Comment. Math. Helvetici 52 (1977) 1–10 Birkhäuser Verlag, Basel

The Kahn–Priddy Theorem and Equivariant Vector Fields on Spheres

L. M. WOODWARD

Introduction

In this paper we give a new proof of the Kahn–Priddy Theorem and show how the theorem may be used to prove results on equivariant vector fields on spheres using a construction from [12]. Let G be a finite group and let W be a (real) G-module. (If G acts trivially and $\dim_{\mathbb{R}} W = n$ we write $\mathbb{R}^n$ in place of W). We may assume that W is equipped with a G-invariant inner product so that the unit sphere $S(W)$ in W may be regarded as a G-space. The G-module W is *G-free* if G acts freely on $S(W)$. (The classification of G-free G-modules is carried out in Chapters 5, 6 and 7 of [11]). Let $F^G(W)$ denote the space of G-maps $f: S(W) \to S(W)$ with the compact open topology. Then there is an inclusion $u: F^G(W) \to F(\mathbb{R}^n)$, where $n = \dim W$, defined by forgetting the G-action. If X, Y are pointed spaces let $[X, Y]$ denote the set of pointed homotopy classes of pointed maps from X to Y. Then taking the identity map in $F^G(W)$ as the base-point we prove the following.

THEOREM A. *Let C_p be the cyclic group of prime order p and let W be a C_p-free C_p-module of dimension n. If X is a connected CW-complex with* $\dim X \leq n-1$ *then the induced map*

$$u_*: [X, F^{C_p}(W)] \to [X, F(\mathbb{R}^n)]$$

is surjective on q-torsion for all primes q which are coprime to $p-1$.

If kW denotes the direct sum of k copies of W then taking the join with the identity map on $S(W)$ gives an inclusion of $F^G(kW)$ in $F^G((k+1)W)$ and if W is G-free we write $F^G = \lim_k F^G(kW)$. As shown in [2], the homotopy type of F^G is independent of the choice of G-free G-module W. When G is the trivial group we write F in place of F^G. The inclusion of $F^G(W)$ in F^G is an n-equivalence, (see Theorem 1.6), so that if X is a connected finite CW-complex then the induced homomorphism $u_*: [X, F^{C_p}] \to [X, F]$ is an epimorphism on q-torsion for all primes q which are coprime to $p-1$. However, by a transfer argument (see [2], Proposition 10.1), u_* is an epimorphism on q-torsion for all primes q which are coprime to p. Since $[X, F]$ is a finite group we have the following theorem which is equivalent to the Kahn–Priddy Theorem as stated in [7].

THEOREM B (Kahn–Priddy). *Let X be a connected finite CW-complex. Then the induced homomorphism*

$$u_*:[X, F^{C_p}] \to [X, F]$$

is surjective.

That Theorem B implies the Kahn–Priddy theorem as stated in [7] may be seen as follows. Recall from [2] that F^{C_p} is homotopy equivalent to $Q(BC_p^+)$, (where X^+ denotes the disjoint union of X with a point and $Q(X^+)$ denotes $\Omega^\infty S^\infty(X^+)$), and that the forgetful map $u: F^{C_p} \to F$ corresponds to the transfer map $\tau: Q(BC_p^+) \to Q(S^0)$. Then, since $Q(X^+) \simeq Q(X) \times Q(S^0)$, we have that

$$\tau_*: \pi_i(Q(BC_p)) \times \pi_i(Q(S^0)) \to \pi_i(Q(S^0)) \qquad (i \geq 1)$$

is given by

$$\tau_*(a, b) = \tau'_*(a) + pb,$$

where $\tau': Q(BC_p) \to Q(S^0)$ is the restriction of τ to $Q(BC_p)$, and τ_* is surjective by Theorem B. Hence $\tau'_*: \pi_i(Q(BC_p)) \to \pi_i(Q(S^0))$ is an epimorphism on p-primary components for $i \geq 1$. An argument showing that the Kahn–Priddy Theorem implies Theorem B is given in [2], (see Theorem 10.2).

By a similar use of the above transfer argument Theorem B can be improved to give the following.

THEOREM C. *Let X be a connected finite CW-complex and let G be a finite group which admits a G-free G-module. Then the induced homomorphism*

$$u_*:[X, F^G] \to [X, F]$$

is surjective on p-torsion for all primes p such that p^2 does not divide the order of G. In particular if the order of G is square-free then u_* is surjective.

Remark. If the order of G is square-free then G is cyclic, (see Theorem 7.4.15 of [11]). I do not know if u_* is surjective in general for G cyclic.

Now let F denote one of the fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of real numbers, complex numbers and quaternions respectively, and let $d = \dim_{\mathbb{R}} F$. If $O_{n,k}$ denotes the Stiefel manifold of orthonormal k-frames in F^n, where F^n is equipped with the usual inner product, then, as is well-known, the map $p_{n,k} : O_{n,k} \to S^{dn-1}$, defined by assigning to a k-frame its first vector, is a fibration. If G is a finite group acting as a group of linear isometries on F^n then both $O_{n,k}$ and S^{dn-1} inherit the structure of G-spaces and $p_{n,k}$ is a G-map. A cross-section of $p_{n,k}$ which is a G-map is called a *G-cross-section.* (Necessary and sufficient conditions for the existence of G-cross-sections in the real case are given in [1]). Let $\Gamma_{n,k} = \Gamma(p_{n,k})$ denote the space of cross-sections of $p_{n,k}$, $\Gamma^G_{n,k}$ the subspace of G-cross-sections, and $v : \Gamma^G_{n,k} \to \Gamma_{n,k}$ the inclusion. Using a construction of [12] we prove the following.

THEOREM D. *Let G be a finite group acting freely as a group of linear isometries on F^n, let W denote the corresponding G-module and suppose $p_{n,k} : O_{n,k} \to S^{dn-1}$ admits a G-cross-section. Then there is a commutative diagram*

$$\begin{array}{ccc} [X, \Gamma^G_{n,k}] & \to & [X \wedge P(F^k), F^G(W)] \\ \downarrow{\scriptstyle v_*} & & \downarrow{\scriptstyle u_*} \\ [X, \Gamma_{n,k}] & \to & [X \wedge P(F^k), F(\mathbb{R}^{dn})], \end{array}$$

where $P(F^k)$ is the projective space associated to F^k. If X is a connected CW-complex the horizontal maps are injections if $\dim X < r$ *and surjections if* $\dim X \leq r$, *where $r = d(n-2k+2)-2$.*

Using Theorems C and D together with the tables of homotopy groups of Stiefel manifolds from [10] we easily deduce the following theorem.

THEOREM E. *Let G be a finite group acting freely as a group of linear isometries on F^n. Suppose that $p_{n,k} : O_{n,k} \to S^{dn-1}$ admits a G-cross-section, and in the real case suppose that $n > 2k$. Then every cross-section is homotopic to a G-cross-section in each of the following cases:*

(a) *if G is of square-free order;*
(b) *if* $|G| \not\equiv 0 \pmod 4$, *and $k = 2, 3, 5$ or 6 when $F = \mathbb{R}$, or $k = 2$ when $F = \mathbb{C}$;*
(c) *if* $|G| \not\equiv 0 \pmod{q^2}$, *where $q = 2, 3$, and $k = 4$ when $F = \mathbb{R}$, or $k = 2$ when* $F = \mathbb{H}$.

Remarks.

1. The condition $n > 2k$ in the real case is always satisfied if $p_{n,k}$ admits a cross-section provided $n \neq 2$, 4, 8 or 16. (See [5]).

2. The main results of this paper have been obtained independently by similar methods by M. C. Crabb (Doctoral thesis, Oxford 1976) and by J. C. Becker (to appear). Similar results in the real case when $G = C_2$ have been proved by different methods by Milgram and Zvengrowski in [8].

1. The Kahn–Priddy Theorem

If X, Y are G-spaces we write $\mathrm{Map}^G(X, Y)$ for the space of G-maps from X to Y. If A is a G-subspace of a G-space X then the inclusion of A in X is a *G-cofibration* if there is a retraction $r: X \times I \to X \times O \cup A \times I$ which is a G-map. The following results are then easily verified.

LEMMA 1.1. *Let A be a G-subspace of a G-space X and suppose the inclusion $j: A \to X$ is a G-cofibration. Then if Y is a G-space the map*

$$j^{\#}: \mathrm{Map}^G(X, Y) \to \mathrm{Map}^G(A, Y)$$

defined by restriction to A is a Hurewicz fibration.

If X is a G-space let X^G denote the subspace of fixed points. Then as an immediate corollary of Lemma 1.1 we have:

LEMMA 1.2. *If X, Y are G-spaces and the inclusion of X^G in X is a G-cofibration then the map*

$$\rho: \mathrm{Map}^G(X, Y) \to \mathrm{Map}(X^G, Y^G)$$

given by restriction to fixed points is a Hurewicz fibration.

The arguments of [9] can be extended to prove the following.

LEMMA 1.3. *If G is a discrete group and X, Y are G-spaces, where X is compact and Y is a simplicial complex on which G acts simplicially, then the space* $\mathrm{Map}^G(X, Y)$ *has the homotopy type of a CW-complex.*

Now suppose that G is a finite group and that W is a G-module. As in the introduction let $F^G(W)$ denote the space of G-maps $f: S(W) \to S(W)$ and if V is a G-submodule of W let $F^G_{(V)}(W)$ denote the subspace of maps whose restriction to $S(V)$ is the identity. Then according to Lemma 1.2 we have a Hurewicz fibration

$$\rho: F^G(W) \to F(W^G),$$

with fibre over the identity map the space $F^G_{(W^G)}(W)$. Furthermore W may be expressed as a direct sum of G-modules $W^G \oplus W'$ where $W'^G = \{0\}$. Taking the join with the identity map on $S(W')$, (and using the fact that $S(W) = S(W^G) * S(W')$), gives a cross-section of $\sigma: F(W^G) \to F^G(W)$ of ρ.

THEOREM 1.4. *The map* $\alpha: F(W^G) \times F^G_{(W^G)}(W) \to F^G(W)$ *defined by* $\alpha(f, f') = \sigma(f) \cdot f'$ *is a homotopy equivalence.*

Proof. According to Lemma 1.3, (and an extension of the argument in the case of $F^G_{(W^G)}(W)$), the spaces $F^G(W)$, $F^G_{(W^G)}(W)$ and $F(W^G)$ each have the homotopy type of a CW-complex. The result now follows from the fact that if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration with cross-section s, where F, E, B each have the homotopy type of a CW-complex and E is an H-space, then the map $\alpha: B \times F \to E$ defined by $\alpha(b, f) = s(b) \cdot i(f)$ is a homotopy equivalence.

If G is a finite group of order m let $\mathbb{R}G$ denote the regular representation of G. Then $(\mathbb{R}G)^G = \mathbb{R}$ and hence $\mathbb{R}G = \mathbb{R} \oplus M$, where M is a G-module with $M^G = \{0\}$, so that $n\mathbb{R}G = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}G = \mathbb{R}^n \oplus nM$ and $(n\mathbb{R}G)^G = \mathbb{R}^n$. Furthermore if W is a G-module of dimension n then W is a direct summand of $n\mathbb{R}G$ since every irreducible G-module is a direct summand of $\mathbb{R}G$.

Let $f: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ be a map. Then the m-fold join

$$f * \cdots * f: S(m\mathbb{R}^n) \to S(m\mathbb{R}^n)$$

is a Σ_m-map where the symmetric group Σ_m acts by permuting the factors in the join. Hence if G is a finite group of order m then, regarding G as a subgroup of Σ_m via the Cayley homomorphism, the above construction defines a map

$$j: F(\mathbb{R}^n) \to F^G(n\mathbb{R}G).$$

Let $r: F^G(n\mathbb{R}G) \to F^G_{(\mathbb{R}^n)}(n\mathbb{R}G)$ be the map defined by taking an inverse to the homotopy equivalence α of Theorem 1.4 and composing with right projection, and let

$$l: F^G_{(\mathbb{R}^n)}(n\mathbb{R}G) \to F(m\mathbb{R}^n)$$

be the map obtained by forgetting the G-action. Then we have a composite map

$$t = lrj : F(\mathbb{R}^n) \to F(m\mathbb{R}^n).$$

Let

$$j^{(m-1)} : F(\mathbb{R}^n) \to F(m\mathbb{R}^n)$$

be defined by $j^{(m-1)}f = f * 1 * \cdots * 1$. Then $j^{(m-1)}$ is homotopic to the map

$$\Delta^{(m-1)} : F(\mathbb{R}^n) \to F(m\mathbb{R}^n)$$

defined by considering $\mathbb{R}^n$ as the diagonal in $m\mathbb{R}^n = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and taking $\Delta^{(m-1)}f = f * 1_{S(\Delta^\perp)}$, where $\Delta^\perp$ is the orthogonal complement of Δ in $m\mathbb{R}^n$. Furthermore if

$$j' : F(\mathbb{R}^n) \to F(m\mathbb{R}^n)$$

is defined by $j'f = 1 * f * \cdots * f$ then the maps $F(\mathbb{R}^n) \to F(m\mathbb{R}^n)$ given by $f \mapsto (j^{(m-1)}f) \cdot (j'f)$ and $f \mapsto (\Delta^{(m-1)}f) \cdot (tf)$ are each homotopic to the diagonal map $f \mapsto f * \cdots * f$.

Recall that if X is a connected CW-complex then $[X, F^G(W)]$ admits a natural group structure and in particular if $\dim X < n$ then $[X, F(\mathbb{R}^n)]$ is an abelian group.

THEOREM 1.5. *Let X be a connected CW-complex and let t_*, $j_*^{(m-1)} : [X, F(\mathbb{R}^n)] \to [X, F(m\mathbb{R}^n)]$ be the homomorphisms induced by the maps t, $j^{(m-1)}$ defined above. Then $t_*([f]) = (m-1)j_*^{(m-1)}([f])$, for each $[f] \in [X, F(\mathbb{R}^n)]$.*

Proof. If $F(\mathbb{R}^n)_{(1)}$ denotes the identity component of $F(\mathbb{R}^n)$ then the maps

$$\lambda_j, \lambda_\Delta : F(\mathbb{R}^n)_{(1)} \times F(m\mathbb{R}^n)_{(1)} \to F(\mathbb{R}^n)_{(1)} \times F(m\mathbb{R}^n)_{(1)}$$

defined by $\lambda_j(f, g) = (f, (j^{(m-1)}f) \cdot g)$ and $\lambda_\Delta(f, g) = (f, (\Delta^{(m-1)}f) \cdot g)$ are homotopy equivalences, and $\lambda_j \simeq \lambda_\Delta$ since $j^{(m-1)} \simeq \Delta^{(m-1)}$. Since each of the maps

$$\lambda_j(1, j'), \lambda_\Delta(1, t) : F(\mathbb{R}^n)_{(1)} \to F(\mathbb{R}^n)_{(1)} \times F(m\mathbb{R}^n)_{(1)}$$

is homotopic to the map $f \mapsto (f, f * \cdots * f)$ it follows that

$$(1, j') \simeq (1, t) : F(\mathbb{R}^n)_{(1)} \to F(\mathbb{R}^n)_{(1)} \times F(m\mathbb{R}^n)_{(1)}$$

and hence $j' \simeq t: F(\mathbb{R}^n)_{(1)} \to F(m\mathbb{R}^n)_{(1)}$. But now if $j_i^{(m-1)}: F(\mathbb{R}^n) \to F(m\mathbb{R}^n)$ is defined by $j_i^{(m-1)} f = 1 * \cdots * f * \cdots * 1$, (with f occurring in the i-th place), then $j_i^{(m-1)} \simeq j^{(m-1)}$ and $j'f = (j_2^{(m-1)} f) \cdots (j_m^{(m-1)} f)$. The result follows.

If V, W are G-modules let $E_{(V)}: F^G(W) \to F^G_{(V)}(V \oplus W)$, $E_V: F^G(W) \to F^G(V \oplus W)$ be the maps defined by taking the join with the identity on $S(V)$.

THEOREM 1.6. *Let V, W be G-modules where W is G-free of dimension n. Then:*

(i) *$E_{(V)}: F^G(W) \to F^G_{(V)}(V \oplus W)$ is an n-equivalence;*
(ii) *if V is G-free then $E_V: F^G(W) \to F^G(V \oplus W)$ is an n-equivalence.*

Proof. (i) Let $D(V)$ be the unit disc in V and define $\mathrm{Maps}_{S(V)}(D(V), S(V \oplus W))$ to be the space of maps $f: D(V) \to S(V \oplus W)$ such that $f \mid S(V)$ is the natural inclusion σ of $S(V)$ in $S(V \oplus W)$. Then $\mathrm{Maps}_{S(V)}(D(V), S(V \oplus W))$ is a G-space and the map $E_{(V)}$ factors as

$$\begin{tikzcd}
\mathrm{Map}^G(S(W), S(W)) \arrow[r, "i_\#"] \arrow[dr, "E_{(V)}"'] & \mathrm{Map}^G(S(W), \mathrm{Maps}_{S(V)}(D(V), S(V \oplus W))) \arrow[d, "h"] \\
& \mathrm{Map}^G_{S(V)}(S(V \oplus W), S(V \oplus W)),
\end{tikzcd}$$

where $i: S(W) \to \mathrm{Maps}_{S(V)}(D(V), S(V \oplus W))$ is the G-map defined by

$$i(w)(v, t) = (v, t, w) \qquad v \in S(V), \quad t \in I, \quad w \in S(W),$$

and h is the map defined by

$$h(f)(v, t, w) = f(w)(v, t) \qquad v \in S(V), \quad t \in I, \quad w \in S(W).$$

Then h is a homomorphism so we need only show that $i_\#$ is an n-equivalence. Now i is a G-map which is a $(2n-1)$-equivalence and for any subgroup H of G with $H \neq \{1\}$ we have $S(W)^H = \phi$ and $\mathrm{Map}^H_{S(V)}(D(V), S(V \oplus W)) = \phi$. The result now follows from §5 of Chapter II of [3].

(ii) The map $\sigma^\#: \mathrm{Map}^G(S(V \oplus W), S(V \oplus W)) \to \mathrm{Map}^G(S(V), S(V \oplus W))$ induced by the inclusion $\sigma: S(V) \to S(V \oplus W)$ is a Hurewicz fibration by Lemma 1.1 with fibre over σ the space $\mathrm{Map}^G_{(S(V))}(S(V \oplus W), S(V \oplus W))$ of maps whose restriction to $S(V)$ is σ. Now any G-map $f: S^r \times S(V) \to S(V \oplus W)$ such that $f \mid z_0 \times S(V) = \sigma$, (where z_0 is the base-point of S^r), is G-homotopic to a map f'

such that $f'(x, v)=\sigma(v)$ if $r\leq n-1$, again by §5 of Chapter II of [3]. Hence $\pi_r(\mathrm{Map}^G_{S(V)}(S(V\oplus W), S(V\oplus W)))=0$ if $r\leq n-1$ and the result follows from the homotopy exact sequence of the fibration $\sigma^{\#}$.

Proof of Theorem A. Let C_p be the cyclic group of prime order p and let W be a C_p-free C_p-module of dimension n. Then $n\mathbb{R}C_p=\mathbb{R}^n\oplus W\oplus W'$ for some C_p-module W' and W' is C_p-free since every irreducible C_p-module is either C_p-free or trivial. Let X be a connected CW-complex of dimension $\leq n-1$ and consider the following commutative diagram:

$$\begin{tikzcd}
 & {[X, F^{C_p}(W)]} \arrow[r, "u_*"] \arrow[d, "(E_{W'})_*"] & {[X, F(\mathbb{R}^n)]} \arrow[dd, "(E_{(p-1)\mathbb{R}^n})_*"] \\
 & {[X, F^{C_p}(W\oplus W')]} \arrow[d, "(E_{(\mathbb{R}^n)})_*"] & \\
{[X, F(\mathbb{R}^n)]} \arrow[r, "(rj)_*"] & {[X, F^{C_p}_{(\mathbb{R}^n)}(n\mathbb{R}C_p)]} \arrow[r, "l_*"] & {[X, F(p\mathbb{R}^n)]}
\end{tikzcd}$$

The vertical maps are all isomorphisms by Theorem 1.6 and on identifying $[X, F(\mathbb{R}^n)]$ with $[X, F(p\mathbb{R}^n)]$ the map $l_*(rj)_*$ corresponds to multiplication by $p-1$ by Lemma 1.5. The result follows.

2. Equivariant Vector Fields on Spheres

Let W be a G-module over the field F with $\dim_F W=n$, let $O_k(W)$ be the Stiefel manifold of orthonormal k-frames in W and let $E_k(W)=\mathrm{Map}^{S(F)}(S(F^k), S(W))$ where $S(F)$ is the topological group of elements of F of unit modulus. Then there is a commutative triangle of G-maps

$$\begin{tikzcd}
O_k(W) \arrow[rr, "i"] \arrow[dr, "p"'] & & E_k(W) \arrow[dl, "q"] \\
 & S(W) &
\end{tikzcd} \tag{2.1}$$

where i is the inclusion defined by considering an orthonormal k-frame in F^n as a linear isometry of F^k into F^n, and q is a Hurewicz fibration given by evaluation on $(1, 0, \ldots, 0)$ in $S(F^k)$. Let $\Gamma(p)$, $\Gamma(q)$ denote the spaces of cross-sections of p, q; $\Gamma^G(p)$, $\Gamma^G(q)$ the subspaces of G-cross-sections. If H denotes the Hopf line bundle over the projective space $P(F^k)$ and L denotes the trivial line bundle then there is a homeomorphism (cf. [12] Lemma 1.2)

$$\phi^G : \mathrm{Map}^G (S(W), E_k(W)) \to \mathrm{Map}^G_{P(F^k)} (S(L \otimes W), S(H \otimes W))$$

under which G-cross-sections correspond to G-fibre homotopy equivalences which are the identity on the parts of the bundles over $P(F^1)$. Suppose p, and hence q, admits a G-cross-section s. Then s defines a G-fibre homotopy equivalence $\phi^G(s)$ and hence a homotopy equivalence

$$\phi^G(s)_\# : \mathrm{Map}^G_{P(F^k)} (S(L \otimes W), S(L \otimes W)) \to \mathrm{Map}_{P(F^k)} (S(L \otimes W), S(H \otimes W)).$$

But since $S(L \otimes W) = P(F^k) \times S(W)$, on taking adjoints we have a homeomorphism between $\mathrm{Map}^G_{P(F^k)} (S(L \otimes W), S(L \otimes W))$ and $\mathrm{Map}\,(P(F^k), F^G(W))$. Putting these equivalences together gives a homotopy equivalence between $\Gamma^G(q)$ and the space of base-point preserving maps $\mathrm{Map}\,((P(F^k), P(F^1)), (F^G(W), 1))$. The construction is natural with respect to G and hence we deduce that there is a commutative diagram

$$\begin{tikzcd} \Gamma^G(p) \arrow[r, "i_\#"] \arrow[d, "v_\#"'] & \Gamma^G(q) \arrow[d, "w_\#"'] & \mathrm{Map}\,((P(F^k), P(F^1)), (F^G(W), 1)) \arrow[l, "\simeq"'] \arrow[d, "u_\#"] \\ \Gamma(p) \arrow[r, "i_\#"] & \Gamma(q) & \mathrm{Map}\,((P(F^k), P(F^1)), (F(\mathbb{R}^{dn}), 1)) \arrow[l, "\simeq"'] \end{tikzcd} \tag{2.2}$$

where v, w are the natural inclusions.

Proof of Theorem D. Suppose W is a G-free G-module. Then from diagram (2.1) we obtain the following diagram of quotient spaces

$$\begin{tikzcd} O_k(W)/G \arrow[rr] \arrow[dr, "\bar{p}"'] & & E_k(W)/G \arrow[dl, "\bar{q}"] \\ & S(W)/G & \end{tikzcd}$$

and $\Gamma^G(p)$, $\Gamma^G(q)$ are homeomorphic to $\Gamma(\bar{p})$, $\Gamma(\bar{q})$ respectively. The map i is a $(2d(n-k+1)-3)$-equivalence by Theorem 6.5 of [4], and hence so is $\bar{\imath}$. Thus it follows from Theorem 3.2 of [6] that $i_\# : \Gamma^G(p) \to \Gamma^G(q)$ is a $(d(n-2k+2)-2)$-equivalence. Theorem D now follows immediately.

Proof of Theorem E. (a): This follows immediately from Theorems C and D together with the fact that if $p_{n,k}$ admits a cross-section then $n > 2(k-1) + 2/d$ in the complex and quaternionic cases, (see [5]).

(b) and (c): In these cases it suffices to note that $\Gamma_{n,k}$ is homotopy equivalent to Map $(S^{dn-1}, O_{n-1,k-1})$ and the only torsion occurring in $\pi_{dn-1}(O_{n-1,k-1})$ is 2-torsion in the cases in (b) and 2-torsion and 3-torsion in the cases in (c).

REFERENCES

[1] J. C. BECKER, *The span of spherical space forms*, Amer. J. Math. *94* (1972), 991–1026.

[2] —— and R. E. SCHULTZ, *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. *44* (1974), 1–34.

[3] G. E. BREDON, *Equivariant cohomology theories*, Springer Lecture Notes in Mathematics No. 34, Berlin 1967.

[4] I. M. JAMES, *The space of bundle maps*, Topology *2* (1963), 45–59.

[5] ——, *The topology of Stiefel manifolds*, London Mathematical Society Lecture Notes in Mathematics.

[6] —— and E. THOMAS, *Note on the classification of cross-sections*, Topology *4* (1966), 351–359.

[7] D. S. KAHN and S. B. PRIDDY, *Applications of the transfer to stable homotopy theory*, Bull. Amer. Math. Soc. *78* (1972), 981–987.

[8] R. J. MILGRAM and P. ZVENGROWSKI, *Skew linearity of r-fields on spheres* (to appear).

[9] J. W. MILNOR, *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. *90* (1959), 272–280.

[10] G. F. PAECHTER, *The groups $\pi_r(V_{n,m})(I)$*, Quart. J. Math. (2) 7 (1956), 249–268.

[11] J. WOLF, *Spaces of constant curvature*, Publish or Perish Inc., Boston 1974.

[12] L. M. WOODWARD, *Vector fields on spheres and a generalisation*, Quart. J. Math. (2) *24* (1973), 357–366.

Dept. of Mathematics,
Univ. of Durham,
Science Laboratories,
South Road,
Durham DH1 3LE.

Received April 13/July 19, 1976.