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**Autor:** Lehto, Olli  
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## Domain constants associated with Schwarzian derivative

OLLI LEHTO

Dedicated to Professor Albert Pfluger on his seventieth birthday

### 1. Definition of the constants

Let  $A$  be a simply connected domain in the extended plane with more than one boundary point. A non-euclidean metric  $\rho(z)|dz|$  of  $A$  is defined by the condition  $\rho(z)|dz| = (1 - |w|^2)^{-1}|dw|$ , where  $z \rightarrow w$  is a conformal mapping of  $A$  onto the unit disc  $D$ . For a function  $\varphi$  holomorphic in  $A$  we introduce the norm

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)| \rho(z)^{-2}.$$

Let  $f$  be a locally injective meromorphic function in  $A$  and  $S_f$  its Schwarzian derivative. At finite points of  $A$  which are not poles of  $f$  we have  $S_f = (f''/f)' - \frac{1}{2}(f''/f)^2$ , and the definition is extended to  $\infty$  and to the poles of  $f$  by means of inversion. Every function which is holomorphic in  $A$  is the Schwarzian of some meromorphic  $f$ . The Schwarzian vanishes identically if and only if  $f$  is a Möbius transformation. A function with a prescribed Schwarzian is determined up to a Möbius transformation.

If  $g : A \rightarrow B$  is a conformal mapping, then

$$|S_f(z) - S_g(z)| \rho_A(z)^{-2} = |S_{f \circ g^{-1}}(\zeta)| \rho_B(\zeta)^{-2}, \quad \zeta = g(z). \quad (1)$$

In particular,

$$\|S_g\|_A = \|S_{g^{-1}}\|_B. \quad (2)$$

We associate with the domain  $A$  the following three constants:

$\sigma_1 = \|S_f\|_A$ , where  $f$  is a conformal map of  $A$  onto a disc,

$\sigma_2 = \sup \{\|S_f\|_A \mid f \text{ univalent in } A\}$ ,

$\sigma_3 = \sup \{a \mid \|S_f\|_A \leq a \text{ implies } f \text{ univalent in } A\}$ .

## 2. Constant $\sigma_1$

In the definition of  $\sigma_1$ , a disc means an ordinary disc or a half-plane. The number  $\sigma_1$  is well defined and equal to 0 if and only if  $A$  itself is a disc. It is well known that  $\sigma_1 \leq 6$ , and the example  $A = \{z \mid 0 < \arg z < k\pi\}$ ,  $1 \leq k \leq 2$ , shows that  $\sigma_1$  can take any value of the closed interval  $[0, 6]$ .

In view of (2), we could define  $\sigma_1$  also with the aid of conformal mappings of a disc onto  $A$ . A further characterization is obtained as follows: Let  $f$  be a conformal mapping of the unit disc  $D$  onto  $A$  and  $h$  a conformal self-mapping of  $D$ , such that  $h(0) = z_0$ . Since  $\rho_D(0) = 1$ , it follows from (1) that

$$|S_f(z_0)| \rho_D(z_0)^{-2} = |S_{f \circ h}(0)|. \quad (3)$$

Hence

$$\sigma_1 = \sup \{|S_f(0)| \mid f: D \rightarrow A \text{ conformal}\}. \quad (4)$$

In some cases, information about the boundary of  $A$  makes it possible to improve the estimate  $\sigma_1 \leq 6$ . Suppose that the boundary of  $A$  is a  $K$ -quasicircle, i.e. the image of a circle under a  $K$ -quasiconformal mapping of the plane. (Quasicircles were first investigated by Pfluger [5].) Then  $\sigma_1 \leq 6(K^2 - 1)/(K^2 + 1)$ .

Another result of this type is that for convex domains  $\sigma_1 \leq 2$ . This follows from known results on the coefficients of univalent functions, see e.g. [6]. We include here a simple proof which also gives the extremals. (Quite recently, Nehari (J. Analyse Math. 30 (1976)) also established this result by using variational techniques.)

**THEOREM 1.** *Let  $f$  be a conformal mapping of a disc onto a convex domain. Then*

$$|S_f(z)| \rho(z)^{-2} \leq 2. \quad (5)$$

*Equality holds if and only if the image domain is bounded by two parallel lines.*

*Proof.* We may assume that  $f$  is a conformal map of the unit disc. In view of (3), inequality (5) follows if we prove that  $|S_f(0)| \leq 2$ . Since we may replace  $f$  by the function  $z \rightarrow cf(ze^{i\varphi})$ ,  $c$  complex,  $\varphi$  real, there is no loss of generality in assuming that  $S_f(0) \geq 0$  and that  $f'(0) = 1$ .

It is well known that  $f'$  admits a representation

$$f'(z) = \exp \left( - \int_0^{2\pi} (\log(1 - ze^{-i\theta}) d\psi(\theta)) \right), \quad (6)$$

where  $\psi$  is increasing and

$$\int_0^{2\pi} d\psi(\theta) = 2.$$

Direct computation yields

$$S_f(0) = \int_0^{2\pi} e^{-2i\theta} d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} e^{-i\theta} d\psi(\theta) \right)^2.$$

Since  $S_f(0)$  is real and  $d\psi(\theta) \geq 0$ , we obtain

$$\begin{aligned} S_f(0) &= \int_0^{2\pi} \cos 2\theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 + \frac{1}{2} \left( \int_0^{2\pi} \sin \theta d\psi(\theta) \right)^2 \\ &\leq \int_0^{2\pi} \cos 2\theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 + \int_0^{2\pi} \sin^2 \theta d\psi(\theta) \\ &= \int_0^{2\pi} \cos^2 \theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 \leq \int_0^{2\pi} \cos^2 \theta d\psi(\theta) \leq 2. \end{aligned}$$

Because  $S_f(0) \geq 0$ , we have proved (5).

Equality holds only if

$$\int_0^{2\pi} \cos^2 \theta d\psi(\theta) = 2, \quad \int_0^{2\pi} \cos \theta d\psi(\theta) = 0.$$

These conditions are fulfilled if and only if  $\psi$  has a jump  $+1$  at the points  $0$  and  $\pi$  and is constant on the intervals  $(0, \pi)$  and  $(\pi, 2\pi)$ . Then  $S_f(0) = 2$ , and it follows from (6) that  $f'(z) = (1 - z^2)^{-1}$ . We conclude that the image of  $D$  is a parallel strip.

### 3. Constant $\sigma_2$

The number  $\sigma_2$  is 6 if  $A$  is a disc and  $\sigma_2 \leq 12$  for all domains  $A$ . In fact, there is a simple relation between  $\sigma_1$  and  $\sigma_2$ :

**THEOREM 2.** *In every domain  $A$ ,*

$$\sigma_2 = \sigma_1 + 6. \quad (7)$$

*Proof.* Let  $f$  be univalent in  $A$  and  $h : D \rightarrow A$  conformal. By (1).

$$\|S_f\|_A = \|S_{f \circ h} - S_h\|_D \leq 6 + \|S_h\|_D = 6 + \sigma_1. \quad (8)$$

In order to show that the estimate  $\|S_f\|_A \leq 6 + \sigma_1$  cannot be improved, let an  $\varepsilon > 0$  be given. Considering (4), we can choose  $h$  such that  $|S_h(0)| > \sigma_1 - \varepsilon$ . The

mapping  $w$ , defined by  $w(z) = z + e^{i\theta}/z$ , is univalent in  $D$ , and

$$S_w(z) = -6e^{i\theta}(e^{i\theta} - z)^{-2}.$$

Set  $f = w \circ h^{-1}$ . Then  $f$  is univalent in  $A$ , and

$$\|S_f\|_A = \|S_w - S_h\|_D \geq |S_w(0) - S_h(0)| = |-6e^{i\theta} - S_h(0)|.$$

By choosing  $\theta$  suitably we obtain

$$\|S_f\|_A \geq 6 + |S_h(0)| > 6 + \sigma_1 - \varepsilon.$$

Combined with (8), this yields (7).

#### 4. Constant $\sigma_3$

In the definition of  $\sigma_3$ , sup can be replaced by max. To prove this, let us suppose that  $f$  is meromorphic in  $A$  with  $\|S_f\|_A = \sigma_3$ . Let  $f_n$ ,  $n = 1, 2, \dots$ , be determined by the condition

$$S_{f_n} = r_n S_f,$$

where  $r_n < 1$  and  $r_n \rightarrow 1$  as  $n \rightarrow \infty$ . All functions  $f_n$  are univalent, and we can normalize them so that they agree with  $f$  at three fixed points of  $A$ . Then the functions  $f_n$  form a normal family, and there is a sub-sequence which converges locally uniformly in  $A$  towards a conformal mapping of  $A$ . This limit function has the same Schwarzian derivative as  $f$ , and it follows that  $f$  is univalent.

If  $A$  is a disc, then  $\sigma_3 = 2$ . This has been known for almost thirty years: the estimate  $\sigma_3 \geq 2$  follows from a theorem of Nehari [4], and examples given by Hille [3] show that  $\sigma_3 \leq 2$ .

There is an intimate connection between the constant  $\sigma_3$  and quasiconformal mappings:  $\sigma_3 > 0$  if and only if the boundary of  $A$  is a quasicircle. The sufficiency of the condition was proved by Ahlfors [1], the necessity by Gehring [2].

#### 5. Universal Teichmüller space

Suppose the domain  $A$  is bounded by a quasicircle. Let  $Q(A)$  be the Banach space consisting of all holomorphic functions of  $A$  with finite norm. We introduce the subsets

$$\Delta(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$

$\Delta_0(A) = \{\varphi = S_f \in \Delta(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$

Both sets are well defined. The set  $\Delta_0(A)$  is called the universal Teichmüller space of  $A$ .

The sets  $\Delta(A)$  and  $\Delta_0(A)$  are connected as follows:

$$\Delta_0(A) = \text{interior of } \Delta(A). \quad (9)$$

This was proved by Gehring [2] in the case where  $A$  is a disc. The same reasoning yields the result for an arbitrary domain  $A$  also.

**THEOREM 3.** *If  $f$  is univalent in  $A$  and  $\|S_f\|_A < \sigma_3$ , then  $f$  can be extended to a quasiconformal mapping of the plane.*

*Proof.* By the remark in Section 4, the closed ball  $\{\varphi \in Q(A) \mid \|\varphi\|_A \leq \sigma_3\}$  is contained in  $\Delta(A)$ . Hence, if  $\|S_f\| < \sigma_3$ , then  $S_f$  is an inner point of  $\Delta(A)$ , and the theorem follows from (9).

In the special case where  $A$  is the upper half-plane  $H$ , we write briefly  $Q$ ,  $\Delta$  and  $\Delta_0$ , without indicating the domain  $H$ .

It is not known whether every point of  $\Delta$  is in the closure of  $\Delta_0$ . We need the following much weaker result:

**LEMMA 1.** *On every sphere  $\|\varphi\| = r$  of  $Q$ ,  $2 \leq r \leq 6$ , there are points of  $\Delta - \Delta_0$  belonging to the closure of  $\Delta_0$ .*

*Proof.* Let  $f$  be a conformal mapping of  $H$  onto a rectilinear quadrilateral  $B$  with symmetry  $f(-\bar{z}) = \bar{f}(z)$ , with vertices at the points  $f(0) = 0$ ,  $f(\pm 1)$ ,  $f(\infty) < 0$ , and with the angles  $\alpha\pi$  at 0,  $1 \leq \alpha < 2$ , and  $(1 - \alpha/2 - \eta)\pi$  at  $f(\pm 1)$ , where  $\eta \geq 0$  is small. If  $\eta = 0$ , then  $f(\infty) = \infty$ , and two sides of  $B$  are half-lines parallel to the real axis.

Direct computation yields

$$\frac{f''(z)}{f'(z)} = \frac{\alpha - 1}{z} - \left(\frac{\alpha}{2} + \eta\right) \left(\frac{1}{z-1} + \frac{1}{z+1}\right).$$

Hence,

$$S_f(z) = \frac{1 - \alpha^2}{2z^2} + \frac{a}{(z-1)^2} + \frac{a}{(z+1)^2} + \frac{b}{z^2 - 1}, \quad (10)$$

where

$$a = \frac{1}{8}(\alpha + 2\eta)(4 - \alpha - 2\eta), \quad b = \frac{1}{4}(\alpha + 2\eta)(4 - 3\alpha + 2\eta). \quad (11)$$

From (10) we see that

$$4y^2 S_f(iy) = 2(\alpha^2 - 1) + o(1), \quad z = x + iy,$$

as  $y \rightarrow 0$ . Because  $\rho_H(z) = (2y)^{-1}$ , it follows that  $\|S_f\|_H \geq 2(\alpha^2 - 1)$ . In particular,  $\|S_f\|_H \rightarrow 6$  as  $\alpha \rightarrow 2$ . On the other hand, for  $\alpha = 1$  the domain  $B$  is convex, and by Theorem 1,  $\|S_f\|_H \leq 2$ .

Suppose, for a moment, that  $\eta = 0$ . From (10) and (11) we deduce, since  $y^2 |z \pm 1|^{-2} \leq 1$ ,  $y^2 |z^2 - 1|^{-1} \leq 1$ , that  $\|S_f\|_H$  depends continuously on  $\alpha$ . Therefore, given any  $r$ ,  $2 \leq r \leq 6$ , there is a quadrilateral  $B$  such that  $\|S_f\|_H = r$ . The boundary of  $B$ , having a cusp at  $\infty$ , is not a quasicircle and so  $S_f \in \Delta - \Delta_0$ .

On the other hand, for  $\eta > 0$  the domain  $B$  is bounded by a quasicircle, and hence  $S_f \in \Delta_0$ . If we write  $f = f_{\alpha, \eta}$ , then it is again immediate from (10) and (11) that for every  $\alpha$ ,

$$\lim_{\eta \rightarrow 0} \|S_{f_{\alpha, \eta}} - S_{f_{\alpha, 0}}\|_H = 0.$$

Consequently, the Schwarzian of  $f_{\alpha, 0}$  is in the closure of  $\Delta_0$ .

## 6. New characterization of $\sigma_3$

Given a domain  $A$  bounded by a quasicircle, let  $f: H \rightarrow A$  be conformal. We let the point  $\varphi_A = S_f$  represent  $A$  in  $\Delta_0$ . Then  $\|\varphi_A\| = \sigma_1$ .

The point  $\varphi_A \in \Delta_0$  is not uniquely determined by  $A$ , nor does a point of  $\Delta_0$  determine a unique domain. We obtain a well defined bijection by identifying two domains if they are equivalent under Möbius transformations, and two points  $S_f$  and  $S_g$  of  $\Delta_0$  if  $g^{-1} \circ f$  is a conformal self-mapping of  $H$ . For our purposes, the choice of the representative of  $A$  in  $\Delta_0$  is immaterial.

**THEOREM 4.** *The constant  $\sigma_3$  of  $A$  is equal to the distance of  $\varphi_A$  to the set  $\Delta - \Delta_0$ .*

*Proof.* Let  $d$  denote the distance between  $\Delta - \Delta_0$  and the point  $\varphi_A = S_h$ , where  $h$  is a conformal map of  $H$  onto  $A$ . Let  $f$  be meromorphic in  $A$ . From

$$\|S_f\|_A = \|S_{f \circ h} - S_h\|_H$$

we see that if  $\|S_f\|_A < d$ , then  $S_{f \circ h} \in \Delta_0$ . But then  $f = (f \circ h) \circ h^{-1}$  is univalent, and consequently  $\sigma_3 \geq d$ .

On the other hand, it follows from Theorem 3 that  $\sigma_3 \leq d$ .

## 7. Estimates for $\sigma_3$

Theorem 4 and Lemma 1 give sharp lower estimates for  $\sigma_3$  if  $\sigma_1$  is given.

**THEOREM 5.** *For domains with given  $\sigma_1$  and bounded by quasicircles,*

$$\min \sigma_3 = 2 - \sigma_1 \quad \text{if} \quad 0 \leq \sigma_1 < 2, \quad (12)$$

$$\inf \sigma_3 = 0 \quad \text{if} \quad 2 \leq \sigma_1 < 6. \quad (13)$$

*Proof.* Suppose first that  $\sigma_1 < 2$ . Since the ball  $\{\varphi \in Q \mid \|\varphi\|_H < 2\}$  lies in  $\Delta_0$ , Theorem 4 yields the lower estimate  $\sigma_3 \geq 2 - \sigma_1$ .

In order to prove that this inequality is sharp, we consider the point  $S_w$ , where  $w$  is the restriction to  $H$  of a branch of the logarithm. Then  $S_w \in \Delta - \Delta_0$  and  $\|S_w\|_H = 2$ . Let  $h$  be determined by the condition  $S_h = rS_w$ ,  $r < 1$ , and set  $A = h(H)$ ,  $f = w \circ h^{-1}$ . From  $\|S_h\|_H < 2$  it follows that  $S_h \in \Delta_0$ , and so  $A$  is bounded by a quasicircle. Furthermore,  $\sigma_1 = \|S_h\|_H = 2r$ , and

$$\|S_f\|_A = \|S_f - S_w\|_H = 2(1 - r) = 2 - \sigma_1.$$

From  $S_w \in \Delta - \Delta_0$  we conclude that  $S_f \in \Delta(A) - \Delta_0(A)$ . Consequently, by Theorem 3,  $\sigma_3 \leq 2 - \sigma_1$ , and (12) follows.

Since  $\sigma_3 = 0$  for a domain not bounded by a quasicircle, equation (13) follows immediately from Theorem 4 and Lemma 1.

The following upper estimates complement Theorem 5.

**THEOREM 6.** *The constant  $\sigma_3$  satisfies the inequality*

$$\sigma_3 \leq \min(2, 6 - \sigma_1).$$

*Proof.* Since  $\Delta$  is contained in the ball of radius 6, the estimate  $\sigma_3 \leq 6 - \sigma_1$  follows immediately from Theorem 4.

In order to prove that

$$\sigma_3 \leq 2 \quad (14)$$

we note that every Jordan domain is Möbius equivalent to a subdomain of  $H$  having 0 and  $\infty$  as boundary points. Therefore, we may assume that  $A$  is such a domain.

Set  $f(z) = \log z$ . From  $S_f(z) = z^{-2}/2$  and  $\rho_A(z) \geq \rho_H(z)$  it follows that

$$|S_f(z)| \rho_A(z)^{-2} \leq 2 \left( \frac{y}{|z|} \right)^2 \leq 2. \quad \bullet$$

Since the boundary of  $f(A)$  is not a Jordan curve,  $S_f \in \Delta(A) - \Delta_0(A)$ . Thus (14) follows from Theorem 3.



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