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Univalent functions and the Schwarzian derivative

F. W. GEHRING⁽¹⁾

Dedicated to Professor A. Pfluger on his seventieth birthday

1. Introduction

This paper is concerned with the problem of extending to an arbitrary simply connected plane domain D the following two well known results relating the univalence of a function f analytic in the unit disk B with the magnitude of its Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2.$$

THEOREM 1. *If f is analytic and univalent in B , then*

$$|S_f(z)| \leq 6(1 - |z|^2)^{-2}$$

in B . The constant 6 is sharp.

THEOREM 2. *If f is analytic with*

$$|S_f(z)| \leq 2(1 - |z|^2)^{-2}$$

in B , then f is univalent in B . The constant 2 is best possible.

Theorem 1 is due to Kraus [7] and Theorem 2 to Nehari [10].

Suppose next that D is a simply connected proper subdomain of the finite

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complex plane \mathbf{C} . Then the hyperbolic metric in D is given by

$$\rho_D(z) = \frac{|g'(z)|}{1 - |g(z)|^2},$$

where g is any conformal mapping of D onto B . The inequality

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leq \rho_D(z) \leq \operatorname{dist}(z, \partial D)^{-1} \quad (1)$$

follows immediately from well known results due to Koebe and Schwarz. (See, for example, page 22 in [12].)

A Jordan curve γ in the extended complex plane $\bar{\mathbf{C}}$ is said to be a K -quasiconformal circle, $1 \leq K < \infty$, if there exists a K -quasiconformal mapping f of $\bar{\mathbf{C}}$ onto $\bar{\mathbf{C}}$ which maps the unit circle onto γ . The curve γ is said to be a quasiconformal circle if it is a K -quasiconformal circle for some K .

The following analogues of Theorems 1 and 2 for simply connected subdomains D of \mathbf{C} are due to Lehto [8] and Ahlfors [1], respectively. See also [3].

THEOREM 3. *If f is analytic and univalent in D , then*

$$|S_f(z)| \leq 12\rho_D(z)^2$$

in D . The constant 12 is sharp.

THEOREM 4. *Suppose that ∂D is a K -quasiconformal circle. Then there exists a positive constant a which depends only on K such that f is univalent in D whenever f is analytic with*

$$|S_f(z)| \leq a\rho_D(z)^2 \quad (2)$$

in D .

Remark. Ahlfors actually proved more than the conclusion given above, namely that one can choose $a = a(K)$ so that f has a quasiconformal extension to $\bar{\mathbf{C}}$ whenever f is analytic and satisfies (2) in D .

In view of the above remark, it is natural to ask if the hypothesis that ∂D be a quasiconformal circle is necessary in Theorem 4. We shall show that this is indeed the case by establishing the following result.

THEOREM 5. *Suppose there exists a positive constant a such that f is*

univalent in D whenever f is analytic with

$$|S_f(z)| \leq a \rho_D(z)^2$$

in D . Then ∂D is a K -quasiconformal circle where K depends only on a .

2. Schwarzian univalence criterion

We obtain Theorem 5 as a corollary of an analogous result for proper subdomains D of \mathbb{C} with arbitrary connectivity. For such domains D we have the following consequence of Theorem 1.

COROLLARY 1. *If f is analytic and univalent in D , then*

$$|S_f(z)| \leq 6 \operatorname{dist}(z, \partial D)^{-2} \quad (3)$$

in D . The constant 6 is best possible.

Proof. Fix $z_0 \in D$, choose r so that $0 < r < \operatorname{dist}(z_0, \partial D)$ and let $g(z) = f(rz + z_0)$. Then g is analytic and univalent in B ,

$$|S_f(z_0)| = |S_g(0)| r^{-2} \leq 6 r^{-2}$$

by Theorem 1, and we obtain (3) for $z = z_0$ by letting $r \rightarrow \operatorname{dist}(z_0, \partial D)$. There is equality in (3) when f is the Koebe function $z(1-z)^{-2}$, $D = B$ and $z = 0$.

Corollary 1 and inequality (1) suggest that $\operatorname{dist}(z, \partial D)^{-1}$ is a reasonable substitute for the hyperbolic metric $\rho_D(z)$ in the case where D is multiply connected.

DEFINITION. *Suppose that D is an arbitrary proper subdomain of \mathbb{C} . We say that D satisfies the Schwarzian univalence criterion if there exists a positive constant a such that f is univalent in D whenever f is analytic with*

$$|S_f(z)| \leq a \operatorname{dist}(z, \partial D)^{-2}$$

in D .

The purpose of this paper is to establish the following result.

THEOREM 6. *If D satisfies the Schwarzian univalence criterion with constant a , then each component of ∂D is either a point or a K -quasiconformal circle where K depends only on a .*

Proof of Theorem 5. Suppose that D is a simply connected proper subdomain of \mathbf{C} which satisfies the hypotheses of Theorem 5. Then by inequality (1), D satisfies the Schwarzian univalence criterion with constant $a/16$. Since ∂D is connected and contains at least two points, Theorem 6 implies that ∂D is a K -quasiconformal circle where K depends only on a .

COROLLARY 2. *Suppose that D is a simply connected proper subdomain of \mathbf{C} . Then D satisfies the Schwarzian univalence criterion if and only if ∂D is a quasiconformal circle.*

Proof. Theorem 4 and inequality (1) imply that D satisfies the Schwarzian univalence criterion whenever ∂D is a quasiconformal circle. The converse follows from Theorem 6.

3. Proof of Theorem 6

The proof of Theorem 6 depends on five lemmas given below. In what follows we let D denote an arbitrary domain in $\bar{\mathbf{C}}$, $B(z_0, r)$ the open disk with center $z_0 \in \mathbf{C}$ and radius $r \in (0, \infty)$, and b a constant in $(1, \infty)$. Next we say that two points z_1, z_2 can be *joined* in a set $E \subset \bar{\mathbf{C}}$ if there exists an arc $\alpha \subset E$ with z_1, z_2 as its endpoints. Finally for each set $E \subset \bar{\mathbf{C}}$ we let ∂E , \bar{E} and $C(E)$ denote respectively the boundary, closure and complement of E in $\bar{\mathbf{C}}$.

LEMMA 1. *Suppose that for some z_0 and r there exist two points in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Then there exist finite points z_1, z_2 in D and w_1, w_2 in $C(D)$ such that*

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D with

$$|h(z_1) - h(z_2) - 2\pi i| \leq \frac{4}{b-1}. \quad (4)$$

Proof. By hypothesis there exist two points z'_1, z'_2 in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Let α' denote the closed segment from z'_1 to z'_2

and let $B_0 = B(z_0, br)$. Since $z'_1, z'_2 \in D$, there exists an open polygonal arc β' from z'_2 to z'_1 in D which meets α' in at most a finite set of points; when $z'_1, z'_2 \neq z_0$, we choose β' so that it lies in $D - \{z_0\}$. Then $\beta' - (\alpha' \cap \beta')$ is the union of a finite number of open subarcs β with endpoints in α' . Since z'_1, z'_2 cannot be joined in $D \cap \bar{B}_0$, we can choose a β whose endpoints cannot be joined in $D \cap \bar{B}_0$. Let z_1 and z_2 denote respectively the terminal and initial points of β , and let α denote the closed segment from z_1 to z_2 . Note that $z_1, z_2 \neq z_0$ whenever $z'_1, z'_2 \neq z_0$.

We want next to find finite points $w_1, w_2 \in C(D)$ so that the function h is analytic in D and satisfies (4). Now z_1 and z_2 are separated in \bar{B}_0 by the closed set $C(D)$. Using Theorem VI.7.1 in [11] it is easy to show that z_1 and z_2 are separated in \bar{B}_0 by a component C_0 of $C(D)$. Let $D_0 = C(C_0)$. Then D_0 is a simply connected domain by Theorem IV.3.3 in [11], $D \subset D_0$, and the points z_1, z_2 cannot be joined in $D_0 \cap \bar{B}_0$. Hence by replacing D by D_0 , we may assume without loss of generality that D is simply connected.

Now $\gamma = \alpha \cup \beta$ is a Jordan curve. Let D_1 and D_2 denote respectively the bounded and unbounded components of $C(\gamma)$. We shall show that there exist points w_1, w_2 such that

$$w_i \in C(D) \cap \partial B_0 \cap D_i \quad (5)$$

for $i = 1, 2$. Fix i . Since z_1, z_2 cannot be joined in $D \cap \bar{B}_0$, β and hence γ must meet ∂B_0 in at least two points. From Kerékjártó's theorem it follows that each component of

$$C(\gamma) \cap C(\partial B_0) = C(\gamma \cup \partial B_0)$$

is a Jordan domain, and hence that each component of $D_i \cap B_0$ is bounded by a Jordan curve. (See page 168 in [11].) Next since D_i is a Jordan domain and since $z_1 \in \partial D_i \cap B_0$, there exists a neighborhood U of z_1 such that points of $D_i \cap U$ can be joined in $D_i \cap B_0$. Hence $D_i \cap U$ is contained in a component D^* of $D_i \cap B_0$,

$$D^* \cap U = D_i \cap U, \quad (6)$$

and ∂D^* is a Jordan curve γ^* .

Choose $z \in \alpha - \{z_1\}$. Since α lies at a positive distance from ∂B_0 , we can choose an open crosscut δ of D_i from z_1 to z which lies in B_0 . Then (6) implies that $\delta \subset D^*$, that $z \in \gamma^*$, and hence that $\alpha \subset \gamma^*$. Thus $\beta^* = \gamma^* - \alpha$ is an open arc joining z_2 to z_1 in \bar{B}_0 , and there exists a point

$$w_i \in \beta^* \cap C(D). \quad (7)$$

Since

$$\gamma^* \subset \partial(D_i \cap B_0) \subset \gamma \cup (\partial B_0 \cap D_i),$$

we have

$$\beta^* \subset \beta \cup (\partial B_0 \cap D_i) \subset D \cup (\partial B_0 \cap D_i), \quad (8)$$

and (5) follows from (7) and (8).

Since D is simply connected, we can define an analytic branch of

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

in D . Then

$$\begin{aligned} h(z_1) - h(z_2) &= \int_{\beta} \frac{dz}{z - w_1} - \int_{\beta} \frac{dz}{z - w_2} \\ &= 2\pi i(n(\gamma, w_1) - n(\gamma, w_2)) - \int_{\alpha} \frac{dz}{z - w_1} + \int_{\alpha} \frac{dz}{z - w_2}, \end{aligned}$$

where $n(\gamma, w_i)$ is the winding number of γ with respect to w_i . Since D_1 is the bounded component of $C(\gamma)$,

$$n(\gamma, w_1) = n = \pm 1, \quad n(\gamma, w_2) = 0,$$

and we have

$$|h(z_1) - h(z_2) - 2n\pi i| \leq \int_{\alpha} \frac{|dz|}{|z - w_1|} + \int_{\alpha} \frac{|dz|}{|z - w_2|}. \quad (9)$$

(See [2].) Then

$$\int_{\alpha} \frac{|dz|}{|z - w_i|} \leq \frac{|z_1 - z_2|}{(b-1)r} \leq \frac{2}{b-1} \quad (10)$$

for $i = 1, 2$, and (4) follows from (9) and (10) when $n = 1$. When $n = -1$, we obtain (4) by interchanging w_1 and w_2 .

LEMMA 2. *Suppose that for some z_0 and r there exist two points in $D - B(z_0, r)$ which cannot be joined in $D - B(z_0, r/b)$. Then the conclusion of Lemma 1 again holds.*

Proof. By hypothesis there exist two points z'_1, z'_2 in $D - B(z_0, r)$ which cannot be joined in $D - B(z_0, r/b)$; we may assume without loss of generality that $z'_1, z'_2 \neq \infty$. Next let Δ and ζ'_i denote the images of D and z'_i under

$$f(z) = \frac{1}{z - z_0} + z_0.$$

Then ζ'_1, ζ'_2 are points in $\Delta \cap \bar{B}(z_0, 1/r)$ which cannot be joined in $\Delta \cap \bar{B}(z_0, b/r)$. By the argument for Lemma 1, there exist finite points

$$\zeta_1, \zeta_2 \in \Delta - \{z_0\}, \quad \omega_1, \omega_2 \in C(\Delta) \cap \partial B(z_0, b/r)$$

such that

$$g(\zeta) = \log \frac{\zeta - \omega_1}{\zeta - \omega_2}$$

is analytic in Δ with

$$|g(\zeta_1) - g(\zeta_2) - 2\pi i| \leq \frac{4}{b-1}.$$

Let z_i, w_i denote the images of ζ_i, ω_i under f^{-1} . Then

$$h(z) = g \circ f(z) + \log \frac{z_0 - w_1}{z_0 - w_2} = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4).

DEFINITION. A set E in $\bar{\mathbf{C}}$ is said to be b -locally connected if for all z_0 and r , points in $E \cap \bar{B}(z_0, r)$ can be joined in $E \cap \bar{B}(z_0, br)$ and points in $E - B(z_0, r)$ can be joined in $E - B(z_0, r/b)$.

See [5] and [6] for other applications of this concept.

LEMMA 3. Suppose that D is a proper subdomain of \mathbf{C} . If D satisfies the Schwarzian univalence criterion for some constant a , then D is b -locally connected where

$$b = \max \left(\frac{5}{a} + 1, 3 \right). \quad (11)$$

Proof. Suppose that D is not b -locally connected. Then there exist $z_0 \in \mathbb{C}$, $r \in (0, \infty)$ and two points in D for which the hypotheses of Lemma 1 or Lemma 2 hold. In either case, we obtain finite points $z_1, z_2 \in D$ and $w_1, w_2 \in C(D)$ such that

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4). Since $b \geq 3$, inequality (4) implies that

$$|h(z_1) - h(z_2)| \geq 2\pi - \frac{4}{b-1} > 4. \quad (12)$$

Now set

$$f(z) = \exp(ch(z)), \quad c = \frac{2\pi i}{h(z_1) - h(z_2)}.$$

Then f is analytic with

$$S_f(z) = \frac{1-c^2}{2} \left(\frac{1}{z-w_1} - \frac{1}{z-w_2} \right)^2$$

in D . Next (4), (11) and (12) imply that

$$2|1-c^2| < \frac{5}{b-1} \leq a,$$

and hence that

$$|S_f(z)| \leq 2|1-c^2| \operatorname{dist}(z, \partial D)^{-2} \leq a \operatorname{dist}(z, \partial D)^{-2}$$

in D . Since D satisfies the univalence criterion, it follows that f must be univalent in D . But

$$\frac{f(z_1)}{f(z_2)} = \exp(c(h(z_1) - h(z_2))) = 1,$$

and we have a contradiction.

LEMMA 4. *Suppose that D is b -locally connected and that ∂D is connected and contains at least two points. Then ∂D is a K -quasiconformal circle where K depends only on b .*

Proof. Suppose that p is a point in \bar{D} . With each neighborhood U of p we associate a second neighborhood V as follows. If $p = z_0 \in \mathbf{C}$, choose $r \in (0, \infty)$ so that $\bar{B}(z_0, br) \subset U$ and let $V = B(z_0, r)$; if $p = \infty$ choose $r \in (0, \infty)$ so that $C(B(0, r/b)) \subset U$ and let $V = C(\bar{B}(0, r))$. In each case, the fact that D is b -locally connected implies that points in $D \cap V$ can be joined in $D \cap U$. Thus D is uniformly locally connected and ∂D is a Jordan curve γ by Theorem VI.16.2 in [11].

We show next that for any pair of finite points $z_1, z_2 \in \gamma$,

$$\min(\text{dia}(\gamma_1), \text{dia}(\gamma_2)) \leq b^2 |z_1 - z_2|, \quad (13)$$

where γ_1, γ_2 denote the components of $\gamma - \{z_1, z_2\}$. By a theorem of Ahlfors, inequality (13) will then imply that γ is a K -quasiconformal circle, where K depends only on b , thus completing the proof. (See, for example, Theorem II.8.6 in [9].)

To this end fix $z_1, z_2 \in \gamma$, set

$$z_0 = \frac{1}{2}(z_1 + z_2), \quad r = \frac{1}{2}|z_1 - z_2|,$$

and suppose that (13) does not hold. Then there exist $t \in (r, \infty)$ and finite points w_1, w_2 such that

$$w_i \in \gamma_i - B(z_0, b^2 t) \quad (14)$$

for $i = 1, 2$. Choose $s \in (r, t)$. Since $z_1, z_2 \in \gamma \cap B(z_0, s)$, we can find for $i = 1, 2$ an endcut α_i of D joining z_i to $z'_i \in D$ in $\bar{B}(z_0, s)$. Next since D is b -locally connected, we can find an arc α_3 joining z'_1 to z'_2 in $D \cap \bar{B}(z_0, bs)$. Then $\alpha_1 \cup \alpha_2 \cup \alpha_3$ contains a crosscut α of D from z_1 to z_2 with

$$\alpha \subset \bar{B}(z_0, bs). \quad (15)$$

By virtue of (14), the same argument can be applied to obtain a crosscut β of D from w_1 to w_2 with

$$\beta \subset C(B(z_0, bt)). \quad (16)$$

But (15) and (16) imply that $\alpha \cap \beta = \emptyset$, contradicting the fact that z_1 and z_2 separate w_1 and w_2 in γ . Thus (13) holds and the proof of Lemma 4 is complete.

LEMMA 5. *Suppose that D is b -locally connected. Then each component of ∂D is either a point or a K -quasiconformal circle where K depends only on b .*

Proof. Let B_0 be a component of ∂D , let C_0 denote the component of $C(D)$ which contains B_0 , and let $D_0 = C(C_0)$. Then D_0 is a domain with $\partial D_0 = B_0$. (See, for example, the proof of Theorem VI.16.3 in [11].) To complete the proof we need only show that D_0 is b -locally connected. For then by Lemma 4, ∂D_0 will be a point or a K -quasiconformal circle where $K = K(b)$.

Fix $z_0 \in \mathbb{C}$ and $r \in (0, \infty)$. Given $z_1, z_2 \in D_0 \cap \bar{B}(z_0, r)$ we must find an arc γ joining these points in $D_0 \cap \bar{B}(z_0, br)$. For this let α be any arc joining z_1 and z_2 in $\bar{B}(z_0, r)$. If $\alpha \subset D_0$, we may take $\gamma = \alpha$. Suppose that $\alpha \not\subset D_0$ and for $i = 1, 2$ let α_i denote the component of $\alpha \cap D_0$ which contains z_i . Then for each i there exists a point w_i such that

$$w_i \in \alpha_i \cap D. \quad (17)$$

If $z_i \in D$, we may take $w_i = z_i$; otherwise $z_i \in C_i$, a component of $C(D)$ different from C_0 , and the fact that

$$\bar{\alpha}_i \cap C_0 \neq \emptyset, \quad \alpha_i \cap C_i \neq \emptyset$$

implies that α_i must meet D and hence contain a point w_i satisfying (17). Since D is b -locally connected and since

$$w_1, w_2 \in \alpha \cap D \subset D \cap \bar{B}(z_0, r),$$

we can join w_1 and w_2 by an arc β in $D \cap \bar{B}(z_0, br)$. Then $\alpha_1 \cup \beta \cup \alpha_2$ will contain an arc γ joining z_1 and z_2 in $D_0 \cap \bar{B}(z_0, br)$.

Next the same argument shows that each pair of points in $D_0 - B(z_0, r)$ can be joined in $D_0 - B(z_0, r/b)$. Hence D_0 is b -locally connected and the proof is complete.

Proof of Theorem 6. Suppose that D is a proper subdomain of \mathbb{C} which satisfies the Schwarzian univalence criterion with constant a . Lemma 3 implies that D is b -locally connected, where b is as in (11). Then Lemma 5 implies that each component of ∂D is either a point or a K -quasiconformal circle, where K depends only on b , and hence only on a .

4. Universal Teichmüller space

We conclude this paper with an application of Theorem 5 to Teichmüller theory.

Let $B_2 = B_2(L, 1)$ denote the Banach space of functions φ analytic in the lower

half plane L with norm

$$\|\varphi\| = \sup_{z \in L} \rho_L(z)^{-2} |\varphi(z)| < \infty,$$

where $\rho_L(z) = \frac{1}{2}|y|^{-1}$ is the hyperbolic metric in L . Next let S denote the family of $\varphi = S_g$ where g is conformal in L , and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ for which g has a quasiconformal extension to $\bar{\mathbb{C}}$. From Theorem 1 it follows that $\|\varphi\| \leq 6$ for all $\varphi \in S$ and hence that $T \subset S \subset B_2$. The set T is the universal Teichmüller space. (See, for example, [4].)

Suppose that $\varphi \in \text{int}(S)$. Then $\varphi = S_g$ where g maps L conformally onto a simply connected subdomain D of \mathbb{C} . In addition, there exists a constant $a > 0$ such that $\psi \in S$ whenever $\|\psi - \varphi\| \leq a$. If f is analytic with

$$|S_f(z)| \leq a \rho_D(z)^2$$

in D , then $\psi = S_{f \circ g}$ is analytic in L , $\|\psi - \varphi\| \leq a$, and hence f is univalent in D . Thus ∂D is a quasiconformal circle by Theorem 5, g has a quasiconformal extension to $\bar{\mathbb{C}}$, and $\varphi \in T$. Hence

$$\text{int}(S) \subset T. \quad (18)$$

Next using the Remark following Theorem 4, Ahlfors showed in [1] that

$$T = \text{int}(T). \quad (19)$$

Combining (18) and (19) we obtain the following result.

COROLLARY 3. *T is the interior of S in B_2 .*

Unfortunately Corollary 3 neither implies nor is implied by the truth of the following interesting conjecture due to Bers. (See, for example, [4].)

CONJECTURE. *S is the closure of T in B_2 .*

Lehto observed in [8] that one would settle the Bers conjecture in the negative if one could find a Jordan domain D and a positive constant a such that ∂D is not a quasiconformal circle and such that f has a quasiconformal extension to $\bar{\mathbb{C}}$

whenever f is analytic with

$$|S_f(z)| \leq a \rho_D(z)^2$$

in D . Theorem 5 shows, however, that no such domain D exists.

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