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Holomorphic Lipschitz functions in balls

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Fix $n > 1$, let B be the open unit ball of \mathbb{C}^n , suppose that f is holomorphic in B and that f satisfies a Lipschitz condition of order $\alpha > 0$. Stein [3] has observed (actually, for domains much more general than B) that f is then, roughly speaking, twice as smooth in the direction of the complex tangents. The present note adds to this that the same conclusion can even be derived from much weaker hypotheses: it is enough to assume that the slice functions f_w of f (see below) form a bounded subset of $\text{Lip } \alpha$. In particular, it is not even necessary to assume that f is continuous on \bar{B} .

For the sake of simplicity, we confine ourselves to the range $0 < \alpha < 1$.

DEFINITIONS. On \mathbb{C}^n there is the inner product $\langle z, w \rangle = \sum z_j \bar{w}_j$ and the associated norm $|z| = \langle z, z \rangle^{1/2}$. Thus $B = \{z : |z| < 1\}$.

For $0 < \alpha < 1$, we let K_α be the set of all $f : \bar{B} \rightarrow \mathbb{C}$ such that

- (i) f is holomorphic in B ,
- (ii) for each $w \in S = \partial B$, the slice function f_w defined by $f_w(\lambda) = f(\lambda w)$ is continuous on the closed unit disc in \mathbb{C} , and satisfies the Lipschitz condition

$$|f_w(e^{i\theta}) - f_w(e^{i\varphi})| \leq |\theta - \varphi|^\alpha \quad (\theta, \varphi \in \mathbb{R}). \quad (1)$$

We say that a C^1 -curve $\gamma : \mathbb{R} \rightarrow S$ is *complex-tangential* if $\langle \gamma'(t), \gamma(t) \rangle = 0$ for every $t \in \mathbb{R}$. We say that γ is *normalized* if $|\gamma'(t)| = 1$, i.e., if γ is parametrized by arc length.

Here are our main results:

THEOREM 1. *If $0 < \alpha < \frac{1}{2}$, there is a constant $A(\alpha) < \infty$ such that the inequality*

$$|f(\gamma(t+h)) - f(\gamma(t))| \leq A(\alpha) |h|^{2\alpha}$$

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holds for all $f \in K_\alpha$, for all complex-tangential normalized curves γ , and for all $t, h \in \mathbf{R}$.

THEOREM 2. If $\frac{1}{2} \leq \alpha < 1$, there is a constant $A(\alpha) < \infty$ such that the inequality

$$|f(\gamma(t+h)) + f(\gamma(t-h)) - 2f(\gamma(t))| \leq A(\alpha) \|\gamma''\|_\infty |h|^{2\alpha}$$

holds for all $f \in K_\alpha$, for all complex-tangential normalized curves of class C^2 , and for all $t, h \in \mathbf{R}$.

Here

$$\|\gamma''\|_\infty = \sup \{|\gamma''(t)| : t \in \mathbf{R}\}.$$

Remarks. (i) In the terminology of [2] and [3], the main point of these two theorems can be briefly stated as follows:

If $\{f_w : w \in S\}$ is a bounded set in Λ_α , then $f \circ \gamma \in \Lambda_{2\alpha}$. It follows that $f \in \Gamma_\alpha$.

(We recall that $\Lambda_\alpha = \text{Lip } \alpha$ when $0 < \alpha < 1$; see Ch. V, §4 of [2]; for Γ_α , see [3].)

(ii) Each $w \in S$ lies on a circle $T_w = \{e^{i\theta}w : \theta \in \mathbf{R}\}$. Our smoothness assumption is imposed on the restrictions of f to these circles. The complex-tangential curves γ (on which f turns out to be “twice as smooth”) are precisely those that are perpendicular (in the sense of the usual real scalar product in $\mathbf{R}^{2n} = \mathcal{C}^n$) to every T_w that they intersect.

(iii) Although the research announcement [3] contains no proofs, it does mention a key fact: the complex-tangential partial derivatives of a holomorphic function in B satisfy more restrictive growth conditions than does the radial derivative. This is also the point of Lemma 2 in the present paper.

THE RADIAL DERIVATIVE Rf . Every f that is holomorphic in B has an expansion $f = \sum F_k$ in which each F_k is a homogeneous polynomial of degree k . Define

$$(Rf)(z) = \sum_{k=0}^{\infty} kF_k(z) \quad (z \in B). \quad (2)$$

Rf is related to the derivative of the slice functions f_w by

$$(Rf)(\lambda w) = \lambda f'_w(\lambda) \quad (w \in S, |\lambda| < 1). \quad (3)$$

For our purposes, Rf is preferable to f'_w since (2) shows that Rf is a holomorphic function in B .

In the following lemmas, $\{e_1, \dots, e_n\}$ will be an orthonormal basis for \mathbb{C}^n , so that $z = \sum z_j e_j$, and we shall write D_j for $\partial/\partial z_j$.

LEMMA 1. *Suppose m_1, \dots, m_n are nonnegative integers, $p = \sum m_j$, $P(D) = D_1^{m_1} \cdots D_n^{m_n}$, and f is holomorphic in B . Then, for $w \in S$, $0 < r < 1$,*

$$\int_0^r [P(D)Rf](tw) t^{p-1} dt = r^p [P(D)f](rw).$$

Proof. By (2), this is an immediate consequence of the fact that $P(D)F_k$ is homogeneous of degree $k - p$ when $k \geq p$, and that $P(D)F_k = 0$ when $k < p$.

LEMMA 2. *Suppose G is holomorphic in B , $\beta \geq 0$, and*

$$|G(z)| \leq (1 - |z|)^{-\beta} \quad (z \in B). \quad (4)$$

Then, for $0 < r < 1$,

$$|(D_2 G)(re_1)| \leq c(\beta)(1 - r)^{-\beta - 1/2} \quad (5)$$

and

$$|(D_2^2 G)(re_1)| \leq c(\beta)(1 - r)^{\beta - 1}, \quad (6)$$

where $c(\beta) < \infty$.

Proof. Put $g(\lambda) = G(re_1 + \lambda e_2)$, if $|\lambda|^2 < 1 - r^2$. Put $p = \{\frac{1}{2}(1 - r^2)\}^{1/2}$. When $|\lambda| = p$ then

$$1 - |re_1 + \lambda e_2|^2 = p^2$$

so that $|g(\lambda)| \leq 2^\beta p^{-2\beta}$. (Note that (4) implies that $|G(z)| \leq 2^\beta (1 - |z|^2)^{-\beta}$.) It follows from the Schwarz lemma that

$$|g'(0)| \leq 2^\beta p^{-2\beta - 1}, \quad |g''(0)| \leq 2^{\beta + 1} p^{-2\beta - 2}. \quad (7)$$

Since $p^2 > \frac{1}{2}(1 - r)$, (5) and (6) follow from (7).

LEMMA 3. *If $0 < \alpha < 1$, there are constants $c_i(\alpha) < \infty$, $1 \leq i \leq 4$, such that every*

$f \in K_\alpha$ satisfies the following inequalities:

$$|(Rf)(z)| \leq c_1(\alpha)(1-|z|)^{\alpha-1} \quad (z \in B). \quad (8)$$

$$|f(w) - f(rw)| \leq c_2(\alpha)(1-r)^\alpha \quad (w \in S, 0 < r < 1). \quad (9)$$

$$|(D_2 f)(re_1)| \leq c_3(\alpha)(1-r)^{\alpha-1/2} \quad (0 < \alpha < \frac{1}{2}). \quad (10)$$

$$|(D_2^2 f)(re_1)| \leq c_4(\alpha)(1-r)^{\alpha-1} \quad (11)$$

Proof. (8) follows from (3) and a classical theorem of Hardy and Littlewood; see, for instance, p. 74 of [1]. It is clear that (8) implies (9). With $\beta = 1 - \alpha$, (8) and Lemma 2 give estimates of $D_2 Rf$ and $D_2^2 Rf$; when these are integrated, Lemma 1 yields (10) and (11).

PROOF OF THEOREMS 1 AND 2. Suppose f and γ are as in the hypotheses. Fix $h \in (0, 1)$, put $r = 1 - h^2$, and define

$$g(t) = f(r\gamma(t)) \quad (t \in \mathbf{R}). \quad (12)$$

By (9), it is enough to show that

$$|g(t+h) - g(t)| \leq A(\alpha)h^{2\alpha} \quad (0 < \alpha < \frac{1}{2}) \quad (13)$$

and

$$|g(t+h) + g(t-h) - 2g(t)| \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha} \quad (0 < \alpha < 1). \quad (14)$$

For any $t_0 \in \mathbf{R}$, our assumptions on γ show that there is a unitary change of variables which makes $\gamma(t_0) = e_1$, $\gamma'(t_0) = e_2$. Then (12) and (10) give

$$|g'(t_0)| = |r(D_2 f)(re_1)| \leq c_3(\alpha)h^{2\alpha-1} \quad (15)$$

if $\alpha < \frac{1}{2}$. This proves (13) and hence Theorem 1.

The left side of (14) is at most $h^2\|g''\|_\infty$. Hence (14) will follow from

$$\|g''\|_\infty \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha-2}. \quad (16)$$

For any $t_0 \in \mathbf{R}$, our preceding change of variables shows that (12) leads to

$$g''(t_0) = r^2(D_2^2 f)(re_1) + r \sum_{j=1}^n (D_j f)(re_1) \gamma''(t_0). \quad (17)$$

By (8) and (10), each derivative of f that occurs in (17) is dominated by $c(\alpha)(1-r)^{\alpha-1} = c(\alpha)h^{2\alpha-2}$. [Note that the right side of (10) can be replaced by $C \log(1/1-r)$ if $\alpha \geq \frac{1}{2}$.] Since differentiation of $\langle \gamma, \gamma \rangle = 1$ gives $\operatorname{Re} \langle \gamma', \gamma \rangle = 0$, hence

$$\operatorname{Re} \langle \gamma'', \gamma \rangle = -\langle \gamma', \gamma' \rangle = -1, \quad (18)$$

we see that $|\gamma''(t_0)| \geq 1$. Hence (17) gives (16). This completes the proof of Theorem 2.

EXAMPLE. Take $n = 2$, define

$$f(z_1, z_2) = \frac{z_2}{z_1} \log \frac{1}{1-z_1}. \quad (19)$$

The singularity at $z_1 = 0$ is removable, and

$$(Rf)(z_1, z_2) = z_2/(1-z_1). \quad (20)$$

Since $|z_2|^2 < 1 - |z_1|^2$, $|(Rf)(z)| < 2^{1/2}(1-|z|)^{-1/2}$. This implies that $Cf \in K_{1/2}$ for some $C > 0$.

Since $t \rightarrow f(\cos t, \sin t)$ is not in $\operatorname{Lip} 1$, we see that Theorem 1 fails when $\alpha = \frac{1}{2}$.

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