**Zeitschrift:** Commentarii Mathematici Helvetici
**Herausgeber:** Schweizerische Mathematische Gesellschaft
**Band:** 52 (1977)

**Artikel:** An extremal problem for harmonic measure.
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**DOI:** https://doi.org/10.5169/seals-40024

Comment. Math. Helvetici **52** (1977) 611–624 Birkhäuser Verlag, Basel

# An extremal problem for harmonic measure

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Dedicated to Professor Albert Pfluger on his Seventieth Birthday

## 1. Introduction

Many inequalities of conformal geometry are based, in essence, on the simple fact that the shortest curve joining two distinct points in the plane is the straight line segment between these points; this is so since a metric is conformal when it is locally Euclidean, i.e. Euclidean in a suitable local uniformizer. The approach to inequalities of this sort is twofold: either for a certain quantity an inequality is sought, and the method of finding it consists in the device of an appropriate conformal metric; or one asks which inequalities will flow out of a given conformal metric. Often, conformal metrics stemming from quadratic differentials are particularly useful, and precise inequalities are obtained in many cases. In this connection we shall consider here problems similar in type to Milloux's *Problem* (see e.g. Nevanlinna [3], p. 72).

For three distinct points $a$, $b$, $c$ in the unit disk $E$ we call $\mathfrak{K}$ the family of all continua $K$ satisfying $\{a, b\} \subset K \subset E - \{c\}$ and which are such that $c$ is contained in the doubly connected component of $E - K$. We write $\omega_K$ for the harmonic measure of $K$ in $E$, and we are interested in sharp inequalities for $\omega_K(c)$ under certain side conditions on $K$. So we ask: for which $K$ is $\omega_K(c)$ extremal in some sense?

Natural candidates for extremality are continua which are obtained in the following way: take a quadratic differential $\sigma$ on the extended plane $\hat{\mathbb{C}}$ with the properties: it is real on the boundary $\partial E$ of $E$, it has in $E$ at most simple poles and these at most at the points $a$, $b$, $c$, and there is a trajectory joining $a$ and $b$. Then the closure of that trajectory is a candidate.

A quadratic differential with the above properties is a "quadratic differential with closed trajectories." For such quadratic differentials on a general Riemann surface $\mathfrak{R}$, Strebel has studied in [4], [5] the extremal properties of the moduli of the induced characteristic ring-domains; the results obtained there will enter here

(Theorem 2), but we first will have to find conditions yielding a "quadratic differential with closed trajectories" of a certain homotopy type (Theorem 1), while thereafter we have to deal also with moduli of ring-domains which are the interior of the closure of the union of two adjacent characteristic ring-domains (Theorems 3, 4, 5).

We shall limit here the discussion to the essentials of the method in order to exhibit its simplicity, so we leave aside for investigation at another place certain special cases and some otherwise interesting questions as, for instance, the problem (posed by Gaier at a colloquium 1970) to determine $\inf\{\omega_K(c);\ K\in\mathfrak{K}\}$; the attempt to solve that problem actually occasioned the present results, which themselves lead to a local solution (Remark 2 at the end) by the use of some further means, to be presented at another place when the general discussion of Gaier's problem is taken up.

## 2. Notation

For two real numbers $x\neq y$, $[x, y]$ is the closed and $\langle x, y\rangle$ is the open interval between them; $[x, y\rangle$ and $\langle x, y]$ are the corresponding half-open intervals. For a set $S$, $\bar{S}$ is its closure and $\partial S$ is its boundary. For two distinct points $a$ and $b$ of the unit disk $E$, $C_{ab}$ is the *Poincaré-circle* (the non-Euclidean straight line) through them in $E$, the *Poincaré-segment* $s_{ab}$ is the closed segment of $C_{ab}$ between $a$ and $b$, $\gamma_{ab}$ is a Jordan arc from $a$ to $b$ along $s_{ab}$.

$a$, $b$, $c$ will always be three distinct points of $E$, fixed up to convenient non-Euclidean translations, such that the Poincaré-circles $C_{ab}$, $C_{ac}$, $C_{bc}$ are likewise distinct; $d$ will be a fourth "variable" point of $E$ which varies mostly on a short Jordan arc through $c$.

$E-(C_{ab}\cup C_{ac}\cup C_{bc})$ has seven components; $E_{abc}$ is the component whose closure does not meet $\partial E$; $L_{ab}(c)$ is the component whose closure is disjoint from $C_{ab}$. $E_{abc}$ is also the simply connected component of $E-(s_{ab}\cup s_{ac}\cup s_{bc})$, it will be called *Poincaré-triangle; $L_{ab}(c)$* is the triangle opposite to the triangle $E_{abc}$ at the vertex $c$, it will be called also *Poincaré-lobe* (with vertex $c$).

For a quadratic differential $\sigma$ on a Riemann surface $\mathfrak{R}$, the set $A$ of its zeros and poles is the *singular set*, $\sigma>0$ determines a directional field on the *regular set* $\mathfrak{R}-A$; a maximal solution curve (considered as set of points) of this directional field is a *trajectory* of $\sigma>0$; two given trajectories are either identical or disjoint. Any trajectory is either a closed analytic Jordan curve or may be parametrized by a injective analytic map $\gamma:\langle 0, 1\rangle\to\mathfrak{R}-A$ with non-vanishing derivative. If in the latter case, $\gamma(t)$ tends in $\mathfrak{R}$ to a limit for $t\to 0$ or $t\to 1$, that limit is necessarily in $A$; if both limits exist we speak of a *two-ended trajectory*, and we call it $T_{uv}$ when

$u$ and $v$ are the two (possibly coinciding) limits. In the case of the quadratic differential

$$\sigma = -\frac{(z-d)(1-\bar{d}z)}{(z-a)(1-\bar{a}z)(z-b)(1-\bar{b}z)(z-c)(1-\bar{c}z)}\,dz^2 = \sigma_d, \qquad z \in \hat{\mathbb{C}}, \tag{2.1}$$

which will be our main tool, we will write also $T_{uv}(d)$ for $T_{uv}$ to indicate the dependence on $d$. Given two singularities $u$ and $v$ of (2.1), at most one trajectory $T_{uv}(d)$ of $\sigma > 0$ exists. $T_c$ is the trajectory of $\sigma_c > 0$ which contains $c$, it is a Jordan curve containing $s_{ab} = \overline{T_{ab}(c)}$ in its interior domain.

When a square root $\sigma^{1/2}$ of a quadratic differential $\sigma$ is integrated along an arc, $\sigma^{1/2}$ is always taken continuously (in the topology of $\hat{\mathbb{C}}$); sometimes a particular branch of $\sigma^{1/2}$ is specified.

We say that $\sigma$ has *regular trajectory structure* if all trajectories $\sigma > 0$ are either closed curves or two-ended trajectories. We shall first give conditions ensuring that $\sigma_d$ of (2.1) does have regular trajectory structure.

## 3. Cases of regular trajectory structure

We observe that for any $d \in E$ the directional field $\sigma_d > 0$ is symmetric with respect to $\partial E$ and that $\partial E$ is a trajectory, but it may happen that $\sigma_d$ fails to have regular trajectory structure (in which case some trajectories of $\sigma_d > 0$ are dense in a certain subdomain of $E$). $\sigma_c$ (where $A = \{a, b, 1/\bar{a}, 1/\bar{b}\}$), however, does have regular trajectory structure: $T_{ab}(c) = s_{ab} - \{a, b\}$, and all other trajectories of $\sigma_c > 0$ in $E$ are closed Jordan curves separating $T_{ab}(c)$ from $\partial E$; we shall find a similar trajectory structure prevailing also for some other $d \in E$:

THEOREM 1. *Let $a$, $b$, $c$ be three distinct points of the unit disk $E$ such that the three Poincaré-circles $C_{ab}$, $C_{ac}$, $C_{bc}$ are distinct; with $\varepsilon$ positive but smaller than the distance between $c$ and $s_{ab}$ let*

$$D_\varepsilon = \left\{ d;\ |d-c| \leqq \varepsilon,\ \mathrm{Im} \int_{\gamma_{ab}} \sigma_d^{1/2} = 0 \right\}. \tag{3.1}$$

*Then for sufficiently small $\varepsilon > 0$*

(a) *$D_\varepsilon$ is the carrier of an analytic Jordan arc lying in $L_{ab}(c) \cup \{c\} \cup E_{abc}$ and meeting $L_{ab}(c)$ as well as $E_{abc}$ which has at $c$ as tangent the line bisecting the angle* *between $C_{ac}$ and $C_{bc}$;*

(b) *$d \in D_\varepsilon$ implies*

(i) *the quadratic differential $\sigma_d$ has regular trajectory structure;*

(ii) *there is a two-ended trajectory $T_{ab}(d) \subset E$, which has the property: for d tending to c within $D_\varepsilon$, $E - \overline{T_{ab}(d)}$ tends to $E - s_{ab}$ in the sense of kernel convergence;*

(iii) *for $d \neq c$, there exist two two-ended trajectories $T_{cd}(d) \subset E$ and $T_{dd}(d) \subset E$ with the property: for d tending to c within $D_\varepsilon - \{c\}$, $E - \overline{T_{cd}(d)}$ tends to $E - \{c\}$ and $E - \overline{T_{dd}(d)}$ tends to $E - T_c$ in the sense of kernel convergence in each case.*

(iv) *for $d \neq c$, $T_d = \overline{T_{dd}(d)}$ is a Jordan curve containing $T_{ab}(d)$ in its interior domain; $T_{cd}(d)$ is contained in the interior or exterior domain of $T_d$ as $d \in L_{ab}(c)$ or $d \in E_{abc}$.*

*Remark.* Statements similar to (ii), (iii) about kernel convergence hold also when $d$ tends within $D_\varepsilon$ to some other point $d_1$ (with essentially the same proof as given below for $d \to c$); the condition: *$\varepsilon$ small,* however, cannot simply be omitted (Lowien [2], Sätze 1, 2).

*Proof of Theorem* 1. We procede in several steps; since a non-Euclidean translation leaves the form (2.1) invariant up to a positive factor we may put the points $a$, $b$, $c$ by a non-Euclidean translation to a convenient position.

I. *Part* (a) *holds.*[1] Since $\sigma_d > 0$ on $\partial E$ we may, by Cauchy's Integral theorem, replace in (3.1) the condition $\operatorname{Im} \int_{\gamma_{ab}} \sigma_d^{1/2} = 0$ by

$$\operatorname{Im} \int_{\gamma_{cd}} \sigma_d^{1/2} = 0 \quad \text{and} \quad d \neq c, \quad \text{or} \quad d = c. \tag{3.2}$$

Defining the integral in (3.2) as function of $d$ continuously near $c$, the implicit function theorem, as follows, leads to statement (a).

We assume $c = 0$, $a > 0$, $\operatorname{Im} b \neq 0$. For $|z| < \min(a, |b|) = \varepsilon_0$ we chose an analytic branch

$$\psi(z) = [ab(1 - z/a)(1 - az)(1 - z/b)(1 - \bar{b}z)]^{-1/2}, \tag{3.3}$$

and when $|d| < \varepsilon_0$ we define $(1 - \bar{d}z)^{1/2}$ by the usual power series. Putting for $0 \neq d = |d|\, e^{i\varphi}$ and $0 \leqq t \leqq |d|$

$$f_d(t) = \psi(te^{i\varphi}) \cdot (1 - \bar{d}te^{i\varphi})^{1/2}, \tag{3.4}$$

[1] Excepting the statement about the tangent, the *given proof* of Lemma 1 in Lowien [2] practically contains (a); the following proof seems simpler though.

and defining for $|d|<\varepsilon_0$ the function $F$ by

$$F(d)=\operatorname{Im}\left[e^{i\varphi}\int_0^{|d|}\left(\frac{|d|-t}{t}\right)^{1/2} f_d(t)\,dt\right],\qquad d=|d|\,e^{i\varphi}\neq 0,$$
$$F(0)=0, \tag{3.5}$$

$F(d)$ is one value of $\operatorname{Im}\int_{\gamma_{cd}}\sigma_d^{1/2}$ for $d\neq c$, and therefore by (3.2), once $\varepsilon$ is sufficiently small, we may in (3.1) replace the condition $\operatorname{Im}\int_{\gamma_{ab}}\sigma_d^{1/2}=0$ also by $F(d)=0$.

Considering $F$ as a function $\phi$ of $x=\operatorname{Re} d$ and $y=\operatorname{Im} d$ we infer from (3.4), (3.5): *$\phi$ is real analytic, $\phi$ vanishes at the origin, the first partial derivatives of $\phi$ at the origin both do not vanish* since $\operatorname{Im} b\neq 0$. Thus $D_\varepsilon$ has a tangent at $c=0$; again by (3.4) and (3.5) this tangent is the line bisecting the angle between $C_{ac}$ and $C_{bc}$ in $L_{ab}(c)\cup\{c\}\cup E_{abc}$, yielding (a) of Theorem 1.

II. *Statement* (ii) *in* (b) *holds.* We assume now $-a=b>0$, $\operatorname{Im} c\neq 0$. Call $E'$ the domain obtained slitting $E$ on the real axis from $-1$ to $a$ and from $b$ to 1; choose $\delta>0$ so small that the trajectory $T$ of $\sigma_c>0$ having $|\sigma_c|^{1/2}$-distance $\delta$ from $[a, b]$ separates $c$ from $[a, b]$ in $E$, call $G$ the interior domain of the Jordan curve $T$ and put $G'=G\cap E'$. Mapping $E'$ onto a rectangle $R$ in the $w$-plane so that the four boundary elements of $E'$ at $\pm 1$ correspond to the geometric vertices of $R$ and that $\langle a, b\rangle$ is mapped onto a horizontal segment, $\sigma_c$ becomes $\lambda\,dw^2$ in the uniformizer $w$ where $\lambda>0$, thus in $R$ the trajectories $\sigma_c>0$ are horizontal lines. Since also $\sigma_d/\sigma_c\to 1$ uniformly in $G$ for $d\to c$, we conclude: once $|d-c|$ is sufficiently small, the trajectory $T_a(d)$ of $\sigma_d>0$ which has the limit point $a$, either leaves $G'$ (when traversed starting from $a$) first at a point $\xi_d\in\langle b, 1\rangle$, or $T_a(d)$ is the two-ended trajectory $T_{ab}(d)\subset G'$ in which case we put $\xi_d=b$; for $d\to c$ in addition, the closure of the piece of $T_a(d)$ between $a$ and $\xi_d$ tends to $[a, b]$ in the sense of kernel convergence of the complement with respect to $E$; we have further

$$\operatorname{Im}\int_{\gamma_{b\xi}}\sigma_d^{1/2}\neq 0\quad\text{for any}\quad \xi\in\langle b, 1\rangle. \tag{3.6}$$

So if $d$ also satisfies the condition $\operatorname{Im}\int_{\gamma_{ab}}\sigma_d^{1/2}=0$, i.e. if $d\in D_\varepsilon$, we have necessarily $\xi_d=b$ since (3.6) precludes $\xi_d\in\langle b, 1\rangle$ because the integral of $\sigma_d^{1/2}$ along a Jordan arc on a trajectory $\sigma_d>0$ is always real. So for $d\in D_\varepsilon$ and $\varepsilon>0$ sufficiently small there is a two-ended trajectory $T_{ab}(d)$, and (ii) in (b) follows.

III. *Statements* (b): (i), (iii), *first part of* (iv), *hold.* With the domain $G$ of step II we choose $\varepsilon_1>0$ so small that $d\in D_{\varepsilon_1}$ implies the existence of a two-ended trajectory $T_{ab}(d)\subset G$. For $d=c$ statements (i), (iii), (iv) are trivially true so we assume $d \in D_{\varepsilon_1} - \{c\}$. We take the conformal map $\chi_d^*$ of the doubly connected domain $E^*(d) = E - \overline{T_{ab}(d)}$ onto $R(d) = \{w; r < |w| < 1\}$ where $r = r(d) \in \langle 0, 1\rangle$, so that $\chi_d^*(c) = c' > 0$ and that the boundary component $\partial E$ of $E^*(d)$ corresponds to the same boundary component of $R(d)$, we define $\chi_c^*$ as the analogous normalized map $E^*(c) = E - s_{ab} \to R(c)$. We lift (the transplanted) $\sigma_d$ to the universal covering surface $\tilde{R}(d)$ of $R(d)$ which we realize as $\{w; \log r < \operatorname{Re} w < 0\} = S(d)$; then the quadratic differential $\tilde{\sigma}_d$ thus obtained on $S(d)$ is positive on $\partial S(d)$ and has after analytic continuation to $\mathbb{C}$ the form $\tilde{\sigma}_d = h(w)\, dw^2$ where $h$ is an elliptic function of order 2 with fundamental periods $2\omega_1 = -2 \log r > 0$, $2\omega_2 = 2\pi i$. We may assume that $c'' = \log c'$ and $-c'' \in \langle 0, \omega_1\rangle$ are the simple poles of $h$ in the fundamental rectangle with center 0, then the simple zeros lie necessarily at some point $d'' \in \langle -\omega_1, 0\rangle$ and at $-d''$ since $h$ is elliptic and negative on $\partial S(d)$, and we obtain with Weierstrass' $p$-function for some $\lambda > 0$

$$h(w) = -\lambda \frac{p(w) - p(d'')}{p(w) - p(c'')}, \qquad w \in \mathbb{C}. \tag{3.7}$$

Since $h$ is thus symmetric with respect to the real axis and is positive between $c''$ and $d''$ we get by projection for the trajectories of $\sigma_d > 0$ on $R(d)$: $\sigma_d$ has regular trajectory structure, and with $\chi_d^*(d) = d' = \exp d'' \in \langle r, 1\rangle$, there is a two-ended trajectory $T_{c'd'} = \langle c', d'\rangle$ and also a two-ended trajectory $T_{d'd'}$ whose closure separates the two components of $\partial R(d)$. Thus in (b) we obtain statement (i); we obtain of statement (iii) the existence of $T_{cd}(d) \subset E$ and $T_{dd}(d) \subset E$, and of statement (iv) the fact that $T_d = \overline{T_{dd}(d)}$ is a Jordan curve containing $T_{ab}(d)$ in its interior domain. We note in particular:

$$\langle c', d'\rangle = \chi_d^*(T_{cd}(d)), \qquad T_{d',d'} = \chi_d^*(T_{dd}(d)). \tag{3.8}$$

By the normalization of $\chi_d^*$, (ii) in (b) now implies for $d$ tending to $c$ within $D_{\varepsilon_1} - \{c\}$: $\chi_d^*$ tends to $\chi_c^*$ uniformly in compact subsets of $E^*(c) = E - s_{ab}$, and the remaining statement in (iii) follows from (3.8).

IV. *The remainder of* (iv) *in* (b) *holds.* In addition to the domain $G$ of step II we choose a disk about $c$ whose closure does not meet $\bar{G}$, and we let $\varepsilon_2 > 0$ be so small that $d \in D_{\varepsilon_2} - \{c\}$ implies the existence of a two-ended trajectory $T_{ab}(d) \subset G$ and of a two-ended trajectory $T_{cd}(d)$ contained in the chosen disk. A computational argument is available to prove the remainder of statement (iv). We take $d \in D_{\varepsilon_2} - \{c\}$ and using the notation of step III, we call $E_1(d)$ the (doubly connected) component of $E^*(d) - T_d$ which has $\overline{T_{ab}(d)}$ on the boundary and $E_2(d)$ the other (doubly connected) component.

All that remains to show is

$$c \in E_1(d) \quad \text{when} \quad d \in L_{ab}(c) \tag{3.9.1}$$

$$c \in E_2(d) \quad \text{when} \quad d \in E_{abc}. \tag{3.9.2}$$

We distinguish in the usual way the two shores of $\overline{T_{cd}(d)}$, $s_{cd}$ and we call $\Gamma^-$, $\gamma^-$ the curve running once along the two shores of $\overline{T_{cd}(d)}$, $s_{cd}$ in the negative sense, i.e. in such a way that $E - \overline{T_{ed}(d)}$, $E - s_{ed}$ lies on the right. In each of the domains $E^*(d) - \overline{T_{cd}(d)}$, $E^*(d) - s_{cd}$ an analytic branch of $\sigma_d^{1/2}$ can be defined, and choosing in each case that branch which is positive on the positively oriented unit circle $\partial E$ we obtain

$$\int_{\Gamma^-} \sigma_d^{1/2} = \int_{\gamma^-} \sigma_d^{1/2}. \tag{3.10}$$

Use of the explicit expression (3.7) yields

$$c \in E_1(d) \Leftrightarrow c'' < d'' \Leftrightarrow \int_{\Gamma} \sigma_d^{1/2} < 0, \tag{3.11.1}$$

$$c \in E_2(d) \Leftrightarrow d'' < c'' \Leftrightarrow \int_{\Gamma^-} \sigma_d^{1/2} > 0, \tag{3.11.2}$$

and a simple computation gives for $d$ sufficiently close to $c$ (whether or not $d \in D_{\varepsilon_2}$)

$$\operatorname{Re} \sigma_d^{1/2} < 0 \quad \text{on } \gamma^- \text{ if } \quad d \in L_{ab}(c), \tag{3.12.1}$$

$$\operatorname{Re} \sigma_d^{1/2} > 0 \quad \text{on } \gamma^- \text{ if } \quad d \in E_{abc}. \tag{3.12.2}$$

Combining (3.10), (3.11), (3.12) we obtain (3.9), completing the proof of Theorem 1.

## 4. Extremal decomposition of the unit disk

If $a$, $b$, $c$ are three distinct points of $E$ and if $K$ is a continuum in the class $\mathfrak{K}$ (see Section 1) then there is a continuous branch $k(z)$ for $z \in K$ of $\arg[(z-c)/(1-\bar{c}z)]$ since $K$ does not separate $c$ and $\partial E$; the variation

$$V(K) = k(b) - k(a) \tag{4.1}$$

of $k$ on $K$ from $a$ to $b$ is independent of the branch chosen, and for two continua $K, K' \in \mathfrak{K}$ the corresponding variations $V(K)$, $V(K')$ differ by an integral multiple of $2\pi$. If $a$, $b$, $c$ are as in Theorem 1, $V(K)$ obviously never is an integral multiple of $\pi$ so it is natural to single out the continua for which $0<|V(K)|<\pi$, $K=s_{ab}$ being one of them.

We take now $a$, $b$, $c$ as in Theorem 1; there exists then $\varepsilon>0$ such that the assertions (a) and (b) of Theorem 1 hold and that $d \in D_\varepsilon$ implies $V(\overline{T_{ab}(d)})=V(s_{ab})$ (recalling $\overline{T_{ab}(c)}=s_{ab}$). Such $\varepsilon>0$ will be called *sufficiently small*, and we assume henceforth $d \in D=D_\varepsilon$ with sufficiently small $\varepsilon$. We put

$$\begin{aligned} &K^*(d)=\overline{T_{ab}(d)}, \\ &K_0^*(d)=\overline{T_{cd}(d)} \cup \overline{T_{dd}(d)} \quad \text{for} \quad d \neq c, \qquad\qquad (4.2)\\ &K_0^*(c)=T_c \quad (=\text{trajectory of } \sigma_c>0 \text{ containing } c). \end{aligned}$$

By Theorem 1(b)(iv), $K_0^*(d)$ separates $\partial E$ and $K^*(d)$; $E-(K_0^*(d) \cup K^*(d))$ consists of two components, each doubly connected, we denote by $E_1^*(d)$ the one which has $K^*(d)$ as one boundary component, and by $E_2^*(d)$ the other which has $\partial E$ as one boundary component.

We denote further by

$\frac{1}{2}l_1(d)$ *the* $|\sigma_d|^{1/2}$*-length of* $K^*(d)$,

$l_2(d)$ *the* $|\sigma_d|^{1/2}$*-length of* $\partial E$,

$h_1(d)$ *the* $|\sigma_d|^{1/2}$*-distance of the boundary components of* $E_1^*(d)$,

$h_2(d)$ *the* $|\sigma_d|^{1/2}$*-distance of the boundary components of* $E_2^*(d)$.

Then we obtain immediately for the moduli $M_1^*(d)$ of $E_1^*(d)$ and $M_2^*(d)$ of $E_2^*(d)$ the expression

$$M_1^*(d)=\frac{h_1(d)}{l_1(d)}, \qquad M_2^*(d)=\frac{h_2(d)}{l_2(d)}. \tag{4.4}$$

[The modulus of a doubly connected domain, whose boundary components are continua, is $(1/2\pi)\log r$, $r>1$, when the domain is conformally equivalent to the annulus $\{z; 1<|z|<r\}$.] We compare now the moduli (4.4) with the moduli obtained when $E$ is decomposed by certain other continua.

We call the continuum $K$ *admissible* if $K \in \mathfrak{K}$ and if $V(K)=V(s_{ab})$; we call the pair $(K, K_0)$ an *admissible pair* of continua if $K$ is an admissible continuum and if $K_0$ is a continuum satisfying $c \in K_0 \subset E-K$ and separating $K$ from $\partial E$. We notice that $(K^*(d), K_0^*(d))$ is an admissible pair for each $d \in D$. For an admissible pair

$(K, K_0)$, $E-(K \cup K_0)$ has exactly two doubly connected components; $E_1(K, K_0)$ will be the one whose boundary meets $K$, $E_2(K, K_0)$ the one whose boundary contains $\partial E$; we let $M_1(K, K_0)$ and $M_2(K, K_0)$ denote the moduli of $E_1(K, K_0)$ and $E_2(K, K_0)$, and we call these *the moduli produced* by the pair $(K, K_0)$.

THEOREM 2. *Let* $(K, K_0)$ *be a pair of admissible continua, and let* $M_1 = M_1(K, K_0)$, $M_2 = M_2(K, K_0)$ *be the moduli produced by* $(K, K_0)$; *let further* $d \in D$. *Then*

$$l_1^2(d)M_1 + l_2^2(d)M_2 \leqq l_1^2(d)M_1^*(d) + l_2^2(d)M_2^*(d); \tag{4.5}$$

*equality holds in* (4.5) *if and only if* $K = K^*(d)$ *as well as* $K_0 = K_0^*(d)$.

*Proof.* The "homotopy-condition" $V(K) = V(s_{ab}) = V(K^*(d))$ together with the fact that $K_0$ contains $c$ and separates $K$ from $\partial E$ ensures that any Jordan curve in $E_i(K, K_0)$, $(i = 1, 2)$, separating the boundary components of $E_i(K, K_0)$ has a $|\sigma_d|^{1/2}$-length of at least $l_i(d)$, and a standard argument of extremal length gives the assertion (see e.g. Strebel [4] section 6).

## 5. Extremal continua for harmonic measure

We will consider the following problem: Let $K$ be an admissible continuum so that the modulus $M(K)$ of the doubly connected component $E(K)$ of $E-K$ equals a given number; denote by $\omega_K(c)$ the value of the harmonic measure $\omega_K$ of $K$ in $E$ at $c$; what are then sharp estimates for $\omega_K(c)$? For this, we shall need a further theorem about the decomposition of an annulus $R$ by trajectories of a quadratic differential which is positive on $\partial R$ and which is regular in $R$ up to at most a simple pole and a simple zero on a radius of $R$.

Let $R = \{w; r < |w| < 1\}$ with $0 < r < 1$, let $r < x < 1$ and $r < y < 1$, and let $f_{xy}: R \to \hat{\mathbb{C}}$ be a meromorphic function with the following properties:

*for* $x \neq y$, $f_{xy}$ *is up to a simple pole at* $x$ *and a simple zero at* $y$ *regular and non-vanishing, for* $x = y$, $f_{xy}$ *is regular and non-vanishing;*

$w^2 f_{xy}(w)$ *has negative boundary values on* $\partial R$. (5.2)

$f_{xy}$ is then determined up to a positive factor $[w^2 f_{xx}(w)$ being a negative constant$]$, and the quadratic differential

$$\sigma_{xy} = f_{xy}(w)\, dw^2 \tag{5.3}$$

has regular trajectory structure. For $x \neq y$ there is a two-ended trajectory $T_{xy}$ (the open interval between $x$ and $y$) and a two-ended trajectory $T_{yy}$ ($\overline{T_{yy}}$ separates the boundary components of $R$). We put

$$\begin{aligned} &\text{for } x \neq y, \qquad K_{xy} = \overline{T_{xy}} \cup \overline{T_{yy}}, \\ &\text{for } x = y, \qquad K_{xx} = \{w; |w| = x\}. \end{aligned} \tag{5.4}$$

$R - K_{xy}$ has then two components, both are doubly connected; we denote by $R_1(x, y)$ the one having $B = \{w; |w| = r\}$ on the boundary and by $M_1(x, y)$ its modulus, and by $R_2(x, y)$ the one having $\partial E$ on the boundary and by $M_2(x, y)$ its modulus. The inequality $M_1(x, y) + M_2(x, y) \leqq (1/2\pi) \log(1/r)$ is well known.

THEOREM 3. *Let* $h: S = \{(x, y); r < x < 1, r < y < 1\} \to \mathbb{R}^2$ *be defined by*

$$h(x, y) = (M_1(x, y), M_2(x, y)), \qquad (x, y) \in S, \tag{5.5}$$

*and let*

$$\Delta = \{(m_1, m_2); m_1 \geqq 0, m_2 \geqq 0, m_1 + m_2 \leqq m\} \tag{5.6}$$

*where* $m = (1/2\pi) \log(1/r)$. *Then*

(i) *$h$ has a continuous extension to the closure $\bar{S}$ of $S$;*

(ii) *denoting the extension again by $h$, $h(\bar{S}) = \Delta$;*

(iii) *the restriction of $h$ to each of the triangles*

$$\bar{S}^+ = \bar{S} \cap \{(x, y); x \geqq y\}, \qquad \bar{S}^- = \bar{S} \cap \{(x, y); x \leqq y\}$$

*is a homeomorphism onto $\Delta$, sense-preserving on $\bar{S}^+$ and sense-reversing on $\bar{S}^-$;*

(iv) *for fixed $x \in \langle r, 1 \rangle$, $h_x: [r, 1] \to \Delta$ given by $h_x(y) = h(x, y)$ for $y \in [r, 1]$, is a Jordan arc with strictly decreasing slope, running from $h_x(r)$ on the vertical side of the triangle $\Delta$ to $h_x(1)$ on the horizontal side of $\Delta$ and touching the slanted side of $\Delta$ at $h_x(x) = ((1/2\pi) \log(x/r), (1/2\pi) \log(1/x))$ with slope $-1$.*

*Proof.* Theorems 1 and 2 of [1], stated for decompositions of a rectangle, give immediately the present Theorem 3 by mapping the rectangle through exponentiation onto a half-annulus and going then to the full annulus by reflection.

THEOREM 4. *Let $a$, $b$, $c$ be as in Theorem 1; let $D = D_\varepsilon$ for sufficiently small*

*$\varepsilon>0$; let $d\in D-\{c\}$ and let $K$ be an admissible continuum with $M(K)=M(K^*(d))$; let $\omega$ and $\omega^*$ be the harmonic measures of $K$ and $K^*(d)$ in $E$. Then*

$$\begin{aligned} &\omega(c)\geqq\omega^*(c) \quad \textit{when} \quad d\in L_{ab}(c),\\ &\omega(c)\leqq\omega^*(c) \quad \textit{when} \quad d\in E_{abc}, \end{aligned} \tag{5.7}$$

*and equality holds in (5.7) if and only if $K=K^*(d)$.*

*Proof.* Let first $d\in L_{ab}(c)$. Denote by $\chi$, $\chi^*$ the functions mapping $E(K)$, $E(K^*(d))$ conformally onto the annulus $R(d)=\{w;\, r<|w|<1\}$, $0<r=r(d)<1$, in such a way that the boundary component $\partial E$ of $E(K)$, $E(K^*(d))$ is invariant and that $\chi(c)=x\in\langle r,1\rangle$, $\chi^*(c)=x^*\in\langle r,1\rangle$. Since then

$$\omega(c)=(\log x)/(\log r), \qquad \omega^*(c)=(\log x^*)/(\log r),$$

we have to show $x\leqq x^*$ and that equality holds only for $K=K^*(d)$. Let therefore $K\neq K^*(d)$. If $K_0$ is a continuum in $E-K$ separating the boundary components and containing $c$, $(K, K_0)$ is an admissible pair of continua, and we obtain for the produced moduli $M_1$, $M_2$ from Theorem 2 the strict inequality

$$l_1^2(d)M_1+l_2^2(d)M_2<l_1^2(d)M_1^*(d)+l_2^2(d)M_2^*(d). \tag{5.8}$$

Putting $y^*=\chi^*(d)$ we obtain $x^*<y^*<1$ (Theorem 1, (b), (iv)), and we have in the notation (5.5)

$$M_1^*(d)=M_1(x^*,y^*), \qquad M_2^*(d)=M_2(x^*,y^*).$$

Choosing $K_0=\chi^{-1}(K_{xy})$, $r<y<1$, we have likewise

$$M_1=M_1(x,y), \qquad M_2=M_2(x,y),$$

and (5.8) becomes

$$l_1^2(d)M_1(x,y)+l_2^2(d)M_2(x,y)<l_1^2(d)M_1(x^*,y^*)+l_2^2(d)M_2(x^*,y^*), \qquad r<y<1. \tag{5.9}$$

Now $x\neq x^*$ follows from (5.9) for $y=y^*$. By Theorem 3, using the notation of Theorem 3 with $m=M(K)=M(K^*(d))$, the straight line

$$L=\{(m_1,m_2);\, l_1^2(d)m_1+l_2^2(d)m_2=l_1^2(d)M_1(x^*,y^*)+l_2^2(d)M_2(x^*,y^*)\}$$

meets $h_{x^*}([x^*, 1])$ at just one point $P^* = (M_1(x^*, y^*), M_2(x^*, y^*))$, and $L$ intercepts the side $s = \{(m_1, m_2); m_1 + m_2 = M(K), m_1 \geqq 0, m_2 \geqq 0\}$ of $\Delta$ at some point $(\xi_0^*, M(K) - \xi_0^*)$ with $(1/2\pi)\log(x^*/r) = \xi^* < \xi_0^* < M_1(x^*, y^*)$, by Theorem 3(iv). $h_{x^*}([x^*, 1])$ is a curve in $\Delta$ joining the point $(\xi^*, M(K) - \xi^*) \in s$ to a point of the $m_1$-axis and $h_x([x, 1])$ is a curve in $\Delta$ joining the point $(\xi, M(K) - \xi) \in s$ to a point of the $m_1$-axis, where $\xi = (1/2\pi)\log(x/r)$. These two curves are disjoint by Theorem 3(iii) since $x \neq x^*$; and since $h_x([x, 1])$ stays below $L$ by (5.9). $\xi < \xi^*$ follows, implying $x < x^*$ as contended, proving the first line of (5.7), and with strict inequality unless $K = K^*(d)$.

In a similar way, (5.7) is proved for $d \in E_{abc}$.

*Remark* 1. The case $d = c$ is left out in Theorem 4 since, as is well known, $K \in \mathfrak{K}$ implies $M(K) \leqq M(s_{ab})$ with strict inequality unless $K = s_{ab} = K^*(c)$.

COROLLARY 1. *If $d_1$ and $d_2$ are distinct points of $D$ lying either both in $L_{ab}(c)$ or both in $E_{abc}$, then $M(K^*(d_1)) \neq M(K^*(d_2))$.*

*Proof.* It is immediate that $K^*(d_1)$ and $K^*(d_2)$ are distinct continua, whence $M(K^*(d_1)) = M(K^*(d_2))$ is incompatible with Theorem 4.

COROLLARY 2. *Let $\delta:[-1, 1] \to D$ be a Jordan arc with carrier $D$ so that $\delta(-1) \in L_{ab}(c)$, $\delta(0) = c$, $\delta(1) \in E_{abc}$. Then $\mu(t) = M(K^*(\delta(t)))$ is continuous, $\mu$ is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, 1]$.*

*Proof.* There is (Theorem 1) a parametrization $\delta$ of $D$ as a Jordan arc in the indicated way. The continuity of $\mu = M(K^* \circ \delta)$ is then obvious, and the stated monotony follows from Corollary 1 since $0 < |t| < 1$ implies $K^*(\delta(t)) \neq K^*(\delta(0)) = s_{ab}$ whence (Remark 1) $\mu(t) < M(s_{ab}) = \mu(0)$.

Putting $t_1 = \mu(-1) - \mu(0) < 0$ and $t_2 = \mu(0) - \mu(1) > 0$ we obtain at once by reparametrization, observing $\delta(t) \in L_{ab}(c)$ or $\delta(t) \in E_{abc}$ as $t < 0$ or $t > 0$.

COROLLARY 3. *There is a parametrization $\delta^*:[t_1, t_2] \to D$ of $D$ as a Jordan arc such that*

$$\begin{aligned} \delta^*(t) &\in L_{ab}(c) \quad \text{for} \quad t_1 \leqq t < 0, \\ &\in E_{abc} \quad \text{for} \quad 0 < t \leqq t_2, \end{aligned}$$

*and that $\mu^* = M(K^* \circ \delta^*)$ satisfies*

$$\mu^*(t) = M(s_{ab}) - |t|, \qquad t_1 \leqq t \leqq t_2.$$

THEOREM 5. *Let $K$ be an admissible continuum and let $0 < M(s_{ab}) - M(K) \leqq \min(-t_1, t_2)$. There is then a unique number $t_1^* \in [t_1, 0]$ and a unique number $t_2^* \in [0, t_2]$ so that*

$$M(K) = M(K^*(\delta^*(t_1^*)) = M(K^*(\delta^*(t_2^*)); \tag{5.10}$$

*if further $\omega$, $\omega_1^*$, $\omega_2^*$ are the harmonic measures of $K$, $K_1^* = K^*(\delta^*(t_1^*))$, $K_2^* = K^*(\delta^*(t_2^*))$, the inequality*

$$\omega_1^*(c) \leqq \omega(c) \leqq \omega_2^*(c) \tag{5.11}$$

*is valid; the first inequality is strict unless $K = K_1^*$, the second inequality is strict unless $K = K_2^*$.*

*Proof.* In (5.10), $t_2^* = M(s_{ab}) - M(K)$ and $t_1^* = -t_2^*$ are uniquely determined by Corollary 3, further $\delta^*(t_1^*) \in L_{ab}(c)$ and $\delta^*(t_2^*) \in E_{abc}$; the remaining statements foltow from Theorem 4.

Incidently, Theorem 5 gives $\omega_1^*(c) < \omega_2^*(c)$.

*Remark* 2. Denoting by $\omega_t$ the harmonic measure of $K^*(\delta^*(t))$ in $E$ for $t \in [t_1, t_2]$, one might ask in general for the dependence of $\omega_t(c)$ on $t$. Actually, $\omega_t(c)$ is in $t$ strictly increasing [to be shown at another place] as might be expected: a simple pole of a quadratic differential pushes away the trajectories while a zero attracts the trajectories. If $d \in L_{ab}(c) \cap D$, the action of the pole $c$ on $T_{ab}(d)$ becomes the more pronounced the more $d$ moves along $D$ away from $c$, so $\omega_t(c)$ will decrease as $t$ decreases from 0; for $d \in E_{abc} \cap D$ the situation is reversed making $\omega_t(c)$ increase as $t$ increases from 0. If $K$ is an admissible continuum, if for some $t^0 \in \langle 0, \min(-t_1, t_2)\rangle$ the inequality $M(s_{ab}) - M(K) \leqq t_0$ is valid and if $\omega$ is the harmonic measure of $K$ in $E$ we have therefore

$$\omega_{-t_0}(c) \leqq \omega(c) \leqq \omega_{t_0}(c), \tag{5.12}$$

and (5.12) has the interpretation: among all such continua $K$, $K^*(\delta^*(-t_0))$ is the continuum conformally farthest away from $c$ and $K^*(\delta^*(t_0))$ is the conformally nearest.

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Received April 30, 1977