathbf{Z}$, and [7, Prop. I.2.1] implies that X/C_p is an integral Poincaré space. ■

Applying [7, Thm. I.3.2] to the double proves

COROLLARY 14. *Suppose a free action of C_k preserves orientation on an integral Poincaré pair $(X, \partial X)$. If the two copies of X are excisive in the double $X\cup_{\partial X}X$ and similarly for the two copies of X/C_k in $X/C_k\cup_{\partial X/C_k}X/C_k$, then $(X/C_k, \partial X/C_k)$ is an integral Poincaré pair and the quotient map $(X,\partial X)\to(X/C_k, \partial X/C_k)$ has degree k.*

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