

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 52 (1977)

**Artikel:** Domain constants associated with Schwarzian derivative.  
**Autor:** Lehto, Olli  
**DOI:** <https://doi.org/10.5169/seals-40023>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 29.12.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Domain constants associated with Schwarzian derivative

OLLI LEHTO

Dedicated to Professor Albert Pfluger on his seventieth birthday

### 1. Definition of the constants

Let  $A$  be a simply connected domain in the extended plane with more than one boundary point. A non-euclidean metric  $\rho(z)|dz|$  of  $A$  is defined by the condition  $\rho(z)|dz| = (1 - |w|^2)^{-1}|dw|$ , where  $z \rightarrow w$  is a conformal mapping of  $A$  onto the unit disc  $D$ . For a function  $\varphi$  holomorphic in  $A$  we introduce the norm

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)| \rho(z)^{-2}.$$

Let  $f$  be a locally injective meromorphic function in  $A$  and  $S_f$  its Schwarzian derivative. At finite points of  $A$  which are not poles of  $f$  we have  $S_f = (f''/f)' - \frac{1}{2}(f''/f)^2$ , and the definition is extended to  $\infty$  and to the poles of  $f$  by means of inversion. Every function which is holomorphic in  $A$  is the Schwarzian of some meromorphic  $f$ . The Schwarzian vanishes identically if and only if  $f$  is a Möbius transformation. A function with a prescribed Schwarzian is determined up to a Möbius transformation.

If  $g : A \rightarrow B$  is a conformal mapping, then

$$|S_f(z) - S_g(z)| \rho_A(z)^{-2} = |S_{f \circ g^{-1}}(\zeta)| \rho_B(\zeta)^{-2}, \quad \zeta = g(z). \quad (1)$$

In particular,

$$\|S_g\|_A = \|S_{g^{-1}}\|_B. \quad (2)$$

We associate with the domain  $A$  the following three constants:

$\sigma_1 = \|S_f\|_A$ , where  $f$  is a conformal map of  $A$  onto a disc,

$\sigma_2 = \sup \{\|S_f\|_A \mid f \text{ univalent in } A\},$

$\sigma_3 = \sup \{a \mid \|S_f\|_A \leq a \text{ implies } f \text{ univalent in } A\}.$

## 2. Constant $\sigma_1$

In the definition of  $\sigma_1$ , a disc means an ordinary disc or a half-plane. The number  $\sigma_1$  is well defined and equal to 0 if and only if  $A$  itself is a disc. It is well known that  $\sigma_1 \leq 6$ , and the example  $A = \{z \mid 0 < \arg z < k\pi\}$ ,  $1 \leq k \leq 2$ , shows that  $\sigma_1$  can take any value of the closed interval  $[0, 6]$ .

In view of (2), we could define  $\sigma_1$  also with the aid of conformal mappings of a disc onto  $A$ . A further characterization is obtained as follows: Let  $f$  be a conformal mapping of the unit disc  $D$  onto  $A$  and  $h$  a conformal self-mapping of  $D$ , such that  $h(0) = z_0$ . Since  $\rho_D(0) = 1$ , it follows from (1) that

$$|S_f(z_0)| \rho_D(z_0)^{-2} = |S_{f \circ h}(0)|. \quad (3)$$

Hence

$$\sigma_1 = \sup \{|S_f(0)| \mid f : D \rightarrow A \text{ conformal}\}. \quad (4)$$

In some cases, information about the boundary of  $A$  makes it possible to improve the estimate  $\sigma_1 \leq 6$ . Suppose that the boundary of  $A$  is a  $K$ -quasicircle, i.e. the image of a circle under a  $K$ -quasiconformal mapping of the plane. (Quasicircles were first investigated by Pfluger [5].) Then  $\sigma_1 \leq 6(K^2 - 1)/(K^2 + 1)$ .

Another result of this type is that for convex domains  $\sigma_1 \leq 2$ . This follows from known results on the coefficients of univalent functions, see e.g. [6]. We include here a simple proof which also gives the extremals. (Quite recently, Nehari (J. Analyse Math. 30 (1976)) also established this result by using variational techniques.)

**THEOREM 1.** *Let  $f$  be a conformal mapping of a disc onto a convex domain. Then*

$$|S_f(z)| \rho(z)^{-2} \leq 2. \quad (5)$$

*Equality holds if and only if the image domain is bounded by two parallel lines.*

*Proof.* We may assume that  $f$  is a conformal map of the unit disc. In view of (3), inequality (5) follows if we prove that  $|S_f(0)| \leq 2$ . Since we may replace  $f$  by the function  $z \rightarrow cf(ze^{i\varphi})$ ,  $c$  complex,  $\varphi$  real, there is no loss of generality in assuming that  $S_f(0) \geq 0$  and that  $f'(0) = 1$ .

It is well known that  $f'$  admits a representation

$$f'(z) = \exp \left( - \int_0^{2\pi} (\log(1 - ze^{-i\theta}) d\psi(\theta)) \right), \quad (6)$$

where  $\psi$  is increasing and

$$\int_0^{2\pi} d\psi(\theta) = 2.$$

Direct computation yields

$$S_f(0) = \int_0^{2\pi} e^{-2i\theta} d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} e^{-i\theta} d\psi(\theta) \right)^2.$$

Since  $S_f(0)$  is real and  $d\psi(\theta) \geq 0$ , we obtain

$$\begin{aligned} S_f(0) &= \int_0^{2\pi} \cos 2\theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 + \frac{1}{2} \left( \int_0^{2\pi} \sin \theta d\psi(\theta) \right)^2 \\ &\leq \int_0^{2\pi} \cos 2\theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 + \int_0^{2\pi} \sin^2 \theta d\psi(\theta) \\ &= \int_0^{2\pi} \cos^2 \theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 \leq \int_0^{2\pi} \cos^2 \theta d\psi(\theta) \leq 2. \end{aligned}$$

Because  $S_f(0) \geq 0$ , we have proved (5).

Equality holds only if

$$\int_0^{2\pi} \cos^2 \theta d\psi(\theta) = 2, \quad \int_0^{2\pi} \cos \theta d\psi(\theta) = 0.$$

These conditions are fulfilled if and only if  $\psi$  has a jump  $+1$  at the points  $0$  and  $\pi$  and is constant on the intervals  $(0, \pi)$  and  $(\pi, 2\pi)$ . Then  $S_f(0) = 2$ , and it follows from (6) that  $f'(z) = (1 - z^2)^{-1}$ . We conclude that the image of  $D$  is a parallel strip.

### 3. Constant $\sigma_2$

The number  $\sigma_2$  is 6 if  $A$  is a disc and  $\sigma_2 \leq 12$  for all domains  $A$ . In fact, there is a simple relation between  $\sigma_1$  and  $\sigma_2$ :

**THEOREM 2.** *In every domain  $A$ ,*

$$\sigma_2 = \sigma_1 + 6. \quad (7)$$

*Proof.* Let  $f$  be univalent in  $A$  and  $h : D \rightarrow A$  conformal. By (1).

$$\|S_f\|_A = \|S_{f \circ h} - S_h\|_D \leq 6 + \|S_h\|_D = 6 + \sigma_1. \quad (8)$$

In order to show that the estimate  $\|S_f\|_A \leq 6 + \sigma_1$  cannot be improved, let an  $\varepsilon > 0$  be given. Considering (4), we can choose  $h$  such that  $|S_h(0)| > \sigma_1 - \varepsilon$ . The

mapping  $w$ , defined by  $w(z) = z + e^{i\theta}/z$ , is univalent in  $D$ , and

$$S_w(z) = -6e^{i\theta}(e^{i\theta} - z)^{-2}.$$

Set  $f = w \circ h^{-1}$ . Then  $f$  is univalent in  $A$ , and

$$\|S_f\|_A = \|S_w - S_h\|_D \geq |S_w(0) - S_h(0)| = |-6e^{i\theta} - S_h(0)|.$$

By choosing  $\theta$  suitably we obtain

$$\|S_f\|_A \geq 6 + |S_h(0)| > 6 + \sigma_1 - \varepsilon.$$

Combined with (8), this yields (7).

#### 4. Constant $\sigma_3$

In the definition of  $\sigma_3$ , sup can be replaced by max. To prove this, let us suppose that  $f$  is meromorphic in  $A$  with  $\|S_f\|_A = \sigma_3$ . Let  $f_n$ ,  $n = 1, 2, \dots$ , be determined by the condition

$$S_{f_n} = r_n S_f,$$

where  $r_n < 1$  and  $r_n \rightarrow 1$  as  $n \rightarrow \infty$ . All functions  $f_n$  are univalent, and we can normalize them so that they agree with  $f$  at three fixed points of  $A$ . Then the functions  $f_n$  form a normal family, and there is a sub-sequence which converges locally uniformly in  $A$  towards a conformal mapping of  $A$ . This limit function has the same Schwarzian derivative as  $f$ , and it follows that  $f$  is univalent.

If  $A$  is a disc, then  $\sigma_3 = 2$ . This has been known for almost thirty years: the estimate  $\sigma_3 \geq 2$  follows from a theorem of Nehari [4], and examples given by Hille [3] show that  $\sigma_3 \leq 2$ .

There is an intimate connection between the constant  $\sigma_3$  and quasiconformal mappings:  $\sigma_3 > 0$  if and only if the boundary of  $A$  is a quasicircle. The sufficiency of the condition was proved by Ahlfors [1], the necessity by Gehring [2].

#### 5. Universal Teichmüller space

Suppose the domain  $A$  is bounded by a quasicircle. Let  $Q(A)$  be the Banach space consisting of all holomorphic functions of  $A$  with finite norm. We introduce the subsets

$$\Delta(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$

$\Delta_0(A) = \{\varphi = S_f \in \Delta(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$

Both sets are well defined. The set  $\Delta_0(A)$  is called the universal Teichmüller space of  $A$ .

The sets  $\Delta(A)$  and  $\Delta_0(A)$  are connected as follows:

$$\Delta_0(A) = \text{interior of } \Delta(A). \quad (9)$$

This was proved by Gehring [2] in the case where  $A$  is a disc. The same reasoning yields the result for an arbitrary domain  $A$  also.

**THEOREM 3.** *If  $f$  is univalent in  $A$  and  $\|S_f\|_A < \sigma_3$ , then  $f$  can be extended to a quasiconformal mapping of the plane.*

*Proof.* By the remark in Section 4, the closed ball  $\{\varphi \in Q(A) \mid \|\varphi\|_A \leq \sigma_3\}$  is contained in  $\Delta(A)$ . Hence, if  $\|S_f\| < \sigma_3$ , then  $S_f$  is an inner point of  $\Delta(A)$ , and the theorem follows from (9).

In the special case where  $A$  is the upper half-plane  $H$ , we write briefly  $Q$ ,  $\Delta$  and  $\Delta_0$ , without indicating the domain  $H$ .

It is not known whether every point of  $\Delta$  is in the closure of  $\Delta_0$ . We need the following much weaker result:

**LEMMA 1.** *On every sphere  $\|\varphi\| = r$  of  $Q$ ,  $2 \leq r \leq 6$ , there are points of  $\Delta - \Delta_0$  belonging to the closure of  $\Delta_0$ .*

*Proof.* Let  $f$  be a conformal mapping of  $H$  onto a rectilinear quadrilateral  $B$  with symmetry  $f(-\bar{z}) = \bar{f}(z)$ , with vertices at the points  $f(0) = 0$ ,  $f(\pm 1)$ ,  $f(\infty) < 0$ , and with the angles  $\alpha\pi$  at 0,  $1 \leq \alpha < 2$ , and  $(1 - \alpha/2 - \eta)\pi$  at  $f(\pm 1)$ , where  $\eta \geq 0$  is small. If  $\eta = 0$ , then  $f(\infty) = \infty$ , and two sides of  $B$  are half-lines parallel to the real axis.

Direct computation yields

$$\frac{f''(z)}{f'(z)} = \frac{\alpha - 1}{z} - \left(\frac{\alpha}{2} + \eta\right) \left(\frac{1}{z-1} + \frac{1}{z+1}\right).$$

Hence,

$$S_f(z) = \frac{1 - \alpha^2}{2z^2} + \frac{a}{(z-1)^2} + \frac{a}{(z+1)^2} + \frac{b}{z^2 - 1}, \quad (10)$$

where

$$a = \frac{1}{8}(\alpha + 2\eta)(4 - \alpha - 2\eta), \quad b = \frac{1}{4}(\alpha + 2\eta)(4 - 3\alpha + 2\eta). \quad (11)$$

From (10) we see that

$$4y^2 S_f(iy) = 2(\alpha^2 - 1) + o(1), \quad z = x + iy,$$

as  $y \rightarrow 0$ . Because  $\rho_H(z) = (2y)^{-1}$ , it follows that  $\|S_f\|_H \geq 2(\alpha^2 - 1)$ . In particular,  $\|S_f\|_H \rightarrow 6$  as  $\alpha \rightarrow 2$ . On the other hand, for  $\alpha = 1$  the domain  $B$  is convex, and by Theorem 1,  $\|S_f\|_H \leq 2$ .

Suppose, for a moment, that  $\eta = 0$ . From (10) and (11) we deduce, since  $y^2 |z \pm 1|^{-2} \leq 1$ ,  $y^2 |z^2 - 1|^{-1} \leq 1$ , that  $\|S_f\|_H$  depends continuously on  $\alpha$ . Therefore, given any  $r$ ,  $2 \leq r \leq 6$ , there is a quadrilateral  $B$  such that  $\|S_f\|_H = r$ . The boundary of  $B$ , having a cusp at  $\infty$ , is not a quasicircle and so  $S_f \in \Delta - \Delta_0$ .

On the other hand, for  $\eta > 0$  the domain  $B$  is bounded by a quasicircle, and hence  $S_f \in \Delta_0$ . If we write  $f = f_{\alpha, \eta}$ , then it is again immediate from (10) and (11) that for every  $\alpha$ ,

$$\lim_{\eta \rightarrow 0} \|S_{f_{\alpha, \eta}} - S_{f_{\alpha, 0}}\|_H = 0.$$

Consequently, the Schwarzian of  $f_{\alpha, 0}$  is in the closure of  $\Delta_0$ .

## 6. New characterization of $\sigma_3$

Given a domain  $A$  bounded by a quasicircle, let  $f: H \rightarrow A$  be conformal. We let the point  $\varphi_A = S_f$  represent  $A$  in  $\Delta_0$ . Then  $\|\varphi_A\| = \sigma_1$ .

The point  $\varphi_A \in \Delta_0$  is not uniquely determined by  $A$ , nor does a point of  $\Delta_0$  determine a unique domain. We obtain a well defined bijection by identifying two domains if they are equivalent under Möbius transformations, and two points  $S_f$  and  $S_g$  of  $\Delta_0$  if  $g^{-1} \circ f$  is a conformal self-mapping of  $H$ . For our purposes, the choice of the representative of  $A$  in  $\Delta_0$  is immaterial.

**THEOREM 4.** *The constant  $\sigma_3$  of  $A$  is equal to the distance of  $\varphi_A$  to the set  $\Delta - \Delta_0$ .*

*Proof.* Let  $d$  denote the distance between  $\Delta - \Delta_0$  and the point  $\varphi_A = S_h$ , where  $h$  is a conformal map of  $H$  onto  $A$ . Let  $f$  be meromorphic in  $A$ . From

$$\|S_f\|_A = \|S_{f \circ h} - S_h\|_H$$

we see that if  $\|S_f\|_A < d$ , then  $S_{f \circ h} \in \Delta_0$ . But then  $f = (f \circ h) \circ h^{-1}$  is univalent, and consequently  $\sigma_3 \geq d$ .

On the other hand, it follows from Theorem 3 that  $\sigma_3 \leq d$ .

## 7. Estimates for $\sigma_3$

Theorem 4 and Lemma 1 give sharp lower estimates for  $\sigma_3$  if  $\sigma_1$  is given.

**THEOREM 5.** *For domains with given  $\sigma_1$  and bounded by quasicircles,*

$$\min \sigma_3 = 2 - \sigma_1 \quad \text{if } 0 \leq \sigma_1 < 2, \quad (12)$$

$$\inf \sigma_3 = 0 \quad \text{if } 2 \leq \sigma_1 < 6. \quad (13)$$

*Proof.* Suppose first that  $\sigma_1 < 2$ . Since the ball  $\{\varphi \in Q \mid \|\varphi\|_H < 2\}$  lies in  $\Delta_0$ , Theorem 4 yields the lower estimate  $\sigma_3 \geq 2 - \sigma_1$ .

In order to prove that this inequality is sharp, we consider the point  $S_w$ , where  $w$  is the restriction to  $H$  of a branch of the logarithm. Then  $S_w \in \Delta - \Delta_0$  and  $\|S_w\|_H = 2$ . Let  $h$  be determined by the condition  $S_h = rS_w$ ,  $r < 1$ , and set  $A = h(H)$ ,  $f = w \circ h^{-1}$ . From  $\|S_h\|_H < 2$  it follows that  $S_h \in \Delta_0$ , and so  $A$  is bounded by a quasicircle. Furthermore,  $\sigma_1 = \|S_h\|_H = 2r$ , and

$$\|S_f\|_A = \|S_f - S_w\|_H = 2(1 - r) = 2 - \sigma_1.$$

From  $S_w \in \Delta - \Delta_0$  we conclude that  $S_f \in \Delta(A) - \Delta_0(A)$ . Consequently, by Theorem 3,  $\sigma_3 \leq 2 - \sigma_1$ , and (12) follows.

Since  $\sigma_3 = 0$  for a domain not bounded by a quasicircle, equation (13) follows immediately from Theorem 4 and Lemma 1.

The following upper estimates complement Theorem 5.

**THEOREM 6.** *The constant  $\sigma_3$  satisfies the inequality*

$$\sigma_3 \leq \min(2, 6 - \sigma_1).$$

*Proof.* Since  $\Delta$  is contained in the ball of radius 6, the estimate  $\sigma_3 \leq 6 - \sigma_1$  follows immediately from Theorem 4.

In order to prove that

$$\sigma_3 \leq 2 \quad (14)$$

we note that every Jordan domain is Möbius equivalent to a subdomain of  $H$  having 0 and  $\infty$  as boundary points. Therefore, we may assume that  $A$  is such a domain.

Set  $f(z) = \log z$ . From  $S_f(z) = z^{-2}/2$  and  $\rho_A(z) \geq \rho_H(z)$  it follows that

$$|S_f(z)| \rho_A(z)^{-2} \leq 2 \left( \frac{y}{|z|} \right)^2 \leq 2. \quad \bullet$$

Since the boundary of  $f(A)$  is not a Jordan curve,  $S_f \in \Delta(A) - \Delta_0(A)$ . Thus (14) follows from Theorem 3.

## REFERENCES

- [1] AHLFORS, L. V., *Quasiconformal reflections*. Acta Math. 109 (1963), 291–301.
- [2] GEHRING, F. W., *Univalent functions and the Schwarzian derivatives*. Comment. Math. Helv. 52 (1977), fasc. 4.
- [3] HILLE, E., *Remarks on a paper by Zeev Nehari*. Bull. Amer. Math. Soc. 55 (1949), 552–553.
- [4] NEHARI Z., *The Schwarzian derivative and schlicht functions*. Bull. Amer. Math. Soc. 55 (1949), 545–551.
- [5] PFLUGER, A., *Über die Konstruktion Riemannscher Flächen durch Verheftung*. J. Indian Math. Soc. 24 (1961), 401–412.
- [6] TRIMBLE, S. 'Y., *A coefficient inequality for convex univalent functions*. Proc. Amer. Math. Soc. 48 (1975), 266–267.

Received April 18, 1977