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An inequality involving the absolute value of an entire function and the counting function of its zeros*

W. H. J. FUCHS

Dedicated to Professor A. Pfluger on his seventieth birthday

The purpose of this paper is to give a proof of

THEOREM 1. *Let $f(z)$ be an entire function of finite, non-integral order λ . If γ satisfies*

$$\max\left(\frac{1}{2}, \lambda\right) < \gamma < [\lambda] + 1; \quad \beta = \pi/2\gamma,$$

then

$$\liminf_{|z| \rightarrow \infty} \frac{\log |f(ze^{i\beta})| + \log |f(ze^{-i\beta})|}{N(|z|, 1/f)} \leq \frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda}.$$

The canonical product $g_\lambda(z)$ with negative zeros and $N(r, 1/g_\lambda) \sim r^\lambda/\lambda$ has the asymptotic behavior [4, p. 232]

$$\log |g_\lambda(re^{i\theta})| = (1 + o(1)) \frac{\pi \cos \lambda\theta}{\sin \pi\lambda} r^\lambda$$

as $r \rightarrow \infty$, uniformly in $|\theta| < \pi - \epsilon < \pi$. Therefore

$$\begin{aligned} & \{\log |g_\lambda(re^{i(\pi-\beta)})| + \log |g_\lambda(re^{-i(\pi-\beta)})|\} / N(r, 1/g_\lambda) \\ & \rightarrow \frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda} \end{aligned}$$

as $r \rightarrow \infty$. This shows that Theorem 1 is best-possible.

It may well be that the assertion of Theorem 1 holds for a wider range of γ , but the method of proof does not give this result.

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We may assume without loss of generality that

$$f(0) = 1.$$

We shall always use the standard notations

$$z = re^{i\theta}, \quad N(r) = N(r, 1/f).$$

Proof of Theorem 1. Theorem 1 is a consequence of

$$\begin{aligned} \log |f(te^{i(\theta+\beta)})| + \log |f(te^{i(\theta-\beta)})| \\ < \left(\frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda} + \eta \right) N(t) \quad (0 \leq \theta < 2\pi, \eta > 0) \end{aligned} \quad (1)$$

where t is a Pólya peak of $N(r)$ satisfying $t > t_0(\eta; N(r))$. It will suffice to prove (1) for $\theta = 0$, the general case then follows by considering $f(ze^{i\varphi})$, φ constant. The Pólya peaks are a sequence of positive numbers t_n tending to ∞ such that for given $\epsilon > 0$ the inequalities

$$N(u) \leq (u/t_n)^{\lambda - \epsilon} N(t_n) \quad (1 < u \leq t_n) \quad (2)$$

$$N(u) \leq (u/t_n)^{\lambda + \epsilon} N(t_n) \quad (t_n < u) \quad (3)$$

hold for all $n > n_0(\epsilon)$. It is well known that Pólya peaks exist [2, p. 103]. Since for a function of non-integral order λ $\limsup N(r)r^{-\lambda+\epsilon} = \infty$ ($\epsilon > 0$), (2) implies that

$$\lim_{n \rightarrow \infty} N(t_n)t_n^{-\lambda+\epsilon} = \infty. \quad (4)$$

Consequently (2) can be replaced by

$$N(u) \leq (u/t_n)^{\lambda - \epsilon} N(t_n) \quad (0 \leq u \leq t_n) \quad (2')$$

for all large t_n , since $N(u) \equiv 0$ near $u = 0$.

Apply Green's Formula

$$\iint_B (u\Delta v - v\Delta u) dA = \int_{\partial B} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds$$

to

$$u = \log |f(z)|, \quad v = \log \left| \frac{z^\gamma + t^\gamma}{z^\gamma - t^\gamma} \right| - \frac{2t^\gamma r^\gamma \cos \gamma\theta}{r^{2\gamma} + t^{2\gamma}} \quad (t > 0)$$

in the domain B obtained from the sector

$$D: \rho < r < R, \quad |\theta| < \beta$$

by removing disks of radius δ centered at t and at the zeros a_ν of $f(z)$. Here ρ and R are chosen so that $|z| = \rho$ and $|z| = R$ do not contain zeros of $f(z)$ and

$$\rho < t < R.$$

On letting $\delta \rightarrow 0$ we obtain

$$\begin{aligned} & \iint_D \log |f(z)| \Delta v(z) dA + \int_{-\beta}^{\beta} \left\{ v(\mathbf{R}e^{i\theta}) \frac{\partial}{\partial r} \log |f(\mathbf{R}e^{i\theta})| \right. \\ & \quad \left. - \log |f(\mathbf{R}e^{i\theta})| \frac{\partial v}{\partial r}(\mathbf{R}e^{i\theta}) \right\} R d\theta \\ & \quad + \int_{-\beta}^{\beta} \left\{ \log |f(\rho e^{i\theta})| \frac{\partial v}{\partial r}(\rho e^{i\theta}) - v(\rho e^{i\theta}) \frac{\partial}{\partial r} \log |f(\rho e^{i\theta})| \right\} \rho d\theta \\ & = 2\pi \log |f(t)| + 2\pi \sum_{a_\nu \in D} v(a_\nu). \end{aligned} \tag{5}$$

The definition of $v(z)$ shows that for $R \rightarrow \infty$, uniformly in $|\theta| \leq \beta$

$$v(\mathbf{R}e^{i\theta}) = O((t/R)^{3\gamma}); \quad \frac{\partial v}{\partial r}(\mathbf{R}e^{i\theta}) = O(t^{3\gamma} R^{-3\gamma-1}). \tag{6}$$

It is known that there are arbitrarily large values of R such that [1]

$$\begin{aligned} \int_0^{2\pi} \left| \frac{\partial}{\partial r} \log |f(\mathbf{R}e^{i\theta})| \right| d\theta & \leq \int_0^{2\pi} \left| \frac{f'}{f}(\mathbf{R}e^{i\theta}) \right| d\theta \\ & < A(\lambda) T(2R, f)/R < A_2(\lambda) R^{\lambda-1+\epsilon}. \end{aligned} \tag{7}$$

Also in $0 \leq \theta < \beta$, $\rho < r < R$, $z \neq t$,

$$\frac{\partial v}{\partial \theta}(\mathbf{r}e^{i\theta}) = -\frac{\partial v}{\partial \theta}(\mathbf{r}e^{-i\theta}) = \frac{2\gamma r^\gamma t^\gamma \sin \gamma\theta}{r^{2\gamma} + t^{2\gamma}} \left\{ 1 - \frac{(r^{2\gamma} + t^{2\gamma})^2}{|z^{2\gamma} - t^{2\gamma}|^2} \right\} \leq 0,$$

and $v(\mathbf{r}e^{i\beta}) = v(\mathbf{r}e^{-i\beta}) = 0$ ($\beta = \pi/2\gamma$).

Therefore $v(z) > 0$ ($z \in D$). Using these observations and the explicit value

$$\Delta v = \frac{16\gamma^2 t^{3\gamma} r^{3\gamma-2} \cos \gamma\theta}{(r^{2\gamma} + t^{2\gamma})^3},$$

we let $R \rightarrow \infty$ and $\rho \rightarrow 0$ in (5) to obtain

$$\log |f(t)| \leq \frac{8\gamma^2 t^{3\gamma}}{\pi} \int_0^\infty \frac{r^{3\gamma-1} dr}{(r^{2\gamma} + t^{2\gamma})^3} \int_{-\beta}^\beta \cos \gamma\theta \log |f(re^{i\theta})| d\theta. \quad (8)$$

Let $h(\theta) = h(\theta + 2\pi)$ be defined by

$$h(\theta) = \begin{cases} \cos \gamma\theta & (|\theta| \leq \beta) \\ 0 & (\beta < |\theta| \leq \pi). \end{cases}$$

Now $h(\theta)$ has the (convergent) Fourier series

$$h(\theta) = \sum_{-\infty}^{\infty} a_k e^{ik\theta} = \sum (\gamma/\pi) \cos k\beta (\gamma^2 - k^2)^{-1} e^{ik\theta}, \quad (9)$$

and if

$$\log |f(re^{i\theta})| = \sum_{-\infty}^{\infty} b_k(r) e^{ik\theta},$$

then, by Parseval's formula,

$$\frac{1}{2\pi} \int_{-\beta}^\beta \cos \gamma\theta \log |f(re^{i\theta})| d\theta = \sum_{-\infty}^{\infty} a_{-k} b_k(r) \leq \sum_{-\infty}^{\infty} |a_{-k}| |b_k(r)|. \quad (10)$$

By an important observation of J. Miles and D. F. Shea [3]

$$\begin{aligned} |b_{-k}(r)| &= |b_k(r)| \leq A_k r^k + \frac{1}{2}k \int_0^r \{(r/u)^k - (u/r)^k\} (N(u)/u) du + N(r) \quad (0 \leq k < \lambda) \\ &\leq \frac{1}{2}k \int_0^r (u/r)^k (N(u)/u) du + \frac{1}{2}k \int_r^\infty (r/u)^k (N(u)/u) du - N(r) \quad (k > \lambda). \end{aligned} \quad (11)$$

Now we choose $\epsilon > 0$ so small that the interval $[\lambda - \epsilon, \lambda + \epsilon]$ does not contain an integer and we choose for $t = t_n$ a Pólya peak satisfying (3) and (2') with this ϵ .

Using the inequalities (3) and (2') for $N(u)$ in (11) leads to the following estimates.

If

$$0 \leq k < \lambda, \quad 0 \leq r \leq t,$$

$$|b_k(r)| \leq A_k r^k + \frac{k^2}{(\lambda - \epsilon)^2 - k^2} N(t)(r/t)^{\lambda - \epsilon} + N(r). \quad (12)$$

If

$$0 \leq k \leq \lambda, \quad t < r,$$

$$|b_k(r)| \leq A_k r^k + \frac{k^2}{(\lambda + \epsilon)^2 - k^2} N(t)(r/t)^{\lambda + \epsilon} + \frac{\epsilon k}{(\lambda - k)^2 - \epsilon^2} N(t)(r/t)^k + N(r). \quad (13)$$

If

$$\lambda < k, \quad 0 \leq r \leq t,$$

$$|b_k(r)| \leq \frac{k^2}{k^2 - (\lambda - \epsilon)^2} N(t)(r/t)^{\lambda - \epsilon} + \frac{\epsilon k}{(k - \lambda)^2 - \epsilon^2} N(t)(r/t)^k - N(r). \quad (14)$$

If

$$\lambda < k, \quad t < r,$$

$$|b_k(r)| \leq \frac{k^2}{k^2 - (\lambda + \epsilon)^2} N(t)(r/t)^{\lambda + \epsilon} + \frac{\epsilon k}{(k + \lambda)^2 - \epsilon^2} N(t)(t/r)^k - N(r). \quad (15)$$

Now replace $f(z)$ by $f_1(z) = f(ze^{i\beta})f(ze^{-i\beta})$. Then

$$\log |f_1(re^{i\theta})| = 2 \sum_{-\infty}^{\infty} \cos k\beta b_k(r) e^{ik\theta}$$

and by (9) and (10)

$$J = \frac{1}{2\pi} \int_{-\beta}^{\beta} \cos \gamma\theta \log |f_1(re^{i\theta})| d\theta \leq (2\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta |\gamma^2 - k^2|^{-1} |b_k(r)|.$$

Using (12) and (14) this leads to

$$J \leq N(t)(r/t)^{\lambda - \epsilon} (2\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta (\gamma^2 - k^2)^{-1} ((\lambda - \epsilon)^2 - k^2)^{-1} k^2$$

$$+ C_1(1 + r^{\lambda}) + \epsilon C_2 N(t) + (2\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta (\gamma^2 - k^2)^{-1} N(r). \quad (0 < r < t) \quad (16)$$

(Note the disappearance of the absolute value signs.)

Similarly, by (13) and (15)

$$J \leq N(t)(r/t)^{\lambda+\epsilon}(2\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta(\gamma^2 - k^2)^{-1}((\lambda + \epsilon)^2 - k^2)^{-1} k^2 \\ + C_1(1 + r^{[\lambda]}) + \epsilon C_3 N(t) + (2\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta(\gamma^2 - k^2)^{-1} N(r). \quad (t < r) \quad (17)$$

By (9)

$$0 = h(\beta) + h(-\beta) = 2(\gamma/\pi) \sum_{-\infty}^{\infty} \cos^2 k\beta(\gamma^2 - k^2)^{-1},$$

so that the terms with $N(r)$ in (16) and (17) disappear.

Using (16) and (17) in (8) (with f_1 in place of f) and dividing by $N(t)$ we have

$$\{\log |f(te^{i\beta})| + \log |f(te^{-i\beta})|\}/N(t) \leq 16\gamma^2 t^{3\gamma} \int_0^t \frac{r^{3\gamma-1}(r/t)^{\lambda-\epsilon} dr}{(r^{2\gamma} + t^{2\gamma})^3} S_1 \\ + 16\gamma^2 t^{3\gamma} \int_t^\infty \frac{r^{3\gamma-1}(r/t)^{\lambda+\epsilon}}{(r^{2\gamma} + t^{2\gamma})^3} S_2 + \epsilon C_4 t^{3\gamma} \int_0^\infty \frac{r^{3\gamma-1} dr}{(r^{2\gamma} + t^{2\gamma})^3} \\ + (C_5/N(t)) t^{3\gamma} \int_0^\infty \frac{r^{3\gamma-1}(1+r^{[\lambda]})}{(r^{2\gamma} + t^{2\gamma})^3} dr, \quad (18)$$

where

$$S_1 = (2\gamma/\pi) \sum_{-\infty}^{\infty} k^2 \cos^2 k\beta(\gamma^2 - k^2)^{-1}((\lambda - \epsilon)^2 - k^2)^{-1} \\ S_2 = (2\gamma/\pi) \sum_{-\infty}^{\infty} k^2 \cos^2 k\beta(\gamma^2 - k^2)^{-1}((\lambda + \epsilon)^2 - k^2)^{-1}.$$

The last two integrals in (18) are less than

$$C_6\epsilon + C_7 t^{[\lambda]}/N(t) < C_8\epsilon,$$

in view of (4).

By an easy application of dominated convergence the first two terms on the right hand side of (18) tend to

$$L = 16\gamma^2 t^{3\gamma} \int_0^\infty \frac{r^{3\gamma-1}(r/t)^\lambda dr}{(r^{2\gamma} + t^{2\gamma})^3} \cdot (2\gamma/\pi) \sum_{-\infty}^{\infty} k^2 \cos^2 k\beta(\gamma^2 - k^2)^{-1}(\lambda^2 - k^2)^{-1}. \quad (19)$$

The sum and the integral can be evaluated explicitly (see below) giving

$$L = \frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda}. \quad (20)$$

(19), (20) and (18) together prove (1) and the Theorem is proved.

The evaluation of the sum in (19) can be accomplished by integrating $z^2(1 + e^{2\beta zi})/\{(\gamma^2 - z^2)(\lambda^2 - z^2)(e^{2\pi iz} - 1)\}$ around a large square and taking real parts. The integral is obtained by differentiating the formula

$$\int_0^\infty \frac{x^{\mu-1}}{x+s} dx = \frac{\pi s^{\mu-1}}{\sin \pi\mu} \quad (0 < \mu < 1)$$

twice with respect to s and noting that the resulting integral formula remains valid in $0 < \mu < 2$ by analytic continuation.

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