Extreme values of the Riemann zeta function

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1. Introduction

In 1928, Titchmarsh [13] (see Titchmarsh [14; Theorem 8.12], Ramachandra [10]) proved that for fixed σ, $\frac{1}{2} \le \sigma < 1$,

$$\log |\zeta(\sigma + it)| = \Omega_+((\log t)^{1-\sigma-\varepsilon}) \tag{1}$$

as t tends to infinity. Previous estimates of this sort had only been established assuming the Riemann Hypothesis (see Landau [6], Bohr–Landau [2], Littlewood [8]). Recently Levinson [7] sharpened (1) by showing that

$$\max_{1 \le t \le T} \log |\zeta(\sigma + it)| > c(\log T)^{1-\sigma}/\log \log T \tag{2}$$

for $\frac{1}{2} \le \sigma < 1$, $T \ge 10$. For $\frac{1}{2} < \sigma < 1$ we give a sharper lower bound than is provided by (2), we show that $|\zeta(s)|$ becomes correspondingly small, and that $\arg \zeta(s)$ becomes correspondingly large in both signs. We write $s = \sigma + it$.

THEOREM 1. *Let* $\frac{1}{2} < \sigma_0 < 1$, $T > T_0(\sigma_0)$. *For any real* θ *there is a* t_0 *such that* $T^{(\sigma_0 - 1/2)/3} \le t_0 \le T$, *and*

$$\operatorname{Re} e^{-i\theta} \log \zeta(s_0) \ge \frac{1}{20} \left(\sigma_0 - \frac{1}{2}\right)^{1/2} (\log T)^{1-\sigma_0}(\log \log T)^{-\sigma_0}. \tag{3}$$

Taking $\theta = 0$, π, $\pm \pi/2$ in the above, we derive the following

COROLLARY. *Let* σ *be fixed,* $\frac{1}{2} < \sigma < 1$. *Then as* t *tends to infinity,*

$$\log |\zeta(s)| = \Omega_+((\log t)^{1-\sigma}(\log \log t)^{-\sigma}), \tag{4}$$

and

$$\arg \zeta(s) = \Omega_\pm((\log t)^{1-\sigma}(\log \log t)^{-\sigma}).\tag{5}$$

In establishing Theorem 1, we actually construct more than $T^{(4/3)-(2\sigma_0-\varepsilon/3)}$ numbers t_0 having the required properties. This is necessitated by our method, since we must construct more candidates for t_0 than there are "bad zeros" of $\zeta(s)$. If we assume the Riemann Hypothesis (RH) then we have no problems with "bad zeros," and in place of (3) we find that if $\frac{1}{2} \le \sigma_0 < 1$ then there is a t_0, $T^{1/6} \le t_0 \le T$, such that

$$\text{Re } e^{-i\theta} \log \zeta(s_0) \ge \frac{1}{20} (\log T)^{1-\sigma_0}(\log \log T)^{-\sigma_0}.\tag{6}$$

In particular, we obtain

THEOREM 2. *Suppose that the Riemann Hypothesis is true. Then*

$$|\zeta(\tfrac{1}{2} + it)| = \Omega\left(\exp\left(\frac{1}{20} (\log t/\log \log t)^{1/2}\right)\right)\tag{7}$$

and

$$S(t) = \Omega_\pm((\log t/\log \log t)^{1/2})\tag{8}$$

as t tends to infinity, where $S(t) = 1/\pi \arg \zeta(\tfrac{1}{2} + it)$.

The best known unconditional estimate corresponding to (7) is obtained by taking $\sigma = \frac{1}{2}$ in (2). With regard to (8), Selberg [12] has shown unconditionally that

$$S(t) = \Omega_\pm((\log t)^{1/3}(\log \log t)^{-7/3}).$$

Selberg [11] remarks that the Bohr–Landau approach, when fully exploited, yields (assuming RH)

$$S(t) = \Omega_\pm((\log t)^{1/2}/\log \log t).$$

The small refinement that (8) gives of this is of interest because our estimates are "likely" to be best possible. To argue that this is the case, suppose that σ is fixed,

$\frac{1}{2} < \sigma < 1$, and that RH holds. Then for $T \le t \le 2T$ we have

$$\log \zeta(s) = \sum_{p \le T} p^{-s} + 0(1)$$

$$= F(t) + 0(1),$$

say. As the numbers p^{-it} have a random appearance, we consider

$$G(\mathbf{\theta}) = \sum_{p \le T} p^{-\sigma} e(\theta_p),$$

where the θ_p are independent, $0 \le \theta_p \le 1$, and $e(\theta) = e^{2\pi i \theta}$. We can *prove* that with great uniformity the distribution function of $F(t)$, for $T \le t \le 2T$, is very close to that of $G(\mathbf{\theta})$, for $\mathbf{\theta} \in \mathbf{T}^{\pi(T)}$. Let us choose "at random" T points $\mathbf{\theta}_1, \dots, \mathbf{\theta}_T$. Then we find that the probability that

$$\max_{1 \le t \le T} |G(\mathbf{\theta}_t)| > C(\log T)^{1-\sigma} (\log \log T)^{-\sigma}$$

is less than T^{-A}, where A becomes large with C. The rapid convergence of the series $\sum 2^{-An}$ leads us to conjecture that

$$|\log \zeta(s)| \ll_\sigma (\log t)^{1-\sigma} (\log \log t)^{-\sigma} \tag{9}$$

for $\frac{1}{2} < \sigma < 1$. The situation in Theorem 2 is more delicate, but we still suggest that for some $c > 0$ and all $t \ge 10$,

$$|\zeta(s)| < \exp (c(\log t/\log \log t)^{1/2}), \tag{10}$$

$$S(t) \ll (\log t/\log \log t)^{1/2}. \tag{11}$$

The estimate (8) was announced previously by the author [9; p. 123]; on that occasion a connection was exhibited between the size of $S(t, \chi)$ and the size of least character non-residues. We repeat the observation that our method can be used to show (assuming RH) that

$$S(t + 1/\log \log t) - S(t) = \Omega_\pm((\log t/\log \log t)^{1/2}).$$

Assuming RH, Selberg (unpublished) has argued from (16) that if $(\log T)^{-1} \le h \le$

$(\log \log T)^{-1}$ then

$$\sup_{T \le t \le 2T} (S(t+h) - S(t)) \ge c(h \log T)^{1/2},$$

$$\inf_{T \le t \le 2T} (S(t+h) - S(t)) \le -c(h \log T)^{1/2}.$$

Selberg's approach differs from ours in that he uses high moments to make his expression large; Dirichlet's theorem does not suffice.

Over a period of years the author has been pleased to have many stimulating discussions with Professor Atle Selberg concerning the subject of this paper.

2. Basic lemmas

We require a modified form of Dirichlet's theorem on simultaneous approximation, which we derive in a standard manner from

LEMMA 1. *Let $\mathscr{C}$ be a convex body in $\mathbf{R}^m$, symmetric about the origin. If $\mu(\mathscr{C}) > 2^m K$ then $\mathscr{C}$ contains at least K distinct pairs of non-zero lattice points $\pm \mathbf{u}_k$, $1 \le k \le K$.*

This form of Minkowski's first main theorem is due to van der Corput [4]; see Cassels [3; Chapter III, Theorem II].

LEMMA 2. *Let $\theta_1, \ldots, \theta_M$ be arbitrary real numbers, and suppose that $0 < \delta < \frac{1}{2}$. There are at least $[\delta^M(R+1)]$ integers r such that $1 \le r \le R$ and $\|r\theta_m\| \le \delta$ for $1 \le m \le M$.*

Here $\|\theta\|$ denotes the distance from θ to the nearest integer, $\|\theta\| = \min |\theta - n|$.

Proof. Let $\mathscr{C}_\varepsilon \subset \mathbf{R}^{M+1}$ be defined as follows: For $\varepsilon \ge 0$ let $\mathbf{x} \in \mathscr{C}_\varepsilon$ if and only if $|x^{(0)}| < (1+\varepsilon)^{-1}(R+1)$, $|x^{(n)} - \theta_m x^{(0)}| \le (1+\varepsilon)\delta$ for $1 \le m \le M$. Thus

$$\mu(\mathscr{C}_\varepsilon) = (1+\varepsilon)^M \delta^M 2^M (R+1),$$

so that if $K = [\delta^M(R+1)]$ and $\varepsilon > 0$ then by Lemma 1 we find that $\mathscr{C}_\varepsilon$ contains K distinct pairs of non-zero lattice points. Letting ε tend to 0, we find that $\mathscr{C}_0$ also contains such points, say $\pm \mathbf{x}_k$, $1 \le k \le K$. The condition $\delta < \frac{1}{2}$ ensures that the leading coordinates $\pm u_k^{(0)}$ are distinct and non-zero; we let r take on the positive values of these leading coordinates.

One can derive Lemma 2 analytically by the method of Blanksby–Montgomery [1; §6].

LEMMA 3. *Suppose that the Dirichlet series* $f(s) = \sum a_n n^{-s}$ *is absolutely convergent for* $\sigma > \sigma_a$. *Suppose also that* $\alpha > 0$ *and that* x *is real. Then for* $c > \sigma_a$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\frac{e^{\alpha s} - e^{-\alpha s}}{s}\right)^2 e^{xs}\, ds = \sum_w a_n w_n, \tag{12}$$

where $w_n = w_n(\alpha, x) = \max(0, \alpha - |x - \log n|)$.

In earlier works on this subject, identities such as (12) have been employed with α large and $x = 0$. To some extent our ability to achieve sharper results is due to the fact that we take $\alpha \approx 1$, and x large.

Proof. In the integral we replace f by its defining sum; we invert the order of summation and integration, and then it suffices to note that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\frac{e^{\alpha s} - e^{-\alpha s}}{s}\right)^2 e^{vs}\, ds = \frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin \alpha t}{t}\right)^2 e^{ivt}\, dt = \max(0, \alpha - |v|),$$

where $v = x - \log n$.

LEMMA 4. *Suppose that* $\frac{1}{2} \le \sigma_0 < 1$, $t_0 \ge 15$, *and that* $\zeta(s) \ne 0$ *for* $\sigma > \sigma_0$, $|t - t_0| \le 2\tau$, *where* $\tau = \tau(t_0) = (\log t_0)^2$. *Then for* $\alpha > 0$, *and real* x,

$$\frac{2}{\pi}\int_{-\tau}^{\tau} \log \zeta(s_0 + it)\left(\frac{\sin \alpha t}{t}\right)^2 e^{ixt}\, dt = \sum \Lambda_1(n) w_n n^{-s_0} + O(e^{|x|+2\alpha}(\log t_0)^{-2}), \tag{13}$$

where $\Lambda_1(n) = \Lambda(n)/\log n$.

Proof. We take $f(s) = \log \zeta(s_0 + s) = \sum \Lambda_1(n) n^{-s_0 - s}$ in Lemma 3. We move the path of integration in (12) to lie on the five line segments determined by the points $1 - i\infty$, $1 - i\tau$, $-i\tau$, $i\tau$, $1 + i\tau$, $1 + i\infty$. Thus our integral is the sum of five integrals I_j, $|j| \le 2$, each over the corresponding segment. The left hand side of (13) is I_0, so to establish (2) we have only to bound $I_{\pm 1}$, $I_{\pm 2}$. From familiar estimates (Titchmarsh [14; Theorem 9.6(B)] we deduce that

$$\log \zeta(s_0 + s) \ll (\log t_0) \log 2/\sigma$$

for s on the horizontal paths of $I_{\pm 1}$. Thus

$$I_{\pm 1} \ll \tau^{-2} e^{|\varkappa|+2\alpha}(\log t_0) \int_0^1 \log 2/\sigma \, d\sigma \ll e^{|\varkappa|+2\alpha}(\log t_0)^{-3}. \tag{14}$$

On the vertical paths of $I_{\pm 2}$ we have $\log \zeta(s_0 + s) \ll 1$, so that

$$I_{\pm 2} \ll e^{|\varkappa|+2\alpha} \int_\tau^\infty t^{-2} \, dt \ll e^{|\varkappa|+2\alpha}(\log t_0)^{-2}. \tag{15}$$

Now (14) and (15) give (13).

Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the zeta function $\zeta(s)$ for which $\beta \geq \sigma$, $0 \leq \gamma \leq T$.

LEMMA 5. *For* $T \geq 10$, $\frac{1}{2} \leq \sigma \leq 1$,

$$N(\sigma, T) \ll T^{3/2-\sigma}(\log T)^5.$$

Proof. This is weaker than the classical estimate

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\alpha)}(\log T)^5$$

of Ingham [5]; see Titchmarsh [14; Theorem 9.19] or Montgomery [9, Theorem 12.1].

3. Proof. of Theorem 1

Let $\alpha = \frac{1}{2}$, and take successively $\varkappa = -\log x$, $\varkappa = 0$, $\varkappa = \log x$, where $x \geq 1$. In the first two cases the sum in (13) is empty. For the three respective values of $\varkappa$ we multiply (13) by $\frac{1}{2}e^{-i\theta}$, 1, $\frac{1}{2}e^{i\theta}$, and sum, to find that

$$\frac{2}{\pi} \int_{-\tau}^{\tau} \log \zeta(s_0 + it)\left(\frac{\sin t/2}{t}\right)^2 (1 + \cos(\theta + t \log x)) \, dt$$

$$\frac{1}{2}e^{i\theta} \sum_{|\log n/x| \leq 1/2} \Lambda_1(n) n^{-s_0}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) + O(x(\log t_0)^{-2}), \tag{16}$$

provided that $\zeta(s) \neq 0$ for

$$\sigma \geq \sigma_0, \qquad |t - t_0| \leq 2(\log t_0)^2. \tag{17}$$

We shall determine t_0 and x so that

$$x \leq (\log t_0)^2, \tag{18}$$

so that $\zeta(s) \neq 0$ in the region (17), and so that

$$\mathrm{Re} \sum_{|\log n/x| \leq 1/2} \Lambda_1(n) n^{-s_0} \left(\frac{1}{2} - \left| \log \frac{n}{x} \right| \right) \geq (\tfrac{1}{8} + o(1)) x^{1-\sigma_0}/\log x. \tag{19}$$

Now

$$\frac{2}{\pi} \int_{-\tau}^{\tau} \left(\frac{\sin t/2}{t} \right)^2 (1 + \cos(\theta + t \log x))\, dt \leq \frac{4}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin t/2}{t} \right)^2 dt = 1,$$

and the first integrand is non-negative, so it then follows from (16) that there is a t_1, $|t_1 - t_0| \leq 2(\log t_0)^2$, for which

$$\mathrm{Re}\, e^{-i\theta} \log \zeta(\sigma_0 + it_1) \geq (\tfrac{1}{8} + o(1)) x^{1-\sigma_0}/\log x. \tag{20}$$

If $\|(2\pi)^{-1} t_0 \log n\| \leq \tfrac{1}{6}$ for all n in (19) for which $\Lambda_1(n) > 0$ then $\mathrm{Re}\, n^{-it_0} \geq \tfrac{1}{2}$, and we obtain (19), since by the prime number theorem

$$\sum_{\log n/x| \leq 1/2} \Lambda_1(n) n^{-\sigma_0} \left(\frac{1}{2} - \left| \log \frac{n}{x} \right| \right) = \left[\left(\frac{2 \sinh(1-\sigma_0)/4}{1-\sigma_0} \right)^2 + o(1) \right] x^{1-\sigma_0}/\log x$$

$$\geq (\tfrac{1}{4} + o(1)) x^{1-\sigma_0}/\log x.$$

Let $T_1 = T^{(\sigma_0 - 1/2)/3}$, $\theta_n = (1/2\pi) T_1 \log n$; we wish to have

$$\|r\theta_n\| \leq \tfrac{1}{6} \qquad \left(\Lambda(n) > 0, \left| \log \frac{n}{x} \right| \leq \tfrac{1}{2} \right), \tag{21}$$

with $1 \leq r \leq = [T/T_1]$. For such r we put $t_0 = rT_1$, so that $T_1 \leq t_0 \leq T$. We appeal to Lemma 3 with

$$\delta = \tfrac{1}{6}, \quad M = \Pi(xe^{1/2}) - \Pi(xe^{-1/2}) \sim c_1 x/\log x,$$

$$c_1 = 2 \sinh \tfrac{1}{2} = 1.042 \cdots. \text{ Take now}$$

$$x = \tfrac{1}{3} c_1^{-1} (\sigma_0 - \tfrac{1}{2})(\log_6 T)(\log \log T);$$

note that this value of x in (20) gives (3), and that (18) is satisfied for all $t_0 = rT_1$. Moreover, this x gives $M \sim \frac{1}{3}(\sigma_0 - \frac{1}{2}) \log_6 T$, so by Lemma 3 we have at least $T^{(4-2\sigma_0-\varepsilon)/3}$ solutions of (21). For distinct $t_0 = rT_1$ we see that the regions (17) are disjoint; thus by Lemma 5, $\zeta(s)$ vanishes at some point in (17) for at most $T^{3/2-\sigma_0+\varepsilon}$ values of r. But $\frac{3}{2} - \sigma_0 + \varepsilon < (4 - 2\sigma_0 - \varepsilon)/3$, so for suitable $t_0 = rT_1$ we have (18), (19), and $\zeta(s) \neq 0$ in (17).

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