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## Univalent functions and the Schwarzian derivative

F. W. Gehring<sup>(1)</sup>

Dedicated to Professor A. Pfluger on his seventieth birthday

### 1. Introduction

This paper is concerned with the problem of extending to an arbitrary simply connected plane domain D the following two well known results relating the univalence of a function f analytic in the unit disk B with the magnitude of its Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2.$$

THEOREM 1. If f is analytic and univalent in B, then

$$|S_f(z)| \le 6(1-|z|^2)^{-2}$$

in B. The constant 6 is sharp.

THEOREM 2. If f is analytic with

$$|S_f(z)| \le 2(1-|z|^2)^{-2}$$

in B, then f is univalent in B. The constant 2 is best possible.

Theorem 1 is due to Kraus [7] and Theorem 2 to Nehari [10]. Suppose next that D is a simply connected proper subdomain of the finite

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complex plane C. Then the hyperbolic metric in D is given by

$$\rho_D(z) = \frac{|g'(z)|}{1 - |g(z)|^2},$$

where g is any conformal mapping of D onto B. The inequality

$$\frac{1}{4}\operatorname{dist}(z,\partial D)^{-1} \le \rho_D(z) \le \operatorname{dist}(z,\partial D)^{-1} \tag{1}$$

follows immediately from well known results due to Koebe and Schwarz. (See, for example, page 22 in [12].)

A Jordan curve  $\gamma$  in the extended complex plane  $\bar{\mathbf{C}}$  is said to be a K-quasiconformal circle,  $1 \le K < \infty$ , if there exists a K-quasiconformal mapping f of  $\bar{\mathbf{C}}$  onto  $\bar{\mathbf{C}}$  which maps the unit circle onto  $\gamma$ . The curve  $\gamma$  is said to be a quasiconformal circle if it is a K-quasiconformal circle for some K.

The following analogues of Theorems 1 and 2 for simply connected subdomains D of  $\mathbb{C}$  are due to Lehto [8] and Ahlfors [1], respectively. See also [3].

THEOREM 3. If f is analytic and univalent in D, then

$$|S_f(z)| \leq 12\rho_D(z)^2$$

in D. The constant 12 is sharp.

THEOREM 4. Suppose that  $\partial D$  is a K-quasiconformal circle. Then there exists a positive constant a which depends only on K such that f is univalent in D whenever f is analytic with

$$|S_{\mathbf{f}}(z)| \le a\rho_{\mathbf{D}}(z)^2 \tag{2}$$

in D.

**Remark.** Ahlfors actually proved more than the conclusion given above, namely that one can choose a = a(K) so that f has a quasiconformal extension to  $\bar{\mathbb{C}}$  whenever f is analytic and satisfies (2) in D.

In view of the above remark, it is natural to ask if the hypothesis that  $\partial D$  be a quasiconformal circle is necessary in Theorem 4. We shall show that this is indeed the case by establishing the following result.

THEOREM 5. Suppose there exists a positive constant a such that f is

univalent in D whenever f is analytic with

$$|S_f(z)| \leq a\rho_D(z)^2$$

in D. Then  $\partial D$  is a K-quasiconformal circle where K depends only on a.

# 2. Schwarzian univalence criterion

We obtain Theorem 5 as a corollary of an analogous result for proper subdomains D of  $\mathbb{C}$  with arbitrary connectivity. For such domains D we have the following consequence of Theorem 1.

COROLLARY 1. If f is analytic and univalent in D, then

$$|S_f(z)| \le 6 \operatorname{dist}(z, \partial D)^{-2} \tag{3}$$

in D. The constant 6 is best possible.

*Proof.* Fix  $z_0 \in D$ , choose r so that  $0 < r < \text{dist}(z_0, \partial D)$  and let  $g(z) = f(rz + z_0)$ . Then g is analytic and univalent in B,

$$|S_f(z_0)| = |S_g(0)|r^{-2} \le 6r^{-2}$$

by Theorem 1, and we obtain (3) for  $z = z_0$  by letting  $r \to \text{dist}(z_0, \partial D)$ . There is equality in (3) when f is the Koebe function  $z(1-z)^{-2}$ , D = B and z = 0.

Corollary 1 and inequality (1) suggest that  $\operatorname{dist}(z, \partial D)^{-1}$  is a reasonable substitute for the hyperbolic metric  $\rho_D(z)$  in the case where D is multiply connected.

DEFINITION. Suppose that D is an arbitrary proper subdomain of  $\mathbb{C}$ . We say that D satisfies the Schwarzian univalence criterion if there exists a positive constant a such that f is univalent in D whenever f is analytic with

$$|S_f(z)| \le a \operatorname{dist}(z, \partial D)^{-2}$$

in D.

The purpose of this paper is to establish the following result.

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THEOREM 6. If D satisfies the Schwarzian univalence criterion with constant a, then each component of  $\partial D$  is either a point or a K-quasiconformal circle where K depends only on a.

**Proof of Theorem 5.** Suppose that D is a simply connected proper subdomain of  $\mathbb{C}$  which satisfies the hypotheses of Theorem 5. Then by inequality (1), D satisfies the Schwarzian univalence criterion with constant a/16. Since  $\partial D$  is connected and contains at least two points, Theorem 6 implies that  $\partial D$  is a K-quasiconformal circle where K depends only on a.

COROLLARY 2. Suppose that D is a simply connected proper subdomain of C. Then D satisfies the Schwarzian univalence criterion if and only if  $\partial D$  is a quasiconformal circle.

**Proof.** Theorem 4 and inequality (1) imply that D satisfies the Schwarzian univalence criterion whenever  $\partial D$  is a quasiconformal circle. The converse follows from Theorem 6.

### 3. Proof of Theorem 6

The proof of Theorem 6 depends on five lemmas given below. In what follows we let D denote an arbitrary domain in  $\overline{\mathbb{C}}$ ,  $B(z_0, r)$  the open disk with center  $z_0 \in \mathbb{C}$  and radius  $r \in (0, \infty)$ , and b a constant in  $(1, \infty)$ . Next we say that two points  $z_1$ ,  $z_2$  can be *joined* in a set  $E \subset \overline{\mathbb{C}}$  if there exists an arc  $\alpha \subset E$  with  $z_1$ ,  $z_2$  as its endpoints. Finally for each set  $E \subset \overline{\mathbb{C}}$  we let  $\partial E$ ,  $\overline{E}$  and C(E) denote respectively the boundary, closure and complement of E in  $\overline{\mathbb{C}}$ .

LEMMA 1. Suppose that for some  $z_0$  and r there exist two points in  $D \cap \overline{B}(z_0, r)$  which cannot be joined in  $D \cap \overline{B}(z_0, br)$ . Then there exist finite points  $z_1, z_2$  in D and  $w_1, w_2$  in C(D) such that

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D with

$$|h(z_1) - h(z_2) - 2\pi i| \le \frac{4}{b-1}. \tag{4}$$

*Proof.* By hypothesis there exist two points  $z_1'$ ,  $z_2'$  in  $D \cap \bar{B}(z_0, r)$  which cannot be joined in  $D \cap \bar{B}(z_0, br)$ . Let  $\alpha'$  denote the closed segment from  $z_1'$  to  $z_2'$ 

and let  $B_0 = B(z_0, br)$ . Since  $z_1'$ ,  $z_2' \in D$ , there exists an open polygonal arc  $\beta'$  from  $z_2'$  to  $z_1'$  in D which meets  $\alpha'$  in at most a finite set of points; when  $z_1'$ ,  $z_2' \neq z_0$ , we choose  $\beta'$  so that it lies in  $D - \{z_0\}$ . Then  $\beta' - (\alpha' \cap \beta')$  is the union of a finite number of open subarcs  $\beta$  with endpoints in  $\alpha'$ . Since  $z_1'$ ,  $z_2'$  cannot be joined in  $D \cap \overline{B_0}$ , we can choose a  $\beta$  whose endpoints cannot be joined in  $D \cap \overline{B_0}$ . Let  $z_1$  and  $z_2$  denote respectively the terminal and initial points of  $\beta$ , and let  $\alpha$  denote the closed segment from  $z_1$  to  $z_2$ . Note that  $z_1$ ,  $z_2 \neq z_0$  whenever  $z_1'$ ,  $z_2' \neq z_0$ .

We want next to find finite points  $w_1, w_2 \in C(D)$  so that the function h is analytic in D and satisfies (4). Now  $z_1$  and  $z_2$  are separated in  $\bar{B}_0$  by the closed set C(D). Using Theorem VI.7.1 in [11] it is easy to show that  $z_1$  and  $z_2$  are separated in  $\bar{B}_0$  by a component  $C_0$  of C(D). Let  $D_0 = C(C_0)$ . Then  $D_0$  is a simply connected domain by Theorem IV.3.3 in [11],  $D \subset D_0$ , and the points  $z_1, z_2$  cannot be joined in  $D_0 \cap \bar{B}_0$ . Hence by replacing D by  $D_0$ , we may assume without loss of generality that D is simply connected.

Now  $\gamma = \alpha \cup \beta$  is a Jordan curve. Let  $D_1$  and  $D_2$  denote respectively the bounded and unbounded components of  $C(\gamma)$ . We shall show that there exist points  $w_1$ ,  $w_2$  such that

$$w_i \in C(D) \cap \partial B_0 \cap D_i \tag{5}$$

for i = 1, 2. Fix i. Since  $z_1$ ,  $z_2$  cannot be joined in  $D \cap \overline{B}_0$ ,  $\beta$  and hence  $\gamma$  must meet  $\partial B_0$  in at least two points. From Kerékjártó's theorem it follows that each component of

$$C(\gamma) \cap C(\partial B_0) = C(\gamma \cup \partial B_0)$$

is a Jordan domain, and hence that each component of  $D_i \cap B_0$  is bounded by a Jordan curve. (See page 168 in [11].) Next since  $D_i$  is a Jordan domain and since  $z_1 \in \partial D_i \cap B_0$ , there exists a neighborhood U of  $z_1$  such that points of  $D_i \cap U$  can be joined in  $D_i \cap B_0$ . Hence  $D_i \cap U$  is contained in a component  $D^*$  of  $D_i \cap B_0$ ,

$$D^* \cap U = D_i \cap U, \tag{6}$$

and  $\partial D^*$  is a Jordan curve  $\gamma^*$ .

Choose  $z \in \alpha - \{z_1\}$ . Since  $\alpha$  lies at a positive distance from  $\partial B_0$ , we can choose an open crosscut  $\delta$  of  $D_i$  from  $z_1$  to z which lies in  $B_0$ . Then (6) implies that  $\delta \subset D^*$ , that  $z \in \gamma^*$ , and hence that  $\alpha \subset \gamma^*$ . Thus  $\beta^* = \gamma^* - \alpha$  is an open arc joining  $z_2$  to  $z_1$  in  $\overline{B_0}$ , and there exists a point

$$w_i \in \beta^* \cap C(D). \tag{7}$$

Since

$$\gamma^* \subset \partial(D_i \cap B_0) \subset \gamma \cup (\partial B_0 \cap D_i),$$

we have

$$\beta^* \subset \beta \cup (\partial B_0 \cap D_i) \subset D \cup (\partial B_0 \cap D_i), \tag{8}$$

and (5) follows from (7) and (8).

Since D is simply connected, we can define an analytic branch of

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

in D. Then

$$h(z_1) - h(z_2) = \int_{\beta} \frac{dz}{z - w_1} - \int_{\beta} \frac{dz}{z - w_2}$$

$$= 2\pi i (n(\gamma, w_1) - n(\gamma, w_2)) - \int_{\alpha} \frac{dz}{z - w_1} + \int_{\alpha} \frac{dz}{z - w_2},$$

where  $n(\gamma, w_i)$  is the winding number of  $\gamma$  with respect to  $w_i$ . Since  $D_1$  is the bounded component of  $C(\gamma)$ ,

$$n(\gamma, w_1) = n = \pm 1, \qquad n(\gamma, w_2) = 0,$$

and we have

$$|h(z_1) - h(z_2) - 2n\pi i| \le \int_{\alpha} \frac{|dz|}{|z - w_1|} + \int_{\alpha} \frac{|dz|}{|z - w_2|}. \tag{9}$$

(See [2].) Then

$$\int_{a} \frac{|dz|}{|z-w_{i}|} \le \frac{|z_{1}-z_{2}|}{(b-1)r} \le \frac{2}{b-1} \tag{10}$$

for i = 1, 2, and (4) follows from (9) and (10) when n = 1. When n = -1, we obtain (4) by interchanging  $w_1$  and  $w_2$ .

LEMMA 2. Suppose that for some  $z_0$  and r there exist two points in  $D-B(z_0, r)$  which cannot be joined in  $D-B(z_0, r/b)$ . Then the conclusion of Lemma 1 again holds.

*Proof.* By hypothesis there exist two points  $z_1'$ ,  $z_2'$  in  $D-B(z_0, r)$  which cannot be joined in  $D-B(z_0, r/b)$ ; we may assume without loss of generality that  $z_1'$ ,  $z_2' \neq \infty$ . Next let  $\Delta$  and  $\zeta_1'$  denote the images of D and  $z_1'$  under

$$f(z) = \frac{1}{z - z_0} + z_0.$$

Then  $\zeta_1'$ ,  $\zeta_2'$  are points in  $\Delta \cap \bar{B}(z_0, 1/r)$  which cannot be joined in  $\Delta \cap \bar{B}(z_0, b/r)$ . By the argument for Lemma 1, there exist finite points

$$\zeta_1, \zeta_2 \in \Delta - \{z_0\}, \quad \omega_1, \omega_2 \in C(\Delta) \cap \partial B(z_0, b/r)$$

such that

$$g(\zeta) = \log \frac{\zeta - \omega_1}{\zeta - \omega_2}$$

is analytic in  $\Delta$  with

$$|g(\zeta_1) - g(\zeta_2) - 2\pi i| \le \frac{4}{b-1}.$$

Let  $z_i$ ,  $w_i$  denote the images of  $\zeta_i$ ,  $\omega_i$  under  $f^{-1}$ . Then

$$h(z) = g \circ f(z) + \log \frac{z_0 - w_1}{z_0 - w_2} = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4).

DEFINITION. A set E in  $\bar{\mathbb{C}}$  is said to be b-locally connected if for all  $z_0$  and r, points in  $E \cap \bar{B}(z_0, r)$  can be joined in  $E \cap \bar{B}(z_0, br)$  and points in  $E - B(z_0, r)$  can be joined in  $E - B(z_0, r/b)$ .

See [5] and [6] for other applications of this concept.

LEMMA 3. Suppose that D is a proper subdomain of C. If D satisfies the Schwarzian univalence criterion for some constant a, then D is b-locally connected where

$$b = \max\left(\frac{5}{a} + 1, 3\right). \tag{11}$$

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**Proof.** Suppose that D is not b-locally connected. Then there exist  $z_0 \in \mathbb{C}$ ,  $r \in (0, \infty)$  and two points in D for which the hypotheses of Lemma 1 or Lemma 2 hold. In either case, we obtain finite points  $z_1, z_2 \in D$  and  $w_1, w_2 \in C(D)$  such that

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4). Since  $b \ge 3$ , inequality (4) implies that

$$|h(z_1) - h(z_2)| \ge 2\pi - \frac{4}{b-1} > 4.$$
 (12)

Now set

$$f(z) = \exp(ch(z)),$$
  $c = \frac{2\pi i}{h(z_1) - h(z_2)}.$ 

Then f is analytic with

$$S_f(z) = \frac{1-c^2}{2} \left( \frac{1}{z-w_1} - \frac{1}{z-w_2} \right)^2$$

in D. Next (4), (11) and (12) imply that

$$2|1-c^2| < \frac{5}{b-1} \le a,$$

and hence that

$$|S_f(z)| \le 2|1-c^2| \operatorname{dist}(z, \partial D)^{-2} \le a \operatorname{dist}(z, \partial D)^{-2}$$

in D. Since D satisfies the univalence criterion, it follows that f must be univalent in D. But

$$\frac{f(z_1)}{f(z_2)} = \exp\left(c(h(z_1) - h(z_2))\right) = 1,$$

and we have a contradiction.

LEMMA 4. Suppose that D is b-locally connected and that  $\partial D$  is connected and contains at least two points. Then  $\partial D$  is a K-quasiconformal circle where K depends only on b.

**Proof.** Suppose that p is a point in  $\overline{D}$ . With each neighborhood U of p we associate a second neighborhood V as follows. If  $p = z_0 \in \mathbb{C}$ , choose  $r \in (0, \infty)$  so that  $\overline{B}(z_0, br) \subset U$  and let  $V = B(z_0, r)$ ; if  $p = \infty$  choose  $r \in (0, \infty)$  so that  $C(B(0, r/b)) \subset U$  and let  $V = C(\overline{B}(0, r))$ . In each case, the fact that D is b-locally connected implies that points in  $D \cap V$  can be joined in  $D \cap U$ . Thus D is uniformly locally connected and  $\partial D$  is a Jordan curve  $\gamma$  by Theorem VI.16.2 in [11].

We show next that for any pair of finite points  $z_1, z_2 \in \gamma$ ,

$$\min (\operatorname{dia}(\gamma_1), \operatorname{dia}(\gamma_2)) \le b^2 |z_1 - z_2|,$$
 (13)

where  $\gamma_1$ ,  $\gamma_2$  denote the components of  $\gamma - \{z_1, z_2\}$ . By a theorem of Ahlfors, inequality (13) will then imply that  $\gamma$  is a K-quasiconformal circle, where K depends only on b, thus completing the proof. (See, for example, Theorem II.8.6 in [9].)

To this end fix  $z_1, z_2 \in \gamma$ , set

$$z_0 = \frac{1}{2}(z_1 + z_2), \qquad r = \frac{1}{2}|z_1 - z_2|,$$

and suppose that (13) does not hold. Then there exist  $t \in (r, \infty)$  and finite points  $w_1$ ,  $w_2$  such that

$$w_i \in \gamma_i - B(z_0, b^2 t) \tag{14}$$

for i = 1, 2. Choose  $s \in (r, t)$ . Since  $z_1, z_2 \in \gamma \cap B(z_0, s)$ , we can find for i = 1, 2 an endcut  $\alpha_i$  of D joining  $z_i$  to  $z_i' \in D$  in  $\overline{B}(z_0, s)$ . Next since D is b-locally connected, we can find an arc  $\alpha_3$  joining  $z_1'$  to  $z_2'$  in  $D \cap \overline{B}(z_0, bs)$ . Then  $\alpha_1 \cup \alpha_2 \cup \alpha_3$  contains a crosscut  $\alpha$  of D from  $z_1$  to  $z_2$  with

$$\alpha \subset \bar{B}(z_0, bs). \tag{15}$$

By virtue of (14), the same argument can be applied to obtain a crosscut  $\beta$  of D from  $w_1$  to  $w_2$  with

$$\beta \subset C(B(z_0, bt)). \tag{16}$$

But (15) and (16) imply that  $\alpha \cap \beta = \emptyset$ , contradicting the fact that  $z_1$  and  $z_2$  separate  $w_1$  and  $w_2$  in  $\gamma$ . Thus (13) holds and the proof of Lemma 4 is complete.

LEMMA 5. Suppose that D is b-locally connected. Then each component of  $\partial D$  is either a point or a K-quasiconformal circle where K depends only on b.

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**Proof.** Let  $B_0$  be a component of  $\partial D$ , let  $C_0$  denote the component of C(D) which contains  $B_0$ , and let  $D_0 = C(C_0)$ . Then  $D_0$  is a domain with  $\partial D_0 = B_0$ . (See, for example, the proof of Theorem VI.16.3 in [11].) To complete the proof we need only show that  $D_0$  is b-locally connected. For then by Lemma 4,  $\partial D_0$  will be a point or a K-quasiconformal circle where K = K(b).

Fix  $z_0 \in \mathbb{C}$  and  $r \in (0, \infty)$ . Given  $z_1, z_2 \in D_0 \cap \bar{B}(z_0, r)$  we must find an arc  $\gamma$  joining these points in  $D_0 \cap \bar{B}(z_0, br)$ . For this let  $\alpha$  be any arc joining  $z_1$  and  $z_2$  in  $\bar{B}(z_0, r)$ . If  $\alpha \subset D_0$ , we may take  $\gamma = \alpha$ . Suppose that  $\alpha \not\subset D_0$  and for i = 1, 2 let  $\alpha_i$  denote the component of  $\alpha \cap D_0$  which contains  $z_i$ . Then for each i there exists a point  $w_i$  such that

$$w_i \in \alpha_i \cap D.$$
 (17)

If  $z_i \in D$ , we may take  $w_i = z_i$ ; otherwise  $z_i \in C_i$ , a component of C(D) different from  $C_0$ , and the fact that

$$\bar{\alpha}_i \cap C_0 \neq \emptyset$$
,  $\alpha_i \cap C_i \neq \emptyset$ 

implies that  $\alpha_i$  must meet D and hence contain a point  $w_i$  satisfying (17). Since D is b-locally connected and since

$$w_1, w_2 \in \alpha \cap D \subset D \cap \bar{B}(z_0, r),$$

we can join  $w_1$  and  $w_2$  by an arc  $\beta$  in  $D \cap \bar{B}(z_0, br)$ . Then  $\alpha_1 \cup \beta \cup \alpha_2$  will contain an arc  $\gamma$  joining  $z_1$  and  $z_2$  in  $D_0 \cap \bar{B}(z_0, br)$ .

Next the same argument shows that each pair of points in  $D_0 - B(z_0, r)$  can be joined in  $D_0 - B(z_0, r/b)$ . Hence  $D_0$  is b-locally connected and the proof is complete.

**Proof of Theorem 6.** Suppose that D is a proper subdomain of  $\mathbb{C}$  which satisfies the Schwarzian univalence criterion with constant a. Lemma 3 implies that D is b-locally connected, where b is as in (11). Then Lemma 5 implies that each component of  $\partial D$  is either a point or a K-quasiconformal circle, where K depends only on D, and hence only on D.

# 4. Universal Teichmüller space

We conclude this paper with an application of Theorem 5 to Teichmüller theory.

Let  $B_2 = B_2(L, 1)$  denote the Banach space of functions  $\varphi$  analytic in the lower

half plane L with norm

$$\|\varphi\| = \sup_{z \in L} \rho_L(z)^{-2} |\varphi(z)| < \infty,$$

where  $\rho_L(z) = \frac{1}{2}|y|^{-1}$  is the hyperbolic metric in L. Next let S denote the family of  $\varphi = S_g$  where g is conformal in L, and let T = T(1) denote the subfamily of those  $\varphi = S_g$  for which g has a quasiconformal extension to  $\bar{\mathbb{C}}$ . From Theorem 1 it follows that  $\|\varphi\| \le 6$  for all  $\varphi \in S$  and hence that  $T \subset S \subset B_2$ . The set T is the universal Teichmüller space. (See, for example, [4].)

Suppose that  $\varphi \in \text{int }(S)$ . Then  $\varphi = S_g$  where g maps L conformally onto a simply connected subdomain D of C. In addition, there exists a constant a > 0 such that  $\psi \in S$  whenever  $\|\psi - \varphi\| \le a$ . If f is analytic with

$$|S_f(z)| \le a\rho_D(z)^2$$

in D, then  $\psi = S_{f \circ g}$  is analytic in L,  $\|\psi - \varphi\| \le a$ , and hence f is univalent in D. Thus  $\partial D$  is a quasiconformal circle by Theorem 5, g has a quasiconformal extension to  $\bar{\mathbb{C}}$ , and  $\varphi \in T$ . Hence

$$\operatorname{int}(S) \subset T.$$
 (18)

Next using the Remark following Theorem 4, Ahlfors showed in [1] that

$$T = \operatorname{int}(T). \tag{19}$$

Combining (18) and (19) we obtain the following result.

COROLLARY 3. T is the interior of S in  $B_2$ .

Unfortunately Corollary 3 neither implies nor is implied by the truth of the following interesting conjecture due to Bers. (See, for example, [4].)

CONJECTURE. S is the closure of T in  $B_2$ .

Lehto observed in [8] that one would settle the Bers conjecture in the negative if one could find a Jordan domain D and a positive constant a such that  $\partial D$  is not a quasiconformal circle and such that f has a quasiconformal extension to  $\bar{\mathbf{C}}$ 

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whenever f is analytic with

$$|S_f(z)| \le a\rho_D(z)^2$$

in D. Theorem 5 shows, however, that no such domain D exists.

#### **REFERENCES**

- [1] AHLFORS, L. V., Quasiconformal reflections, Acta Math. 109 (1963) 291-301.
- [2] ----, Complex analysis, McGraw-Hill, New York 1966.
- [3] Bers, L., A non-standard integral equation with applications to quasiconformal mappings, *Acta Math.* 116 (1966) 113-134.
- [4] —, Uniformization, moduli, and Kleinian groups, Bull. London Math. Soc. 4 (1972) 257-300.
- [5] GEHRING, F. W., Extension of quasiconformal mappings in three space, J. d'Analyse Math. 14 (1965) 171-182.
- [6] —, Quasiconformal mappings of slit domains in three space, J. Math. Mech. 18 (1969) 689-703.
- [7] Kraus, W., Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung, Mitt. Math. Sem. Giessen 21 (1932) 1-28.
- [8] Lehto, O., Quasiconformal mappings in the plane, Lecture Notes 14, Univ. of Maryland 1975.
- [9] LEHTO, O. and VIRTANEN, K. I., Quasiconformal mappings in the plane, Springer-Verlag, Berlin-Heidelberg-New York 1973.
- [10] Nehari, Z., The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949) 545-551.
- [11] NEWMAN, M. H. A., Elements of the topology of plane sets of points, Cambridge Univ. Press 1954.
- [12] POMMERENKE, C., Univalent functions, Vandenhoeck and Ruprecht, Göttingen 1975.

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