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Explicit imbedding of the (punctured) disc into $\mathbf{C}^2$

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1. According to a basic result of Bishop [3] and Narasimhan [7], every n-dimensional Stein manifold can be imbedded (via a holomorphic, proper, one-to-one, non-singular mapping) as a closed complex submanifold of $\mathbf{C}^{2n+1}$. Forster [4] has shown that the dimension $2n+1$ can be replaced by $2n$ when $n \geq 2$. Applied to open Riemann surfaces ($n = 1$), these results yield imbeddings into $\mathbf{C}^3$. It seems likely however that every open Riemann surface can be imbedded into $\mathbf{C}^2$. That this is the case for the open unit disc was proved by Kasahara and Nishino [5] by an argument that employs the well-known mapping of Fatou (and Bieberbach). Laufer [6] observed that their idea can be adapted to imbed into $\mathbf{C}^2$ every planar annulus both of whose boundary components do not degenerate to a point; i.e., after a conformal map, annuli of the form $\{z \in \mathbf{C} : r < |z| < 1\}$ for $r > 0$. The case of the punctured disc $\Delta = \{z \in \mathbf{C} : 0 < |z| < 1\}$ was left open by Laufer and does not seem to be amenable to the technique of [5] and [6].

In this note we shall give an explicit imbedding of Δ into $\mathbf{C}^2$. Namely, we write down two mapping functions, the elliptic modular function λ together with a simple rational expression of λ and its derivative λ' (transplanted from the Poincaré upper half plane to Δ) and then we verify that they do yield an imbedding. The second function will have a pole at the origin. By a minor modification, we remove this pole and get an explicit imbedding of the open unit disc U into $\mathbf{C}^2$. This is of interest because of the indirectness of the Kasahara–Nishino imbedding. Other proper holomorphic mappings of U into $\mathbf{C}^2$, in which one of the components is a universal covering map of a plane domain, were constructed in [2], but these will not generally be one-to-one.

2. We shall begin by recalling a few basic facts about the elliptic modular function λ[1]. Let $\Pi = \{\tau \in \mathbf{C} : \operatorname{Im} \tau > 0\}$ be the Poincaré upper half plane and put $\Omega = \{\tau \in \Pi : 0 < \operatorname{Re} \tau < 1 \text{ and } |\tau - \frac{1}{2}| > \frac{1}{2}\}$. Let λ be the conformal map of Ω onto Π such that $\lambda(0) = 1$, $\lambda(1) = \infty$ and $\lambda(\infty) = 0$. Then, by reflection, λ can be extended over the whole of Π and represents Π as the universal covering surface of $\mathbf{C} \setminus \{0, 1\}$. The group G of covering transformations is the set of linear transformations $S(\tau) = (a\tau + b)/(c\tau + d)$ where a and d are odd integers, b and c are even

integers and $ad-bc=1$. We write $\tau_1 \equiv \tau_2 \pmod G$ if and only if there exists $S \in G$ with $S\tau_2 = \tau_1$. Then $\lambda(\tau_1) = \lambda(\tau_2)$ if and only if $\tau_1 \equiv \tau_2 \pmod G$. Thus $\lambda \circ S = \lambda$ for $S \in G$ and, differentiating, we get $\lambda' \circ S \cdot S' = \lambda'$ for $S \in G$. Note, for $S = (a\tau + b)/(c\tau + d) \in G$, that $S'(\tau) = 1/(c\tau + d)^2$. We shall also use the functional equation $\lambda(-1/\tau) = 1 - \lambda(\tau)$. This can be seen by observing, via a reflection in the imaginary axis, that $\tau \to \lambda(-1/\tau)$ maps Ω to the lower half plane, hence $\tau \to 1 - \lambda(-1/\tau)$ maps Ω to Π and agrees with λ at the vertices of Ω, consequently the two mappings are the same.

Let $H = \{S \in G : S(z) \equiv z + 2n,\ n \in \mathbf{Z}\}$ be the subgroup of translations in G. Consider the map $E : \Pi \to \Delta$ given by $E(\tau) = e^{i\pi\tau}$. Then $E(\tau_1) = E(\tau_2)$ if and only if $\tau_1 \equiv \tau_2 \pmod H$. This allows us to identify Δ with the quotient space Π/H. Observe, for $S(\tau) = (a\tau + b)/(c\tau + d) \in G$, that $S \in H$ if and only if $c = 0$ (because $c = 0$ implies $ad = 1$ implies $a/d = 1$). If $S \in H$, then $S' \equiv 1$ and so $\lambda' \circ S \equiv \lambda'$; i.e., λ and λ' are both well-defined on Π/H. We can now state our first result. Let $f = \lambda'/(\lambda^2(1-\lambda))$.

THEOREM 1. *The mapping $F = (\lambda, f)$ defines a proper, one-to-one, non-singular, holomorphic imbedding of Π/H into $\mathbf{C}^2$.*

Observe that F is non-singular because $\lambda' \neq 0$ on Π. We first verify that F is one-to-one.

LEMMA 1. (a) *The functions λ and λ' separate the points of Π/H.* (b) *F is one-to-one on Π/H.*

Proof. (a). We suppose (i) $\lambda(\tau_1) = \lambda(\tau_2)$ and (ii) $\lambda'(\tau_1) = \lambda'(\tau_2)$ and must show that $\tau_1 \equiv \tau_2 \pmod H$. By (i), there is $S \in G$ such that $S(\tau_2) = \tau_1$. In $\lambda'(S(\tau))\, S'(\tau) = \lambda'(\tau)$ put $\tau = \tau_2$ and get $\lambda'(\tau_1)\ S'(\tau_2) = \lambda'(\tau_2)$. Using (ii) we conclude $S'(\tau_2) = 1$. Writing $S = (a\tau + b)/(c\tau + d)$ we have $(c\tau_2 + d)^2 = 1$; i.e., $c\tau_2 + d = \pm 1$. Taking imaginary parts yields $c \operatorname{Im} \tau_2 = 0$. Hence $c = 0$ and so $S \in H$; i.e., $\tau_1 \equiv \tau_2 \pmod H$. (b) If $F(\tau_1) = F(\tau_2)$, then $\lambda(\tau_1) = \lambda(\tau_2)$ and $\lambda'(\tau_1) = \lambda^2(\tau_1)(1 - \lambda(\tau_1))\, f(\tau_1) = \lambda^2(\tau_2)\ (1 - \lambda(\tau_2))\ f(\tau_2) = \lambda'(\tau_2)$. Therefore $\tau_1 \equiv \tau_2 \pmod H$.

3. We shall collect a few elementary facts needed to verify that F is proper. For $\tau \in \Pi$ with $0 \le \operatorname{Re} \tau \le 2$, define $\tau^* = 2 - \bar{\tau}$, $0 \le \operatorname{Re} \tau^* \le 2$. By reflection of points of Ω in the line $\operatorname{Re}\tau = 1$ and differentiation we get

LEMMA 2. *For $0 \le \operatorname{Re}\tau \le 2$, $\lambda(\tau) = \bar{\lambda}(\tau^*)$ and $\lambda'(\tau) = -\bar{\lambda}'(\tau^*)$.*

LEMMA 3. ("Schwarz lemma"). *For* $S \in G$ *and* $\tau_1, \tau_2 \in \Pi$, *if* $S\tau_2 = \tau_1$, *then* $|S'(\tau_2)| = \operatorname{Im} \tau_1 / \operatorname{Im} \tau_2$.

Proof. By reflection in the real axis, $S\bar{\tau}_2 = \bar{\tau}_1$. Hence

$$\frac{S - \tau_1}{S - \bar{\tau}_1} = k\frac{\tau - \tau_2}{\tau - \bar{\tau}_2}.$$

As S is real on the real axis we get $|k| = 1$. Now differentiate this relation and put $\tau = \tau_2$.

LEMMA 4. *As* τ *converges to* ∞ *in* $\bar{\Omega}$, λ'/λ *converges to* $i\pi$. *Consequently* λ'/λ^2 *converges to* ∞.

Proof. The map $\tau \to e^{i\pi\tau}$ is a one-to-one mapping of Ω into Π and the image contains $\Pi \cap \{z \in \mathbf{C}: |z| < \varepsilon\}$ for some $\varepsilon > 0$. The map $e^{i\pi\tau} \circ (\lambda \mid \Omega)^{-1}$ is defined on Π, is real on the real axis, and, by reflection, extends to be an analytic function Ψ defined near $z = 0$. By the argument principle, $\Psi'(0) \neq 0$. Thus we have $(e^{i\pi\tau} \circ \lambda^{-1})(z) = \Psi(z)$ near $z = 0$. Putting $z = \lambda(\tau)$, we get $e^{i\pi\tau} = \Psi(\lambda(\tau))$. For the inverse $\sigma = \Psi^{-1}$ with $\sigma(0) = 0$ and $\sigma'(0) = \beta \neq 0$, we have $\sigma(e^{i\pi\tau}) = \lambda(\tau)$. Differentiating, $\sigma'(e^{i\pi\tau})\, e^{i\pi\tau} i\pi = \lambda'(\tau)$. Write $\sigma(z) = \beta z \delta(z)$ with $\delta(0) = 1$. Then $(\lambda'/\lambda)(\tau) = i\pi \circ \sigma'(e^{i\pi\tau})/(\beta \cdot \delta(e^{i\pi\tau}))$. As $\tau \to \infty$ in $\bar{\Omega}$, $e^{i\pi\tau} \to 0$ and so $\sigma'(e^{i\pi\tau}) \to \beta$ and $\delta(e^{i\pi\tau}) \to 1$. Thus $\lambda'/\lambda \to i\pi$.

LEMMA 5. *As* τ *converges to* 0 *in* $\bar{\Omega}$, $\tau^2\lambda'/(1-\lambda)$ *converges to* $-i\pi$. *Consequently* $\lambda'/(1-\lambda)$ *converges to* ∞.

Proof. Differentiating the functional equation $\lambda(\tau) = 1 - \lambda(-\tau^{-1})$ we get $\lambda'(\tau) = -\tau^{-2}\lambda'(-\tau^{-1})$. Thus $\tau^2\lambda'(\tau)/(1-\lambda(\tau)) = -\lambda'(-\tau^{-1})/\lambda(-\tau^{-1})$. Reflecting in the imaginary axis gives $\lambda(\zeta) = \bar{\lambda}(-\bar{\zeta})$ and $\lambda'(\zeta) = -\bar{\lambda}'(-\bar{\zeta})$. Thus, putting $t = 1/\bar{\tau}$ we have

$$\frac{\tau^2\lambda'(\tau)}{1-\lambda(\tau)} = \overline{\left(\frac{\lambda'(t)}{\lambda(t)}\right)}.$$

As $\tau \to 0$ in $\bar{\Omega}$, $t \to \infty$ in $\bar{\Omega}$ and $(\lambda'/\lambda)(t) \to i\pi$ by Lemma 4.

4. We can now prove that $F = (\lambda, f)$ is a proper mapping of Π/H into $\mathbf{C}^2$. We argue by contradiction and suppose that there is a sequence $\{z_j\} \subseteq \Pi$ with $z_j \to \partial(\Pi/H)$ and $M > 0$ such that (i) $|\lambda(z_j)| \leq M$ and (ii) $|f(z_j)| \leq M$. Since F has period 2 we may assume that $0 \leq \operatorname{Re} z_j < 2$. By Lemma 2, $\{z_j^*\}$ is a sequence in Π

with $z_j^* \to \partial(\Pi/H)$ and such that (i) and (ii) hold for z_j^* in place of z_j. Thus it is no loss of generality to assume that there are $\tau_j \in \bar{\Omega} \cap \Pi$ such that $\tau_j \equiv z_j \pmod{G}$. So there is $S_j \in G$ with $S_j(z_j) = \tau_j$.

If $z_j \to \infty$, then $z_j = \tau_j$ is in $\bar{\Omega}$ and so $f(z_j) \to \infty$ by Lemma 4, contradicting (ii).

Otherwise, we have $\operatorname{Im} z_j \to 0$. Since $\{\lambda(z_j)\}$ is bounded, we may pass to a subsequence and suppose that $\lambda(z_j) \to \alpha \in \mathbf{C}$. We consider three cases:

Case 1. $\alpha \neq 0$, $\alpha \neq 1$. Then, after possibily passing to a subsequence, there is a $\tau \in \bar{\Omega} \cap \Pi$ such that $\tau_j \to \tau$ and $\lambda(\tau) = \alpha$. Then $\lambda'(z_j) = \lambda'(\tau_j) \cdot S_j'(z_j)$. But $|S_j'(z_j)| = \operatorname{Im} \tau_j / \operatorname{Im} z_j \to \infty$ since $\operatorname{Im} \tau_j \to \operatorname{Im} \tau \neq 0$ and $\operatorname{Im} z_j \to 0$, while $\lambda'(\tau_j) \to \lambda'(\tau) \neq 0$. Thus $\lambda'(z_j) \to \infty$ which implies $f(z_j) \to \infty$, another contradiction.

Case 2. $\alpha = 0$. Then $\tau_j \to \infty$ in $\bar{\Omega}$. We have $(\lambda'/\lambda)(z_j) = (\lambda'/\lambda)(\tau_j) \cdot S_j'(z_j)$. By Lemma 4, $(\lambda'/\lambda)(\tau_j) \to i\pi$. Also $|S'(z_j)| = (\operatorname{Im} \tau_j / \operatorname{Im} z_j) \to \infty$. Thus $(\lambda'/\lambda)(z_j) \to \infty$ which implies $f(z_j) \to \infty$.

Case 3. $\alpha = 1$. Then $\tau_j \to 0$. From $\lambda'(z_j) = \lambda'(\tau_j) \cdot S'(z_j)$ we get

$$\left|\frac{\lambda'(z_j)}{1-\lambda(z_j)}\right| = \left|\frac{\lambda'(\tau_j)}{1-\lambda(\tau_j)}\right| \cdot \left|\frac{\operatorname{Im} \tau_j}{\operatorname{Im} z_j}\right|$$
$$= \left|\frac{\tau_j^2 \lambda'(\tau_j)}{1-\lambda(\tau_j)}\right| \cdot \left|\frac{\operatorname{Im} \tau_j}{\tau_j}\right| \cdot \left|\frac{1}{\tau_j \operatorname{Im} z_j}\right|.$$

By Lemma 5, the first factor on the right converges to π. Since 0 is a cusp of Ω, it is easy to check that the second factor converges to 1 as $\tau_j \to 0$. We conclude that $(\lambda'/(1-\lambda))(z_j) \to \infty$, in contradiction to (ii). This completes the proof of Theorem 1.

5. Finally we alter our mapping in order to get an imbedding of the open unit disc U. Because λ and λ' are H-automorphic, $G = (\lambda, \lambda'/(\lambda(1-\lambda))$ gives a well-defined mapping of Π/H into $\mathbf{C}^2$. Note that $G = (\lambda, \lambda f)$ while $F = (\lambda, f)$. An obvious modification of the proof of Lemma 4(b) shows that G is one-to-one on Π/H. From $z = e^{i\pi\tau}$, we get a well-defined one-to-one holomorphic map $\tilde{G}: \Delta \to \mathbf{C}^2$ given by $\tilde{G}(z) = G(\log z / i\pi)$. Write $\tilde{G} = (g_1, g_2)$.

LEMMA 6. *$\tilde{G}$ extends to be analytic at the origin with $\tilde{G}(0) = (0, i\pi)$.*

Proof. We restrict $\tau \in \Pi$ to $0 \leq \operatorname{Re} \tau < 2$ and observe that this region maps onto Δ via $z = e^{i\pi\tau}$. Put $\Omega^* = \{\tau^* : \tau \in \Omega\}$. By Lemma 2, $(\lambda'/\lambda) = -\overline{(\lambda'/\lambda)}(\tau^*)$. Since $\tau^* \to \infty$ in Ω as $\tau \to \infty$ in Ω^*, we conclude from Lemma 4 that $(\lambda'/\lambda)(\tau) \to -(\overline{i\pi}) = i\pi$ as $\tau \to \infty$ in Ω^*. Hence $(\lambda'/\lambda)(\tau) \to i\pi$ as $\tau \to \infty$ in $\bar{\Omega} \cup \Omega^*$. Now as $z = e^{i\pi\tau} \to 0$

in Δ, $\tau \to \infty$ in $\bar{\Omega} \cup \Omega^*$ and so $g_1(z) = \lambda(\tau) \to 0$ and $g_2(z) = (\lambda'(\tau)/\lambda(\tau)) \cdot (1/(1-\lambda(\tau))) \to i\pi \cdot 1$, Thus g_1 and g_2 have removable singularities at $z = 0$.

Henceforth we shall consider $\tilde{G}$ as a mapping on U.

THEOREM 2. *$\tilde{G}$ is a proper, one-to-one, non-singular, holomorphic mapping of U into $\mathbf{C}^2$.*

LEMMA 7. *$\tilde{G}$ separates the points of U.*

Proof. Suppose, for z_1 and z_2 in U, that (i) $g_1(z_1) = g_1(z_2)$ and (ii) $g_2(z_1) = g_2(z_2)$. Observe that if $g_1(z) = 0$ for $z \in U$ then, since $\lambda \neq 0$ on Π, we have $z = 0$. Thus if $g_1(z_1) = 0$, then $g_1(z_2) = 0$ and $z_1 = 0$, $z_2 = 0$. If $g_1(z_1) \neq 0$, then $g_1(z_2) \neq 0$ and so z_1 and z_2 are in Δ. Since we have already observed that $\tilde{G}$ separates the points of Δ, we conclude that $z_1 = z_2$ in either case.

LEMMA 8. *$\tilde{G}$ is non-singular on U.*

Proof. We show that $g_1' \neq 0$ on U. We have, for $z \neq 0$, $g_1(z) = \lambda(\log z/\pi i)$. Hence $g_1'(z) = (\pi i z)^{-1} \lambda'(\log z/\pi i)$ and so $g_1' \neq 0$ on Δ. Putting $z = e^{i\pi\tau}$ we have $g_1'(z) = (i\pi e^{i\pi\tau})^{-1} \lambda'(\tau) = \sigma'(z)$ in the notation of the proof of Lemma 4. As $z \to 0$, $g_1'(z) \to \sigma'(0) \neq 0$.

Finally we show that $\tilde{G}$ is proper. Returning to the half plane, it is enough (because of Lemma 2) to show: If $\{z_j\} \subseteq \Pi$, $0 \leq \operatorname{Re} z_j \leq 1$, and $\operatorname{Im} z_j \to 0$, then $G(z_j) \to \infty$. Arguing by contradiction, as before, we may suppose (a) $\lambda(z_j) \to \alpha \in \mathbf{C}$ and (b) $|\lambda'(z_j)/(\lambda(z_j)(1-\lambda(z_j)))| \leq M$.

We reconsider the proof of the properness of F with its three cases. If $\alpha \neq 0, 1$, we saw that $\lambda'(z_j) \to \infty$, if $\alpha = 0$, we had $\lambda'(z_j)/\lambda(z_j) \to \infty$ and if $\alpha = 1$, we got $\lambda'(z_j)/(1-\lambda(z_j)) \to \infty$. In every case there is a contradiction to (b). This completes the proof of Theorem 2.

As a final remark we observe that if $V = F(\Pi/H)$ and $W = \tilde{G}(U)$ are the two image submanifolds in $\mathbf{C}^2$, then the inclusion map $\Delta \subset U$ induces a map $V \to W$ which is the restriction to V of the Cremona transformation $(z, w) \mapsto (z, zw)$ of $\mathbf{C}^2$.

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