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Domain constants associated with Schwarzian derivative

OLLI LEHTO

Dedicated to Professor Albert Pfluger on his seventieth birthday

1. Definition of the constants

Let A be a simply connected domain in the extended plane with more than one boundary point. A non-euclidean metric $\rho(z)|dz|$ of A is defined by the condition $\rho(z)|dz| = (1-|w|^2)^{-1}|dw|$, where $z \to w$ is a conformal mapping of A onto the unit disc D. For a function φ holomorphic in A we introduce the norm

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)| \, \rho(z)^{-2}.$$

Let f be a locally injective meromorphic function in A and S_f its Schwarzian derivative. At finite points of A which are not poles of f we have $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$, and the definition is extended to ∞ and to the poles of f by means of inversion. Every function which is holomorphic in A is the Schwarzian of some meromorphic f. The Schwarzian vanishes identically if and only if f is a Möbius transformation. A function with a prescribed Schwarzian is determined up to a Möbius transformation.

If $g: A \rightarrow B$ is a conformal mapping, then

$$|S_f(z) - S_g(z)| \rho_A(z)^{-2} = |S_{f \circ g^{-1}}(\zeta)| \rho_B(\zeta)^{-2}, \qquad \zeta = g(z).$$
 (1)

In particular,

$$||S_{g}||_{A} = ||S_{g^{-1}}||_{B}.$$
 (2)

We associate with the domain A the following three constants:

 $\sigma_1 = ||S_f||_A$, where f is a conformal map of A onto a disc,

 $\sigma_2 = \sup \{ ||S_f||_A \mid f \text{ univalent in } A \},$

 $\sigma_3 = \sup \{a \mid ||S_f||_A \le a \text{ implies } f \text{ univalent in } A\}.$

2. Constant σ_1

In the definition of σ_1 , a disc means an ordinary disc or a half-plane. The number σ_1 is well defined and equal to 0 if and only if A itself is a disc. It is well known that $\sigma_1 \le 6$, and the example $A = \{z \mid 0 < \arg z < k\pi\}, 1 \le k \le 2$, shows that σ_1 can take any value of the closed interval [0, 6].

In view of (2), we could define σ_1 also with the aid of conformal mappings of a disc onto A. A further characterization is obtained as follows: Let f be a conformal mapping of the unit disc D onto A and h a conformal self-mapping of D, such that $h(0) = z_0$. Since $\rho_D(0) = 1$, it follows from (1) that

$$|S_f(z_0)| \rho_D(z_0)^{-2} = |S_{f \circ h}(0)|. \tag{3}$$

Hence

$$\sigma_1 = \sup \{ |S_f(0)| \mid f: D \to A \text{ conformal} \}. \tag{4}$$

In some cases, information about the boundary of A makes it possible to improve the estimate $\sigma_1 \le 6$. Suppose that the boundary of A is a K-quasicircle, i.e. the image of a circle under a K-quasiconformal mapping of the plane. (Quasicircles were first investigated by Pfluger [5].) Then $\sigma_1 \le 6(K^2 - 1)/(K^2 + 1)$.

Another result of this type is that for convex domains $\sigma_1 \le 2$. This follows from known results on the coefficients of univalent functions, see e.g. [6]. We include here a simple proof which also gives the extremals. (Quite recently, Nehari (J. Analyse Math. 30 (1976)) also established this result by using variational techniques.)

THEOREM 1. Let f be a conformal mapping of a disc onto a convex domain. Then

$$|S_f(z)| \rho(z)^{-2} \le 2.$$
 (5)

Equality holds if and only if the image domain is bounded by two parallel lines.

Proof. We may assume that f is a conformal map of the unit disc. In view of (3), inequality (5) follows if we prove that $|S_f(0)| \le 2$. Since we may replace f by the function $z \to cf(ze^{i\varphi})$, c complex, φ real, there is no loss of generality in assuming that $S_f(0) \ge 0$ and that f'(0) = 1.

It is well known that f' admits a representation

$$f'(z) = \exp\left(-\int_0^{2\pi} (\log\left(1 - ze^{-i\theta}\right) d\psi(\theta))\right),\tag{6}$$

where ψ is increasing and

$$\int_0^{2\pi} d\psi(\theta) = 2.$$

Direct computation yields

$$S_f(0) = \int_0^{2\pi} e^{-2i\theta} d\psi(\theta) - \frac{1}{2} \left(\int_0^{2\pi} e^{-i\theta} d\psi(\theta) \right)^2.$$

Since $S_f(0)$ is real and $d\psi(\theta) \ge 0$, we obtain

$$\begin{split} S_f(0) &= \int_0^{2\pi} \cos 2\theta \, d\psi(\theta) - \frac{1}{2} \left(\int_0^{2\pi} \cos \theta \, d\psi(\theta) \right)^2 + \frac{1}{2} \left(\int_0^{2\pi} \sin \theta \, d\psi(\theta) \right)^2 \\ &\leq \int_0^{2\pi} \cos 2\theta \, d\psi(\theta) - \frac{1}{2} \left(\int_0^{2\pi} \cos \theta \, d\psi(\theta) \right)^2 + \int_0^{2\pi} \sin^2 \theta \, d\psi(\theta) \\ &= \int_0^{2\pi} \cos^2 \theta \, d\psi(\theta) - \frac{1}{2} \left(\int_0^{2\pi} \cos \theta \, d\psi(\theta) \right)^2 \leq \int_0^{2\pi} \cos^2 \theta \, d\psi(\theta) \leq 2. \end{split}$$

Because $S_f(0) \ge 0$, we have proved (5).

Equality holds only if

$$\int_0^{2\pi} \cos^2\theta \ d\psi(\theta) = 2, \int_0^{2\pi} \cos \theta \ d\psi(\theta) = 0.$$

These conditions are fulfilled if and only if ψ has a jump +1 at the points 0 and π and is constant on the intervals $(0, \pi)$ and $(\pi, 2\pi)$. Then $S_f(0) = 2$, and it follows from (6) that $f'(z) = (1 - z^2)^{-1}$. We conclude that the image of D is a parallel strip.

3. Constant σ_2

The number σ_2 is 6 if A is a disc and $\sigma_2 \le 12$ for all domains A. In fact, there is a simple relation between σ_1 and σ_2 :

THEOREM 2. In every domain A,

$$\sigma_2 = \sigma_1 + 6. \tag{7}$$

Proof. Let f be univalent in A and $h: D \rightarrow A$ conformal. By (1).

$$||S_f||_A = ||S_{f \circ h} - S_h||_D \le 6 + ||S_h||_D = 6 + \sigma_1.$$
(8)

In order to show that the estimate $||S_f||_A \le 6 + \sigma_1$ cannot be improved, let an $\varepsilon > 0$ be given. Considering (4), we can choose h such that $|S_h(0)| > \sigma_1 - \varepsilon$. The

mapping w, defined by $w(z) = z + e^{i\theta}/z$, is univalent in D, and

$$S_w(z) = -6e^{i\theta}(e^{i\theta} - z)^{-2}$$
.

Set $f = w \circ h^{-1}$. Then f is univalent in A, and

$$||S_f||_A = ||S_w - S_h||_D \ge |S_w(0) - S_h(0)| = |-6e^{i\theta} - S_h(0)|.$$

By choosing θ suitably we obtain

$$||S_f||_A \ge 6 + |S_h(0)| > 6 + \sigma_1 - \varepsilon.$$

Combined with (8), this yields (7).

4. Constant σ_3

In the definition of σ_3 , sup can be replaced by max. To prove this, let us suppose that f is meromorphic in A with $||S_f||_A = \sigma_3$. Let f_n , $n = 1, 2, \ldots$, be determined by the condition

$$S_{f_n}=r_nS_f,$$

where $r_n < 1$ and $r_n \to 1$ as $n \to \infty$. All functions f_n are univalent, and we can normalize them so that they agree with f at three fixed points of A. Then the functions f_n form a normal family, and there is a sub-sequence which converges locally uniformly in A towards a conformal mapping of A. This limit function has the same Schwarzian derivative as f, and it follows that f is univalent.

If A is a disc, then $\sigma_3 = 2$. This has been known for almost thirty years: the estimate $\sigma_3 \ge 2$ follows from a theorem of Nehari [4], and examples given by Hille [3] show that $\sigma_3 \le 2$.

There is an intimate connection between the constant σ_3 and quasiconformal mappings: $\sigma_3 > 0$ if and only if the boundary of A is a quasicircle. The sufficiency of the condition was proved by Ahlfors [1], the necessity by Gehring [2].

5. Universal Teichmüller space

Suppose the domain A is bounded by a quasicircle. Let Q(A) be the Banach space consisting of all holomorphic functions of A with finite norm. We introduce the subsets

$$\Delta(A) = \{ \varphi = S_f \mid f \text{ univalent in } A \},$$

 $\Delta_0(A) = \{ \varphi = S_f \in \Delta(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane} \}.$

Both sets are well defined. The set $\Delta_0(A)$ is called the universal Teichmüller space of A.

The sets $\Delta(A)$ and $\Delta_0(A)$ are connected as follows:

$$\Delta_0(A) = \text{interior of } \Delta(A).$$
 (9)

This was proved by Gehring [2] in the case where A is a disc. The same reasoning yields the result for an arbitrary domain A also.

THEOREM 3. If f is univalent in A and $||S_f||_A < \sigma_3$, then f can be extended to a quasiconformal mapping of the plane.

Proof. By the remark in Section 4, the closed ball $\{\varphi \in Q(A) \mid ||\varphi||_A \leq \sigma_3\}$ is contained in $\Delta(A)$. Hence, if $||S_f|| < \sigma_3$, then S_f is an inner point of $\Delta(A)$, and the theorem follows from (9).

In the special case where A is the upper half-plane H, we write briefly Q, Δ and Δ_0 , without indicating the domain H.

It is not known whether every point of Δ is in the closure of Δ_0 . We need the following much weaker result:

LEMMA 1. On every sphere $\|\varphi\| = r$ of Q, $2 \le r \le 6$, there are points of $\Delta - \Delta_0$ belonging to the closure of Δ_0 .

Proof. Let f be a conformal mapping of H onto a rectilinear quadrilateral B with symmetry $f(-\bar{z}) = \bar{f}(z)$, with vertices at the points f(0) = 0, $f(\pm 1)$, $f(\infty) < 0$, and with the angles $\alpha \pi$ at 0, $1 \le \alpha < 2$, and $(1 - \alpha/2 - \eta)\pi$ at $f(\pm 1)$, where $\eta \ge 0$ is small. If $\eta = 0$, then $f(\infty) = \infty$, and two sides of B are half-lines parallel to the real axis.

Direct computation yields

$$\frac{f''(z)}{f'(z)} = \frac{\alpha - 1}{z} - \left(\frac{\alpha}{2} + \eta\right) \left(\frac{1}{z - 1} + \frac{1}{z - 1}\right).$$

Hence,

$$S_f(z) = \frac{1 - \alpha^2}{2z^2} + \frac{a}{(z - 1)^2} + \frac{a}{(z + 1)^2} + \frac{b}{z^2 - 1},\tag{10}$$

where

$$a = \frac{1}{8}(\alpha + 2\eta)(4 - \alpha - 2\eta), \qquad b = \frac{1}{4}(\alpha + 2\eta)(4 - 3\alpha + 2\eta). \tag{11}$$

From (10) we see that

$$4y^2S_f(iy) = 2(\alpha^2 - 1) + o(1), \qquad z = x + iy,$$

as $y \to 0$. Because $\rho_H(z) = (2y)^{-1}$, it follows that $||S_f||_H \ge 2(\alpha^2 - 1)$. In particular, $||S_f||_H \to 6$ as $\alpha \to 2$. On the other hand, for $\alpha = 1$ the domain B is convex, and by Theorem 1, $||S_f||_H \le 2$.

Suppose, for a moment, that $\eta = 0$. From (10) and (11) we deduce, since $y^2 |z \pm 1|^{-2} \le 1$, $y^2 |z^2 - 1|^{-1} \le 1$, that $||S_f||_H$ depends continuously on α . Therefore, given any $r, 2 \le r \le 6$, there is a quadrilateral B such that $||S_f||_H = r$. The boundary of B, having a cusp at ∞ , is not a quasicircle and so $S_f \in \Delta - \Delta_0$.

On the other hand, for $\eta > 0$ the domain B is bounded by a quasicircle, and hence $S_f \in \Delta_0$. If we write $f = f_{\alpha,\eta}$, then it is again immediate from (10) and (11) that for every α ,

$$\lim_{n\to 0} \|S_{f_{\alpha,n}} - S_{f_{\alpha,0}}\|_{H} = 0.$$

Consequently, the Schwarzian of $f_{\alpha,0}$ is in the closure of Δ_0 .

6. New characterization of σ_3

Given a domain A bounded by a quasicircle, let $f: H \to A$ be conformal. We let the point $\varphi_A = S_f$ represent A in Δ_0 . Then $\|\varphi_A\| = \sigma_1$.

The point $\varphi_A \in \Delta_0$ is not uniquely determined by A, nor does a point of Δ_0 determine a unique domain. We obtain a well defined bijection by identifying two domains if they are equivalent under Möbius transformations, and two points S_f and S_g of Δ_0 if g^{-1} of is a conformal self-mapping of H. For our purposes, the choice of the representative of A in Δ_0 is immaterial.

THEOREM 4. The constant σ_3 of A is equal to the distance of φ_A to the set $\Delta - \Delta_0$.

Proof. Let d denote the distance between $\Delta - \Delta_0$ and the point $\varphi_A = S_h$, where h is a conformal map of H onto A. Let f be meromorphic in A. From

$$||S_f||_A = ||S_{f \circ h} - S_h||_H$$

we see that if $||S_f||_A < d$, then $S_{f \circ h} \in \Delta_0$. But then $f = (f \circ h) \circ h^{-1}$ is univalent, and consequently $\sigma_3 \ge d$.

On the other hand, it follows from Theorem 3 that $\sigma_3 \le d$.

7. Estimates for σ_3

Theorem 4 and Lemma 1 give sharp lower estimates for σ_3 if σ_1 is given.

THEOREM 5. For domains with given σ_1 and bounded by quasicircles,

$$\min \sigma_3 = 2 - \sigma_1 \quad \text{if} \quad 0 \le \sigma_1 < 2, \tag{12}$$

$$\inf \sigma_3 = 0 \quad \text{if} \quad 2 \le \sigma_1 < 6. \tag{13}$$

Proof. Suppose first that $\sigma_1 < 2$. Since the ball $\{ \varphi \in Q \mid ||\varphi||_H < 2 \}$ lies in Δ_0 , Theorem 4 yields the lower estimate $\sigma_3 \ge 2 - \sigma_1$.

In order to prove that this inequality is sharp, we consider the point S_w , where w is the restriction to H of a branch of the logarithm. Then $S_w \in \Delta - \Delta_0$ and $||S_w||_H = 2$. Let h be determined by the condition $S_h = rS_w$, r < 1, and set A = h(H), $f = w \circ h^{-1}$. From $||S_h||_H < 2$ it follows that $S_h \in \Delta_0$, and so A is bounded by a quasicircle. Furthermore, $\sigma_1 = ||S_h||_H = 2r$, and

$$||S_f||_A = ||S_f - S_w||_H = 2(1-r) = 2-\sigma_1.$$

From $S_w \in \Delta - \Delta_0$ we conclude that $S_f \in \Delta(A) - \Delta_0(A)$. Consequently, by Theorem 3, $\sigma_3 \leq 2 - \sigma_1$, and (12) follows.

Since $\sigma_3 = 0$ for a domain not bounded by a quasicircle, equation (13) follows immediately from Theorem 4 and Lemma 1.

The following upper estimates complement Theorem 5.

THEOREM 6. The constant σ_3 satisfies the inequality

$$\sigma_3 \leq \min{(2, 6 - \sigma_1)}.$$

Proof. Since Δ is contained in the ball of radius 6, the estimate $\sigma_3 \leq 6 - \sigma_1$ follows immediately from Theorem 4.

In order to prove that

$$\sigma_3 \leq 2 \tag{14}$$

we note that every Jordan domain is Möbius equivalent to a subdomain of H having 0 and ∞ as boundary points. Therefore, we may assume that A is such a domain.

Set $f(z) = \log z$. From $S_f(z) = z^{-2}/2$ and $\rho_A(z) \ge \rho_H(z)$ it follows that

$$|S_f(z)| \rho_A(z)^{-2} \le 2\left(\frac{y}{|z|}\right)^2 \le 2.$$

Since the boundary of f(A) is not a Jordan curve, $S_f \in \Delta(A) - \Delta_0(A)$. Thus (14) follows from Theorem 3.

REFERENCES

- [1] AHLFORS, L. V., Quasiconformal reflections. Acta Math. 109 (1963), 291-301.
- [2] GEHRING, F. W., Univalent functions and the Schwarzian derivatives. Comment. Math. Helv. 52 (1977), fasc. 4.
- [3] HILLE, E., Remarks on a paper by Zeev Nehari. Bull. Amer. Math. Soc. 55 (1949), 552-553.
- [4] NEHARI Z., The Schwarzian derivative and schlicht functions. Bull. Amer. Math. Soc. 55 (1949), 545-551.
- [5] PFLUGER, A., Über die Konstruktion Riemannscher Flächen durch Verheftung. J. Indian Math. Soc. 24 (1961), 401-412.
- [6] TRIMBLE, S. Y., A coefficient inequality for convex univalent functions. Proc. Amer. Math. Soc. 48 (1975), 266-267.

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