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## A remark on Abel's Theorem and the mapping of linear series

HENRIK H. MARTENS

Let  $X$  and  $Y$  be closed Riemann surfaces of positive genera, and let  $\phi : X \rightarrow Y$  be a holomorphic map onto.

*Remark.* If  $D = \sum m_i Q_i$  is a positive divisor of degree  $n$  and (projective) dimension  $r$  on  $X$ , then  $\phi(D) = \sum m_i \phi(Q_i)$  is a positive divisor of degree  $n$  and (projective) dimension  $\geq r$  on  $Y$ . If the complete linear series determined by  $D$  is without fixed points, then so is that determined by  $\phi(D)$ .

*Proof.* An arbitrary positive divisor of degree  $r$  on  $Y$  may be written as  $\phi(D')$  where  $D'$  is a positive divisor of degree  $r$  on  $X$ . If  $D$  is of dimension  $r$ , there is a positive divisor  $D''$  on  $X$  such that  $D \sim D' + D''$  (linear equivalence). Connecting the points of  $D$  and  $D' + D''$  by curves and applying Abel's theorem to the pullbacks to  $X$  of the holomorphic differentials on  $Y$ , we immediately see that  $\phi(D) \sim \phi(D') + \phi(D'')$  on  $Y$ . Since  $\phi(D')$  was an arbitrary positive divisor of degree  $r$  on  $Y$ , we conclude that  $\phi(D)$  is of dimension  $\geq r$ .

The preimage under  $\phi$  of the set of points occurring in  $\phi(D)$  is a finite set of points on  $X$ . If  $D$  determines a complete linear series without fixed points, then we can find a positive divisor  $D'$  on  $X$  such that  $D' \sim D$ , and no point in the preimage occurs in  $D'$ . Then, since  $\phi(D) \sim \phi(D')$  on  $Y$ , the complete linear series determined by  $\phi(D)$  must be without fixed points.

**COROLLARY 1.** *If  $X$  can be displayed as an  $n$ -sheeted covering of the Riemann sphere, then so can  $Y$ . In particular, if  $X$  is hyperelliptic then  $Y$  is hyperelliptic or elliptic.*

*Proof.*  $X$  can be displayed as an  $n$ -sheeted covering of the Riemann Sphere if and only if it admits a positive divisor of degree  $n$  and dimension  $\geq 1$  that determines a complete linear series without fixed points.

**COROLLARY 2.** *If  $D$  and  $\phi(D)$  both are divisors of dimension 1, then the branch points of the coverings of the Riemann sphere determined by  $D$  are mapped on the branch points of the coverings determined by  $\phi(D')$  without reduction of branching order.*

*Proof.* Under the stated assumption, the branch points of any cover are precisely those that appear with multiplicity  $\geq 2$  in the divisors of the linear series.

**APPLICATION.** We consider the case when  $X$  is hyperelliptic. Then  $Y$  is hyperelliptic or elliptic, and in any case a display of  $Y$  as a 2-sheeted cover of the Riemann sphere must have  $2h + 2$  branch points, where  $h$  is the genus of  $Y$ . If  $g$  is the genus of  $X$ , the  $2g + 2$  branch points of  $X$  must map on the branch points of  $Y$ , whence  $2g + 2 \leq m(2h + 2)$  or

$$g \leq m(h + 1) - 1$$

where  $m$  is the degree of  $\phi$ . Thus, for instance, as noted by Accola and Farkas, a hyperelliptic surface of genus  $> 3$  cannot be a 2-sheeted covering of an elliptic surface. Combining the above formula with the Riemann–Hurwitz relation, we get

$$m(h - 1) + 1 + \frac{1}{2}b_\phi = g \leq m(h + 1) - 1$$

or

$$b_\phi \leq 4(m - 1)$$

where  $b_\phi$  is the total branch order of  $\phi$ .

(If only  $Y$  is known to be hyperelliptic, certain restrictions are imposed on  $X$ . Thus, if  $Y$  is of genus  $\geq 3$ , no function of order 3 can exist on  $Y$  and hence not on  $X$ . The above formulas can be generalized to non-hyperelliptic cases).

Each divisor of degree 2 of the hyperelliptic series on  $X$  is mapped on a divisor of degree 2 in the image series on  $Y$ . Hence a sheet interchange on  $X$  followed by  $\phi$  is equivalent to  $\phi$  followed by a sheet interchange on  $Y$ . Thus, the function field on  $Y$  pulls back to a subfield on  $X$  invariant under the hyperelliptic involution.

(The above argument can be carried out for complete, irreducible algebraic curves, over a general groundfield, whose canonical images in the jacobian variety are invariant (modulo translation) under the involution  $u \rightarrow (-u)$ . This yields a geometric interpretation of a result of Tamme [*Ein Satz über Hyperelliptische Funktionenkörper*, J. Reine Angew. Math. 257 (1972) 217–220]. Of course, our main remark can also be proved in the more general context).

The result for hyperelliptic surfaces given in Corollary was mentioned by R. D. M. Accola, *Advances in the Theory of Riemann Surfaces*, Ann. of Math. Studies, No. 66, Princeton Univ. Press, Princeton N.J. 1971, pp. 7–18. A special case of our main remark was proven by M. Newman, Math. Ann. 196 (1972) 198–217.

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