

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 53 (1978)

Artikel: Holomorphic Lipschitz functions in balls.
Autor: Rudin, Walter
DOI: <https://doi.org/10.5169/seals-40759>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Holomorphic Lipschitz functions in balls

WALTER RUDIN

Fix $n > 1$, let B be the open unit ball of \mathcal{C}^n , suppose that f is holomorphic in B and that f satisfies a Lipschitz condition of order $\alpha > 0$. Stein [3] has observed (actually, for domains much more general than B) that f is then, roughly speaking, twice as smooth in the direction of the complex tangents. The present note adds to this that the same conclusion can even be derived from much weaker hypotheses: it is enough to assume that the slice functions f_w of f (see below) form a bounded subset of $\text{Lip } \alpha$. In particular, it is not even necessary to assume that f is continuous on \bar{B} .

For the sake of simplicity, we confine ourselves to the range $0 < \alpha < 1$.

DEFINITIONS. On \mathcal{C}^n there is the inner product $\langle z, w \rangle = \sum z_j \bar{w}_j$ and the associated norm $|z| = \langle z, z \rangle^{1/2}$. Thus $B = \{z : |z| < 1\}$.

For $0 < \alpha < 1$, we let K_α be the set of all $f: \bar{B} \rightarrow \mathcal{C}$ such that

- (i) f is holomorphic in B ,
- (ii) for each $w \in S = \partial B$, the slice function f_w defined by $f_w(\lambda) = f(\lambda w)$ is continuous on the closed unit disc in \mathcal{C} , and satisfies the Lipschitz condition

$$|f_w(e^{i\theta}) - f_w(e^{i\varphi})| \leq |\theta - \varphi|^\alpha \quad (\theta, \varphi \in \mathbf{R}). \quad (1)$$

We say that a C^1 -curve $\gamma: \mathbf{R} \rightarrow S$ is *complex-tangential* if $\langle \gamma'(t), \gamma(t) \rangle = 0$ for every $t \in \mathbf{R}$. We say that γ is *normalized* if $|\gamma'(t)| = 1$, i.e., if γ is parametrized by arc length.

Here are our main results:

THEOREM 1. *If $0 < \alpha < \frac{1}{2}$, there is a constant $A(\alpha) < \infty$ such that the inequality*

$$|f(\gamma(t+h)) - f(\gamma(t))| \leq A(\alpha) |h|^{2\alpha}$$

This research was partially supported by NSF Grant MPS 75-06687.

holds for all $f \in K_\alpha$, for all complex-tangential normalized curves γ , and for all $t, h \in \mathbf{R}$.

THEOREM 2. *If $\frac{1}{2} \leq \alpha < 1$, there is a constant $A(\alpha) < \infty$ such that the inequality*

$$|f(\gamma(t+h)) + f(\gamma(t-h)) - 2f(\gamma(t))| \leq A(\alpha) \|\gamma''\|_\infty |h|^{2\alpha}$$

holds for all $f \in K_\alpha$, for all complex-tangential normalized curves of class C^2 , and for all $t, h \in \mathbf{R}$.

Here

$$\|\gamma''\|_\infty = \sup \{|\gamma''(t)| : t \in \mathbf{R}\}.$$

Remarks. (i) In the terminology of [2] and [3], the main point of these two theorems can be briefly stated as follows:

If $\{f_w : w \in S\}$ is a bounded set in Λ_α , then $f \circ \gamma \in \Lambda_{2\alpha}$. It follows that $f \in \Gamma_\alpha$.

(We recall that $\Lambda_\alpha = \text{Lip } \alpha$ when $0 < \alpha < 1$; see Ch. V, §4 of [2]; for Γ_α , see [3].)

(ii) Each $w \in S$ lies on a circle $T_w = \{e^{i\theta} w : \theta \in \mathbf{R}\}$. Our smoothness assumption is imposed on the restrictions of f to these circles. The complex-tangential curves γ (on which f turns out to be “twice as smooth”) are precisely those that are perpendicular (in the sense of the usual real scalar product in $\mathbf{R}^{2n} = \mathcal{C}^n$) to every T_w that they intersect.

(iii) Although the research announcement [3] contains no proofs, it does mention a key fact: the complex-tangential partial derivatives of a holomorphic function in B satisfy more restrictive growth conditions than does the radial derivative. This is also the point of Lemma 2 in the present paper.

THE RADIAL DERIVATIVE Rf . Every f that is holomorphic in B has an expansion $f = \sum F_k$ in which each F_k is a homogeneous polynomial of degree k . Define

$$(Rf)(z) = \sum_{k=0}^{\infty} k F_k(z) \quad (z \in B). \quad (2)$$

Rf is related to the derivative of the slice functions f_w by

$$(Rf)(\lambda w) = \lambda f'_w(\lambda) \quad (w \in S, |\lambda| < 1). \quad (3)$$

For our purposes, Rf is preferable to f'_w since (2) shows that Rf is a holomorphic function in B .

In the following lemmas, $\{e_1, \dots, e_n\}$ will be an orthonormal basis for \mathbb{C}^n , so that $z = \sum z_j e_j$, and we shall write D_j for $\partial/\partial z_j$.

LEMMA 1. Suppose m_1, \dots, m_n are nonnegative integers, $p = \sum m_j$, $P(D) = D_1^{m_1} \cdots D_n^{m_n}$, and f is holomorphic in B . Then, for $w \in S$, $0 < r < 1$,

$$\int_0^r [P(D)Rf](tw)t^{p-1} dt = r^p [P(D)f](rw).$$

Proof. By (2), this is an immediate consequence of the fact that $P(D)F_k$ is homogeneous of degree $k - p$ when $k \geq p$, and that $P(D)F_k = 0$ when $k < p$.

LEMMA 2. Suppose G is holomorphic in B , $\beta \geq 0$, and

$$|G(z)| \leq (1 - |z|)^{-\beta} \quad (z \in B). \quad (4)$$

Then, for $0 < r < 1$,

$$|(D_2 G)(re_1)| \leq c(\beta)(1 - r)^{-\beta-1/2} \quad (5)$$

and

$$|(D_2^2 G)(re_1)| \leq c(\beta)(1 - r)^{\beta-1}, \quad (6)$$

where $c(\beta) < \infty$.

Proof. Put $g(\lambda) = G(re_1 + \lambda e_2)$, if $|\lambda|^2 < 1 - r^2$. Put $p = \{\frac{1}{2}(1 - r^2)\}^{1/2}$. When $|\lambda| = p$ then

$$1 - |re_1 + \lambda e_2|^2 = p^2$$

so that $|g(\lambda)| \leq 2^\beta p^{-2\beta}$. (Note that (4) implies that $|G(z)| \leq 2^\beta (1 - |z|^2)^{-\beta}$.) It follows from the Schwarz lemma that

$$|g'(0)| \leq 2^\beta p^{-2\beta-1}, \quad |g''(0)| \leq 2^{\beta+1} p^{-2\beta-2}. \quad (7)$$

Since $p^2 > \frac{1}{2}(1 - r)$, (5) and (6) follow from (7).

LEMMA 3. If $0 < \alpha < 1$, there are constants $c_i(\alpha) < \infty$, $1 \leq i \leq 4$, such that every

$f \in K_\alpha$ satisfies the following inequalities:

$$|(Rf)(z)| \leq c_1(\alpha)(1-|z|)^{\alpha-1} \quad (z \in B). \quad (8)$$

$$|f(w) - f(rw)| \leq c_2(\alpha)(1-r)^\alpha \quad (w \in S, 0 < r < 1). \quad (9)$$

$$|(D_2f)(re_1)| \leq c_3(\alpha)(1-r)^{\alpha-1/2} \quad (0 < \alpha < \frac{1}{2}). \quad (10)$$

$$|(D_2^2f)(re_1)| \leq c_4(\alpha)(1-r)^{\alpha-1} \quad (11)$$

Proof. (8) follows from (3) and a classical theorem of Hardy and Littlewood; see, for instance, p. 74 of [1]. It is clear that (8) implies (9). With $\beta = 1 - \alpha$, (8) and Lemma 2 give estimates of D_2Rf and D_2^2Rf ; when these are integrated, Lemma 1 yields (10) and (11).

PROOF OF THEOREMS 1 AND 2. Suppose f and γ are as in the hypotheses. Fix $h \in (0, 1)$, put $r = 1 - h^2$, and define

$$g(t) = f(r\gamma(t)) \quad (t \in \mathbf{R}). \quad (12)$$

By (9), it is enough to show that

$$|g(t+h) - g(t)| \leq A(\alpha)h^{2\alpha} \quad (0 < \alpha < \frac{1}{2}) \quad (13)$$

and

$$|g(t+h) + g(t-h) - 2g(t)| \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha} \quad (0 < \alpha < 1). \quad (14)$$

For any $t_0 \in \mathbf{R}$, our assumptions on γ show that there is a unitary change of variables which makes $\gamma(t_0) = e_1$, $\gamma'(t_0) = e_2$. Then (12) and (10) give

$$|g'(t_0)| = |r(D_2f)(re_1)| \leq c_3(\alpha)h^{2\alpha-1} \quad (15)$$

if $\alpha < \frac{1}{2}$. This proves (13) and hence Theorem 1.

The left side of (14) is at most $h^2\|g''\|_\infty$. Hence (14) will follow from

$$\|g''\|_\infty \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha-2}. \quad (16)$$

For any $t_0 \in \mathbf{R}$, our preceding change of variables shows that (12) leads to

$$g''(t_0) = r^2(D_2^2f)(re_1) + r \sum_{j=1}^n (D_jf)(re_1)\gamma''(t_0). \quad (17)$$

By (8) and (10), each derivative of f that occurs in (17) is dominated by $c(\alpha)(1-r)^{\alpha-1} = c(\alpha)h^{2\alpha-2}$. [Note that the right side of (10) can be replaced by $C \log(1/1-r)$ if $\alpha \geq \frac{1}{2}$.] Since differentiation of $\langle \gamma, \gamma \rangle = 1$ gives $\operatorname{Re} \langle \gamma', \gamma \rangle = 0$, hence

$$\operatorname{Re} \langle \gamma'', \gamma \rangle = -\langle \gamma', \gamma' \rangle = -1, \quad (18)$$

we see that $|\gamma''(t_0)| \geq 1$. Hence (17) gives (16). This completes the proof of Theorem 2.

EXAMPLE. Take $n = 2$, define

$$f(z_1, z_2) = \frac{z_2}{z_1} \log \frac{1}{1-z_1}. \quad (19)$$

The singularity at $z_1 = 0$ is removable, and

$$(Rf)(z_1, z_2) = z_2/(1-z_1). \quad (20)$$

Since $|z_2|^2 < 1 - |z_1|^2$, $|(Rf)(z)| < 2^{1/2}(1-|z|)^{-1/2}$. This implies that $Cf \in K_{1/2}$ for some $C > 0$.

Since $t \rightarrow f(\cos t, \sin t)$ is not in Lip 1, we see that Theorem 1 fails when $\alpha = \frac{1}{2}$.

REFERENCES

- [1] DUREN P. L., *Theory of H^p -Spaces*, Academic Press, 1970.
- [2] STEIN E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [3] STEIN E. M., *Singular integrals and estimates for the Cauchy-Riemann equations*, Bull. Amer. Math. Soc. 79 (1973), 440-445.

*University of Wisconsin
Madison*

Received March 31, 1977

