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Extreme values of the Riemann zeta function

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1. Introduction

In 1928, Titchmarsh [13] (see Titchmarsh [14; Theorem 8.12], Ramachandra [10]) proved that for fixed σ, $\frac{1}{2}\le\sigma<1$,

$$\log|\zeta(\sigma+it)|=\Omega_+((\log t)^{1-\sigma-\varepsilon}) \tag{1}$$

as t tends to infinity. Previous estimates of this sort had only been established assuming the Riemann Hypothesis (see Landau [6], Bohr–Landau [2], Littlewood [8]). Recently Levinson [7] sharpened (1) by showing that

$$\max_{1\le t\le T}\log|\zeta(\sigma+it)|>c(\log T)^{1-\sigma}/\log\log T \tag{2}$$

for $\frac{1}{2}\le\sigma<1$, $T\ge 10$. For $\frac{1}{2}<\sigma<1$ we give a sharper lower bound than is provided by (2), we show that $|\zeta(s)|$ becomes correspondingly small, and that $\arg\zeta(s)$ becomes correspondingly large in both signs. We write $s=\sigma+it$.

THEOREM 1. *Let $\frac{1}{2}<\sigma_0<1$, $T>T_0(\sigma_0)$. For any real θ there is a t_0 such that $T^{(\sigma_0-1/2)/3}\le t_0\le T$, and*

$$\operatorname{Re} e^{-i\theta}\log\zeta(s_0)\ge\frac{1}{20}\left(\sigma_0-\frac{1}{2}\right)^{1/2}(\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0}. \tag{3}$$

Taking $\theta=0$, π, $\pm\pi/2$ in the above, we derive the following

COROLLARY. *Let σ be fixed, $\frac{1}{2}<\sigma<1$. Then as t tends to infinity,*

$$\log|\zeta(s)|=\Omega_+((\log t)^{1-\sigma}(\log\log t)^{-\sigma}), \tag{4}$$

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and

$$\arg \zeta(s) = \Omega_{\pm}((\log t)^{1-\sigma}(\log\log t)^{-\sigma}). \tag{5}$$

In establishing Theorem 1, we actually construct more than $T^{(4/3)-(2\sigma_0-\varepsilon/3)}$ numbers t_0 having the required properties. This is necessitated by our method, since we must construct more candidates for t_0 than there are "bad zeros" of $\zeta(s)$. If we assume the Riemann Hypothesis (RH) then we have no problems with "bad zeros," and in place of (3) we find that if $\frac{1}{2} \le \sigma_0 < 1$ then there is a t_0, $T^{1/6} \le t_0 \le T$, such that

$$\operatorname{Re} e^{-i\theta} \log \zeta(s_0) \ge \frac{1}{20} (\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0}. \tag{6}$$

In particular, we obtain

THEOREM 2. *Suppose that the Riemann Hypothesis is true. Then*

$$|\zeta(\tfrac{1}{2}+it)| = \Omega\left(\exp\left(\frac{1}{20}(\log t/\log\log t)^{1/2}\right)\right) \tag{7}$$

and

$$S(t) = \Omega_{\pm}((\log t/\log\log t)^{1/2}) \tag{8}$$

as t tends to infinity, where $S(t) = 1/\pi \arg \zeta(\frac{1}{2}+it)$.

The best known unconditional estimate corresponding to (7) is obtained by taking $\sigma = \frac{1}{2}$ in (2). With regard to (8), Selberg [12] has shown unconditionally that

$$S(t) = \Omega_{\pm}((\log t)^{1/3}(\log\log t)^{-7/3}).$$

Selberg [11] remarks that the Bohr–Landau approach, when fully exploited, yields (assuming RH)

$$S(t) = \Omega_{\pm}((\log t)^{1/2}/\log\log t).$$

The small refinement that (8) gives of this is of interest because our estimates are "likely" to be best possible. To argue that this is the case, suppose that σ is fixed,

$\frac{1}{2}<\sigma<1$, and that RH holds. Then for $T\leq t\leq 2T$ we have

$$\log \zeta(s)=\sum_{p\leq T} p^{-s}+0(1)$$

$$=F(t)+0(1),$$

say. As the numbers p^{-it} have a random appearance, we consider

$$G(\boldsymbol{\theta})=\sum_{p\leq T} p^{-\sigma}e(\theta_p),$$

where the θ_p are independent, $0\leq\theta_p\leq 1$, and $e(\theta)=e^{2\pi i\theta}$. We can *prove* that with great uniformity the distribution function of $F(t)$, for $T\leq t\leq 2T$, is very close to that of $G(\boldsymbol{\theta})$, for $\boldsymbol{\theta}\in\mathbf{T}^{\pi(T)}$. Let us choose "at random" T points $\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T$. Then we find that the probability that

$$\max_{1\leq t\leq T}|G(\boldsymbol{\theta}_t)|>C(\log T)^{1-\sigma}(\log\log T)^{-\sigma}$$

is less than T^{-A}, where A becomes large with C. The rapid convergence of the series $\sum 2^{-An}$ leads us to conjecture that

$$|\log \zeta(s)|\ll_\sigma (\log t)^{1-\sigma}(\log\log t)^{-\sigma} \tag{9}$$

for $\frac{1}{2}<\sigma<1$. The situation in Theorem 2 is more delicate, but we still suggest that for some $c>0$ and all $t\geq 10$,

$$|\zeta(s)|<\exp\left(c(\log t/\log\log t)^{1/2}\right), \tag{10}$$

$$S(t)\ll(\log t/\log\log t)^{1/2}. \tag{11}$$

The estimate (8) was announced previously by the author [9; p. 123]; on that occasion a connection was exhibited between the size of $S(t,\chi)$ and the size of least character non-residues. We repeat the observation that our method can be used to show (assuming RH) that

$$S(t+1/\log\log t)-S(t)=\Omega_\pm((\log t/\log\log t)^{1/2}).$$

Assuming RH, Selberg (unpublished) has argued from (16) that if $(\log T)^{-1}\leq h\leq$

$(\log\log T)^{-1}$ then

$$\sup_{T\le t\le 2T} (S(t+h)-S(t)) \ge c(h\log T)^{1/2},$$

$$\inf_{T\le t\le 2T} (S(t+h)-S(t)) \le -c(h\log T)^{1/2}.$$

Selberg's approach differs from ours in that he uses high moments to make his expression large; Dirichlet's theorem does not suffice.

Over a period of years the author has been pleased to have many stimulating discussions with Professor Atle Selberg concerning the subject of this paper.

2. Basic lemmas

We require a modified form of Dirichlet's theorem on simultaneous approximation, which we derive in a standard manner from

LEMMA 1. *Let $\mathscr{C}$ be a convex body in $\mathbf{R}^m$, symmetric about the origin. If $\mu(\mathscr{C})>2^m K$ then $\mathscr{C}$ contains at least K distinct pairs of non-zero lattice points $\pm\mathbf{u}_k$, $1\le k\le K$.*

This form of Minkowski's first main theorem is due to van der Corput [4]; see Cassels [3; Chapter III, Theorem II].

LEMMA 2. *Let $\theta_1,\ldots,\theta_M$* be arbitrary real numbers, and suppose that $0<\delta<\frac{1}{2}$. *There are at least $[\delta^M(R+1)]$ integers r such that $1\le r\le R$ and $\|r\theta_m\|\le\delta$ for $1\le m\le M$.*

Here $\|\theta\|$ denotes the distance from θ to the nearest integer, $\|\theta\|=\min|\theta-n|$.

Proof. Let $\mathscr{C}_\varepsilon\subset\mathbf{R}^{M+1}$ be defined as follows: For $\varepsilon\ge 0$ let $\mathbf{x}\in\mathscr{C}_\varepsilon$ if and only if $|\mathbf{x}^{(0)}|<(1+\varepsilon)^{-1}(R+1)$, $|\mathbf{x}^{(n)}-\theta_m\mathbf{x}^{(0)}|\le(1+\varepsilon)\delta$ for $1\le m\le M$. Thus

$$\mu(\mathscr{C}_\varepsilon)=(1+\varepsilon)^M\delta^M 2^M(R+1),$$

so that if $K=[\delta^M(R+1)]$ and $\varepsilon>0$ then by Lemma 1 we find that $\mathscr{C}_\varepsilon$ contains K distinct pairs of non-zero lattice points. Letting ε tend to 0, we find that $\mathscr{C}_0$ also contains such points, say $\pm\mathbf{x}_k$, $1\le k\le K$. The condition $\delta<\frac{1}{2}$ ensures that the leading coordinates $\pm\mathbf{u}_k^{(0)}$ are distinct and non-zero; we let r take on the positive values of these leading coordinates.

One can derive Lemma 2 analytically by the method of Blanksby–Montgomery [1; §6].

LEMMA 3. *Suppose that the Dirichlet series $f(s)=\sum a_n n^{-s}$ is absolutely convergent for $\sigma>\sigma_a$. Suppose also that $\alpha>0$ and that $\varkappa$ is real. Then for $c>\sigma_a$,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 e^{\varkappa s}\,ds=\sum_{w} a_n w_n, \tag{12}$$

where $w_n=w_n(\alpha,\varkappa)=\max(0,\alpha-|\varkappa-\log n|)$.

In earlier works on this subject, identities such as (12) have been employed with α large and $\varkappa=0$. To some extent our ability to achieve sharper results is due to the fact that we take $\alpha\approx 1$, and $\varkappa$ large.

Proof. In the integral we replace f by its defining sum; we invert the order of summation and integration, and then it suffices to note that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 e^{\nu s}\,ds=\frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin\alpha t}{t}\right)^2 e^{i\nu t}\,dt=\max(0,\alpha-|\nu|),$$

where $\nu=\varkappa-\log n$.

LEMMA 4. *Suppose that $\frac{1}{2}\le\sigma_0<1$, $t_0\ge 15$, and that $\zeta(s)\ne 0$ for $\sigma>\sigma_0$, $|t-t_0|\le 2\tau$, where $\tau=\tau(t_0)=(\log t_0)^2$. Then for $\alpha>0$, and real $\varkappa$,*

$$\frac{2}{\pi}\int_{-\tau}^{\tau}\log\zeta(s_0+it)\left(\frac{\sin\alpha t}{t}\right)^2 e^{i\varkappa t}\,dt=\sum\Lambda_1(n)w_n n^{-s_0}+O(e^{|\varkappa|+2\alpha}(\log t_0)^{-2}), \tag{13}$$

where $\Lambda_1(n)=\Lambda(n)/\log n$.

Proof. We take $f(s)=\log\zeta(s_0+s)=\sum\Lambda_1(n)n^{-s_0-s}$ in Lemma 3. We move the path of integration in (12) to lie on the five line segments determined by the *points* $1-i\infty$, $1-i\tau$, $-i\tau$, $i\tau$, $1+i\tau$, $1+i\infty$. Thus our integral is the sum of five integrals I_j, $|j|\le 2$, each over the corresponding segment. The left hand side of (13) is I_0, so to establish (2) we have only to bound $I_{\pm1}$, $I_{\pm2}$. From familiar estimates (Titchmarsh [14; Theorem 9.6(B)] we deduce that

$$\log\zeta(s_0+s)\ll(\log t_0)\log 2/\sigma$$

for s on the horizontal paths of $I_{\pm 1}$. Thus

$$I_{\pm 1} \ll \tau^{-2} e^{|\varkappa|+2\alpha}(\log t_0) \int_0^1 \log 2/\sigma \, d\sigma \ll e^{|\varkappa|+2\alpha}(\log t_0)^{-3}. \tag{14}$$

On the vertical paths of $I_{\pm 2}$ we have $\log \zeta(s_0 + s) \ll 1$, so that

$$I_{\pm 2} \ll e^{|\varkappa|+2\alpha} \int_T^\infty t^{-2} \, dt \ll e^{|\varkappa|+2\alpha}(\log t_0)^{-2}. \tag{15}$$

Now (14) and (15) give (13).

Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the zeta function $\zeta(s)$ for which $\beta \geq \sigma$, $0 \leq \gamma \leq T$.

LEMMA 5. *For* $T \geq 10$, $\frac{1}{2} \leq \sigma \leq 1$,

$$N(\sigma, T) \ll T^{3/2-\sigma}(\log T)^5.$$

Proof. This is weaker than the classical estimate

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\alpha)}(\log T)^5$$

of Ingham [5]; see Titchmarsh [14; Theorem 9.19] or Montgomery [9, Theorem 12.1].

3. Proof. of Theorem 1

Let $\alpha = \frac{1}{2}$, and take successively $\varkappa = -\log x$, $\varkappa = 0$, $\varkappa = \log x$, where $x \geq 1$. In the first two cases the sum in (13) is empty. For the three respective values of $\varkappa$ we multiply (13) by $\frac{1}{2}e^{-i\theta}$, 1, $\frac{1}{2}e^{i\theta}$, and sum, to find that

$$\frac{2}{\pi} \int_{-T}^{T} \log \zeta(s_0 + it)\left(\frac{\sin t/2}{t}\right)^2 (1 + \cos(\theta + t \log x) \, dt$$
$$\tfrac{1}{2}e^{i\theta} \sum_{|\log n/x| \leq 1/2} \Lambda_1(n) n^{-s_0}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) + O(x(\log t_0)^{-2}), \tag{16}$$

provided that $\zeta(s) \neq 0$ for

$$\sigma \geq \sigma_0, \qquad |t - t_0| \leq 2(\log t_0)^2. \tag{17}$$

We shall determine t_0 and x so that

$$x \le (\log t_0)^2, \tag{18}$$

so that $\zeta(s) \neq 0$ in the region (17), and so that

$$\operatorname{Re} \sum_{|\log n/x| \le 1/2} \Lambda_1(n) n^{-s_0}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) \ge (\tfrac{1}{8} + o(1)) x^{1-\sigma_0}/\log x. \tag{19}$$

Now

$$\frac{2}{\pi}\int_{-T}^{T} \left(\frac{\sin t/2}{t}\right)^2 (1 + \cos(\theta + t \log x))\, dt \le \frac{4}{\pi}\int_{-\infty}^{+\infty} \left(\frac{\sin t/2}{t}\right)^2 dt = 1,$$

and the first integrand is non-negative, so it then follows from (16) that there is a t_1, $|t_1 - t_0| \le 2(\log t_0)^2$, for which

$$\operatorname{Re} e^{-i\theta} \log \zeta(\sigma_0 + it_1) \ge (\tfrac{1}{8} + o(1)) x^{1-\sigma_0}/\log x. \tag{20}$$

If $\|(2\pi)^{-1} t_0 \log n\| \le \frac{1}{6}$ for all n in (19) for which $\Lambda_1(n) > 0$ then $\operatorname{Re} n^{-it_0} \ge \frac{1}{2}$, and we obtain (19), since by the prime number theorem

$$\sum_{\log n/x| \le 1/2} \Lambda_1(n) n^{-\sigma_0}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) = \left[\left(\frac{2 \sinh (1-\sigma_0)/4}{1-\sigma_0}\right)^2 + o(1)\right] x^{1-\sigma_0}/\log x$$
$$\ge (\tfrac{1}{4} + o(1)) x^{1-\sigma_0}/\log x.$$

Let $T_1 = T^{(\sigma_0 - 1/2)/3}$, $\theta_n = (1/2\pi) T_1 \log n$; we wish to have

$$\|r\theta_n\| \le \tfrac{1}{6} \qquad \left(\Lambda(n) > 0, \left|\log \frac{n}{x}\right| \le \tfrac{1}{2}\right), \tag{21}$$

with $1 \le r \le = [T/T_1]$. For such r we put $t_0 = rT_1$, so that $T_1 \le t_0 \leqq T$. We appeal to Lemma 3 with

$$\delta = \tfrac{1}{6},\ M = \Pi(xe^{1/2}) - \Pi(xe^{-1/2}) \sim c_1 x/\log x,$$

$c_1 = 2 \sinh \frac{1}{2} = 1.042 \cdots$. Take now

$$x = \tfrac{1}{3} c_1^{-1} (\sigma_0 - \tfrac{1}{2})(\log_6 T)(\log \log T);$$

note that this value of x in (20) gives (3), and that (18) is satisfied for all $t_0 = rT_1$. Moreover, this x gives $M \sim \frac{1}{3}(\sigma_0 - \frac{1}{2}) \log_6 T$, so by Lemma 3 we have at least $T^{(4-2\sigma_0-\varepsilon)/3}$ solutions of (21). For distinct $t_0 = rT_1$ we see that the regions (17) are disjoint; thus by Lemma 5, $\zeta(s)$ vanishes at some point in (17) for at most $T^{3/2-\sigma_0+\varepsilon}$ values of r. But $\frac{3}{2} - \sigma_0 + \varepsilon < (4 - 2\sigma_0 - \varepsilon)/3$, so for suitable $t_0 = rT_1$ we have (18), (19), and $\zeta(s) \neq 0$ in (17).

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