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Explicit imbedding of the (punctured) disc into C²

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1. According to a basic result of Bishop [3] and Narasimhan [7], every n-dimensional Stein manifold can be imbedded (via a holomorphic, proper, one-to-one, non-singular mapping) as a closed complex submanifold of \mathbb{C}^{2n+1} . Forster [4] has shown that the dimension 2n+1 can be replaced by 2n when $n \ge 2$. Applied to open Riemann surfaces (n=1), these results yield imbeddings into \mathbb{C}^3 . It seems likely however that every open Riemann surface can be imbedded into \mathbb{C}^2 . That this is the case for the open unit disc was proved by Kasahara and Nishino [5] by an argument that employs the well-known mapping of Fatou (and Bieberbach). Laufer [6] observed that their idea can be adapted to imbed into \mathbb{C}^2 every planar annulus both of whose boundary components do not degenerate to a point; i.e., after a conformal map, annuli of the form $\{z \in \mathbb{C}: r < |z| < 1\}$ for r > 0. The case of the punctured disc $\Delta = \{z \in \mathbb{C}: 0 < |z| < 1\}$ was left open by Laufer and does not seem to be amenable to the technique of [5] and [6].

In this note we shall give an explicit imbedding of Δ into \mathbb{C}^2 . Namely, we write down two mapping functions, the elliptic modular function λ together with a simple rational expression of λ and its derivative λ' (transplanted from the Poincaré upper half plane to Δ) and then we verify that they do yield an imbedding. The second function will have a pole at the origin. By a minor modification, we remove this pole and get an explicit imbedding of the open unit disc U into \mathbb{C}^2 . This is of interest because of the indirectness of the Kasahara-Nishino imbedding. Other proper holomorphic mappings of U into \mathbb{C}^2 , in which one of the components is a universal covering map of a plane domain, were constructed in [2], but these will not generally be one-to-one.

2. We shall begin by recalling a few basic facts about the elliptic modular function $\lambda[1]$. Let $\Pi = \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$ be the Poincaré upper half plane and put $\Omega = \{\tau \in \Pi : 0 < \text{Re } \tau < 1 \text{ and } |\tau - \frac{1}{2}| > \frac{1}{2}\}$. Let λ be the conformal map of Ω onto Π such that $\lambda(0) = 1$, $\lambda(1) = \infty$ and $\lambda(\infty) = 0$. Then, by reflection, λ can be extended over the whole of Π and represents Π as the universal covering surface of $\mathbb{C} \setminus \{0, 1\}$. The group G of covering transformations is the set of linear transformations $S(\tau) = (a\tau + b)/(c\tau + d)$ where a and d are odd integers, b and c are even

540 H. ALEXANDER

integers and ad-bc=1. We write $\tau_1\equiv\tau_2\pmod G$ if and only if there exists $S\in G$ with $S\tau_2=\tau_1$. Then $\lambda(\tau_1)=\lambda(\tau_2)$ if and only if $\tau_1\equiv\tau_2\pmod G$. Thus $\lambda\circ S=\lambda$ for $S\in G$ and, differentiating, we get $\lambda'\circ S\cdot S'=\lambda'$ for $S\in G$. Note, for $S=(a\tau+b)/(c\tau+d)\in G$, that $S'(\tau)=1/(c\tau+d)^2$. We shall also use the functional equation $\lambda(-1/\tau)=1-\lambda(\tau)$. This can be seen by observing, via a reflection in the imaginary axis, that $\tau\to\lambda(-1/\tau)$ maps Ω to the lower half plane, hence $\tau\to1-\lambda(-1/\tau)$ maps Ω to Π and agrees with λ at the vertices of Ω , consequently the two mappings are the same.

Let $H = \{S \in G : S(z) \equiv z + 2n, n \in \mathbb{Z}\}$ be the subgroup of translations in G. Consider the map $E : \Pi \to \Delta$ given by $E(\tau) = e^{i\pi\tau}$. Then $E(\tau_1) = E(\tau_2)$ if and only if $\tau_1 \equiv \tau_2 \pmod{H}$. This allows us to identify Δ with the quotient space Π/H . Observe, for $S(\tau) = (a\tau + b)/(c\tau + d) \in G$, that $S \in H$ if and only if c = 0 (because c = 0 implies ad = 1 implies a/d = 1). If $S \in H$, then $S' \equiv 1$ and so $\lambda' \circ S \equiv \lambda'$; i.e., λ and λ' are both well-defined on Π/H . We can now state our first result. Let $f = \lambda'/(\lambda^2(1-\lambda))$.

THEOREM 1. The mapping $F = (\lambda, f)$ defines a proper, one-to-one, non-singular, holomorphic imbedding of Π/H into \mathbb{C}^2 .

Observe that F is non-singular because $\lambda' \neq 0$ on Π . We first verify that F is one-to-one.

LEMMA 1. (a) The functions λ and λ' separate the points of Π/H . (b) F is one-to-one on Π/H .

Proof. (a). We suppose (i) $\lambda(\tau_1) = \lambda(\tau_2)$ and (ii) $\lambda'(\tau_1) = \lambda'(\tau_2)$ and must show that $\tau_1 \equiv \tau_2 \pmod{H}$. By (i), there is $S \in G$ such that $S(\tau_2) = \tau_1$. In $\lambda'(S(\tau)) = S'(\tau) = \lambda'(\tau)$ put $\tau = \tau_2$ and get $\lambda'(\tau_1) S'(\tau_2) = \lambda'(\tau_2)$. Using (ii) we conclude $S'(\tau_2) = 1$. Writing $S = (a\tau + b)/(c\tau + d)$ we have $(c\tau_2 + d)^2 = 1$; i.e., $c\tau_2 + d = \pm 1$. Taking imaginary parts yields $c \text{ Im } \tau_2 = 0$. Hence c = 0 and so $S \in H$; i.e., $\tau_1 \equiv \tau_2 \pmod{H}$. (b) If $F(\tau_1) = F(\tau_2)$, then $\lambda(\tau_1) = \lambda(\tau_2)$ and $\lambda'(\tau_1) = \lambda^2(\tau_1) (1 - \lambda(\tau_1)) f(\tau_1) = \lambda^2(\tau_2) (1 - \lambda(\tau_2)) f(\tau_2) = \lambda'(\tau_2)$. Therefore $\tau_1 \equiv \tau_2 \pmod{H}$.

3. We shall collect a few elementary facts needed to verify that F is proper. For $\tau \in \Pi$ with $0 \le \text{Re } \tau \le 2$, define $\tau^* = 2 - \bar{\tau}$, $0 \le \text{Re } \tau^* \le 2$. By reflection of points of Ω in the line $\text{Re}\tau = 1$ and differentiation we get

LEMMA 2. For $0 \le \text{Re}\tau \le 2$, $\lambda(\tau) = \overline{\lambda}(\tau^*)$ and $\lambda'(\tau) = -\overline{\lambda'}(\tau^*)$.

LEMMA 3. ("Schwarz lemma"). For $S \in G$ and $\tau_1, \tau_2 \in \Pi$, if $S\tau_2 = \tau_1$, then $|S'(\tau_2)| = \text{Im } \tau_1/\text{Im } \tau_2$.

Proof. By reflection in the real axis, $S\bar{\tau}_2 = \bar{\tau}_1$. Hence

$$\frac{S-\tau_1}{S-\bar{\tau}_1}=k\frac{\tau-\tau_2}{\tau-\bar{\tau}_2}.$$

As S is real on the real axis we get |k| = 1. Now differentiate this relation and put $\tau = \tau_2$.

LEMMA 4. As τ converges to ∞ in $\bar{\Omega}$, λ'/λ converges to $i\pi$. Consequently λ'/λ^2 converges to ∞ .

Proof. The map $\tau \to e^{i\pi\tau}$ is a one-to-one mapping of Ω into Π and the image contains $\Pi \cap \{z \in \mathbb{C} : |z| < \varepsilon\}$ for some $\varepsilon > 0$. The map $e^{i\pi\tau} \circ (\lambda \mid \Omega)^{-1}$ is defined on Π , is real on the real axis, and, by reflection, extends to be an analytic function Ψ defined near z = 0. By the argument principle, $\Psi'(0) \neq 0$. Thus we have $(e^{i\pi\tau} \circ \lambda^{-1})(z) = \Psi(z)$ near z = 0. Putting $z = \lambda(\tau)$, we get $e^{i\pi\tau} = \Psi(\lambda(\tau))$. For the inverse $\sigma = \Psi^{-1}$ with $\sigma(0) = 0$ and $\sigma'(0) = \beta \neq 0$, we have $\sigma(e^{i\pi\tau}) = \lambda(\tau)$. Differentiating, $\sigma'(e^{i\pi\tau}) = \lambda'(\tau)$. Write $\sigma(z) = \beta z \delta(z)$ with $\delta(0) = 1$. Then $(\lambda'/\lambda)(\tau) = i\pi \circ \sigma'(e^{i\pi\tau})/(\beta \cdot \delta(e^{i\pi\tau}))$. As $\tau \to \infty$ in $\bar{\Omega}$, $e^{i\pi\tau} \to 0$ and so $\sigma'(e^{i\pi\tau}) \to \beta$ and $\delta(e^{i\pi\tau}) \to 1$. Thus $\lambda'/\lambda \to i\pi$.

LEMMA 5. As τ converges to 0 in $\bar{\Omega}$, $\tau^2 \lambda'/(1-\lambda)$ converges to $-i\pi$. Consequently $\lambda'/(1-\lambda)$ converges to ∞ .

Proof. Differentiating the functional equation $\lambda(\tau) = 1 - \lambda(-\tau^{-1})$ we get $\lambda'(\tau) = -\tau^{-2}\lambda'(-\tau^{-1})$. Thus $\tau^2\lambda'(\tau)/(1-\lambda(\tau)) = -\lambda'(-\tau^{-1})/\lambda(-\tau^{-1})$. Reflecting in the imaginary axis gives $\lambda(\zeta) = \bar{\lambda}(-\bar{\zeta})$ and $\lambda'(\zeta) = -\bar{\lambda}'(-\bar{\zeta})$. Thus, putting $t = 1/\bar{\tau}$ we have

$$\frac{\tau^2 \lambda'(\tau)}{1 - \lambda(\tau)} = \left(\frac{\overline{\lambda'(t)}}{\lambda(t)}\right).$$

As $\tau \to 0$ in $\bar{\Omega}$, $t \to \infty$ in $\bar{\Omega}$ and $(\lambda'/\lambda)(t) \to i\pi$ by Lemma 4.

4. We can now prove that $F = (\lambda, f)$ is a proper mapping of Π/H into \mathbb{C}^2 . We argue by contradiction and suppose that there is a sequence $\{z_j\} \subseteq \Pi$ with $z_j \to \partial(\Pi/H)$ and M > 0 such that (i) $|\lambda(z_j)| \le M$ and (ii) $|f(z_j)| \le M$. Since F has period 2 we may assume that $0 \le \text{Re } z_j < 2$. By Lemma 2, $\{z_j^*\}$ is a sequence in Π

with $z_i^* \to \partial(\Pi/H)$ and such that (i) and (ii) hold for z_i^* in place of z_i . Thus it is no loss of generality to assume that there are $\tau_i \in \bar{\Omega} \cap \Pi$ such that $\tau_i \equiv z_i \pmod{G}$. So there is $S_i \in G$ with $S_i(z_i) = \tau_i$.

If $z_i \to \infty$, then $z_i = \tau_i$ is in $\bar{\Omega}$ and so $f(z_i) \to \infty$ by Lemma 4, contradicting (ii).

Otherwise, we have Im $z_i \to 0$. Since $\{\lambda(z_i)\}$ is bounded, we may pass to a subsequence and suppose that $\lambda(z_i) \to \alpha \in \mathbb{C}$. We consider three cases:

Case 1. $\alpha \neq 0$, $\alpha \neq 1$. Then, after possibily passing to a subsequence, there is a $\tau \in \bar{\Omega} \cap \Pi$ such that $\tau_j \to \tau$ and $\lambda(\tau) = \alpha$. Then $\lambda'(z_j) = \lambda'(\tau_j) \cdot S_j'(z_j)$. But $|S_j'(z_j)| = \text{Im } \tau_j/\text{Im } z_j \to \infty$ since $\text{Im } \tau_j \to \text{Im } \tau \neq 0$ and $\text{Im } z_j \to 0$, while $\lambda'(\tau_j) \to \lambda'(\tau) \neq 0$. Thus $\lambda'(z_j) \to \infty$ which implies $f(z_j) \to \infty$, another contradiction.

Case 2. $\alpha = 0$. Then $\tau_j \to \infty$ in $\bar{\Omega}$. We have $(\lambda'/\lambda)(z_j) = (\lambda'/\lambda)(\tau_j) \cdot S_j'(z_j)$. By Lemma 4, $(\lambda'/\lambda)(\tau_j) \to i\pi$. Also $|S'(z_j)| = (\operatorname{Im} \tau_j/\operatorname{Im} z_j) \to \infty$. Thus $(\lambda'/\lambda)(z_j) \to \infty$ which implies $f(z_j) \to \infty$.

Case 3. $\alpha = 1$. Then $\tau_i \to 0$. From $\lambda'(z_i) = \lambda'(\tau_i) \cdot S'(z_i)$ we get

$$\left| \frac{\lambda'(z_i)}{1 - \lambda(z_i)} \right| = \left| \frac{\lambda'(\tau_i)}{1 - \lambda(\tau_i)} \right| \cdot \left| \frac{\operatorname{Im} \tau_i}{\operatorname{Im} z_i} \right|$$

$$= \left| \frac{\tau_i^2 \lambda'(\tau_i)}{1 - \lambda(\tau_i)} \right| \cdot \left| \frac{\operatorname{Im} \tau_i}{\tau_i} \right| \cdot \left| \frac{1}{\tau_i \operatorname{Im} z_i} \right|.$$

By Lemma 5, the first factor on the right converges to π . Since 0 is a cusp of Ω , it is easy to check that the second factor converges to 1 as $\tau_i \to 0$. We conclude that $(\lambda'/(1-\lambda))(z_i) \to \infty$, in contradiction to (ii). This completes the proof of Theorem 1.

5. Finally we alter our mapping in order to get an imbedding of the open unit disc U. Because λ and λ' are H-automorphic, $G = (\lambda, \lambda'/(\lambda(1-\lambda)))$ gives a well-defined mapping of Π/H into \mathbb{C}^2 . Note that $G = (\lambda, \lambda f)$ while $F = (\lambda, f)$. An obvious modification of the proof of Lemma 4(b) shows that G is one-to-one on Π/H . From $z = e^{i\pi\tau}$, we get a well-defined one-to-one holomorphic map $\tilde{G}: \Delta \to \mathbb{C}^2$ given by $\tilde{G}(z) = G(\log z/i\pi)$. Write $\tilde{G} = (g_1, g_2)$.

LEMMA 6. \tilde{G} extends to be analytic at the origin with $\tilde{G}(0) = (0, i\pi)$.

Proof. We restrict $\tau \in \Pi$ to $0 \le \text{Re } \tau < 2$ and observe that this region maps onto Δ via $z = e^{i\pi\tau}$. Put $\Omega^* = \{\tau^* : \tau \in \Omega\}$. By Lemma 2, $(\lambda'/\lambda) = -(\overline{\lambda'/\lambda})(\tau^*)$. Since $\tau^* \to \infty$ in Ω as $\tau \to \infty$ in Ω^* , we conclude from Lemma 4 that $(\lambda'/\lambda)(\tau) \to -(\overline{i\pi}) = i\pi$ as $\tau \to \infty$ in Ω^* . Hence $(\lambda'/\lambda)(\tau) \to i\pi$ as $\tau \to \infty$ in $\overline{\Omega} \cup \Omega^*$. Now as $z = e^{i\pi\tau} \to 0$

in Δ , $\tau \to \infty$ in $\bar{\Omega} \cup \Omega^*$ and so $g_1(z) = \lambda(\tau) \to 0$ and $g_2(z) = (\lambda'(\tau)/\lambda(\tau)) \cdot (1/(1-\lambda(\tau))) \to i\pi \cdot 1$, Thus g_1 and g_2 have removable singularities at z = 0.

Henceforth we shall consider \tilde{G} as a mapping on U.

THEOREM 2. \tilde{G} is a proper, one-to-one, non-singular, holomorphic mapping of U into \mathbb{C}^2 .

LEMMA 7. \tilde{G} separates the points of U.

Proof. Suppose, for z_1 and z_2 in U, that (i) $g_1(z_1) = g_1(z_2)$ and (ii) $g_2(z_1) = g_2(z_2)$. Observe that if $g_1(z) = 0$ for $z \in U$ then, since $\lambda \neq 0$ on Π , we have z = 0. Thus if $g_1(z_1) = 0$, then $g_1(z_2) = 0$ and $z_1 = 0$, $z_2 = 0$. If $g_1(z_1) \neq 0$, then $g_1(z_2) \neq 0$ and so z_1 and z_2 are in Δ . Since we have already observed that \tilde{G} separates the points of Δ , we conclude that $z_1 = z_2$ in either case.

LEMMA 8. \tilde{G} is non-singular on U.

Proof. We show that $g_1' \neq 0$ on U. We have, for $z \neq 0$, $g_1(z) = \lambda(\log z/\pi i)$. Hence $g_1'(z) = (\pi i z)^{-1} \lambda'(\log z/\pi i)$ and so $g_1' \neq 0$ on Δ . Putting $z = e^{i\pi\tau}$ we have $g_1'(z) = (i\pi e^{i\pi\tau})^{-1} \lambda'(\tau) = \sigma'(z)$ in the notation of the proof of Lemma 4. As $z \to 0$, $g_1'(z) \to \sigma'(0) \neq 0$.

Finally we show that \tilde{G} is proper. Returning to the half plane, it is enough (because of Lemma 2) to show: If $\{z_i\}\subseteq \Pi$, $0 \le \text{Re } z_i \le 1$, and $\text{Im } z_i \to 0$, then $G(z_i) \to \infty$. Arguing by contradiction, as before, we may suppose (a) $\lambda(z_i) \to \alpha \in \mathbb{C}$ and (b) $|\lambda'(z_i)/(\lambda(z_i)(1-\lambda(z_i)))| \le M$.

We reconsider the proof of the properness of F with its three cases. If $\alpha \neq 0$, 1, we saw that $\lambda'(z_i) \to \infty$, if $\alpha = 0$, we had $\lambda'(z_i)/\lambda(z_i) \to \infty$ and if $\alpha = 1$, we got $\lambda'(z_i)/(1-\lambda(z_i)) \to \infty$. In every case there is a contradiction to (b). This completes the proof of Theorem 2.

As a final remark we observe that if $V = F(\Pi/H)$ and $W = \tilde{G}(U)$ are the two image submanifolds in \mathbb{C}^2 , then the inclusion map $\Delta \subset U$ induces a map $V \to W$ which is the restriction to V of the Cremona transformation $(z, w) | \to (z, zw)$ of \mathbb{C}^2 .

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