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Autor: Ruscheweyh, Stephan

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Linear operators between classes of prestarlike functions

STEPHAN RUSCHEWEYH

1. Introduction

Let $\mathcal{A}$ denote the set of analytic functions $f(z)$ in the unit disc $\Delta := \{z \mid |z|<1\}$ normalized by $f(0)=0$, $f'(0 \neq 0$. A function $f \in \mathcal{A}$ is called starlike of order α, $\alpha \leq 1$, if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha, \qquad z \in \Delta,$$

and $\mathcal{S}_\alpha$ stands for the collection of these functions. Furthermore let

$$\frac{z}{(1-z)^{2(1-\alpha)}} = \sum_{k=1}^{\infty} \gamma(\alpha, k) z^k, \qquad \alpha \leq 1.$$

In a recent paper T. J. Suffridge [12] proved the following remarkable results.

THEOREM A. *If* $\alpha \leq 1$ *and*

$$f(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) b_k z^k$$

are starlike of order α *then the same is true for*

$$h(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) a_k b_k z^k.$$

THEOREM B. *Let* $\alpha \leq \beta \leq 1$ *and* $\sum_{k=1}^{\infty} \gamma(\alpha, k) a_k z^k \in \mathcal{S}_\alpha$. *Then* $\sum_{k=1}^{\infty} \gamma(\beta, k) a_k z^k \in \mathcal{S}_\beta$.

These theorems contain many previous results, in particular the well known Pólya–Schoenberg conjecture (the case $\alpha = 0$ of Theorem A, see [2], [3]). They also provide a useful instrument to solve certain extremal problems for analytic functions in Δ (see [7], [8], [9]).

Let $\mathscr{C}_\alpha$, $\alpha \le 1$, be the class of functions $f \in \mathscr{A}$ close-to-convex of type α, i.e. it exists $g \in \mathscr{S}_\alpha$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \qquad z \in \Delta.$$

In [3] the case $\alpha = 0$ of Theorem C was established, which is of similar character but not contained in Theorems A, B.

THEOREM C. *If* $\alpha \le 1$ *and*

$$f(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) a_k z^k \in \mathscr{C}_\alpha, \qquad g(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) b_k z^k \in \mathscr{S}_\alpha$$

then

$$h(z) = \sum_{k=1}^{\infty} \gamma(\alpha, k) a_k b_k z^k \in \mathscr{C}_\alpha.$$

For $\alpha \neq 0$ Theorem C appears to be unknown.

Two other results of different type are also of great importance in this context.

THEOREM D. ([3]). *Let* $f \in \mathscr{S}_{1/2}$. *Then for every* $z_0 \in \Delta$

$$\operatorname{Re} \frac{z_0}{f(z_0)} \frac{f(z) - f(z_0)}{z - z_0} > \frac{1}{2}, \qquad z \in \Delta.$$

THEOREM E. (Suffridge [13]). *Let* $f \in \mathscr{A}$ *be univalent in* Δ *with* $f(\Delta)$ *convex. Then for every* $z_0 \in \Delta$

$$z \frac{f(z) - f(z_0)}{z - z_0} \in \mathscr{S}_{1/2}.$$

In the present paper we deal with the classes $\mathscr{R}_\alpha \subset \mathscr{A}$ of prestarlike functions of order α. A function $f \in \mathcal{A}$ is called *prestarlike of order* α, $\alpha \le 1$, if and only if[1]

$$\begin{cases} \operatorname{Re} \dfrac{f(z)}{zf'(0)} > \dfrac{1}{2}, & z \in \Delta, \quad \text{for} \quad \alpha = 1, \\ \dfrac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}_\alpha, & \text{for} \quad \alpha < 1. \end{cases}$$

In this notation (slightly extended versions of) Theorems A, B, C can be stated as:

THEOREM A′. *For $f, g \in \mathcal{R}_\alpha$, $\alpha \le 1$, we have $f * g \in \mathcal{R}_\alpha$.*

THEOREM B′. *For $\alpha \le \beta \le 1$ we have $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.*

THEOREM C′. *For $f \in \mathcal{C}_\alpha$, $g \in \mathcal{R}_\alpha$, $\alpha \le 1$, we have $f * g \in \mathcal{C}_\alpha$.*

Since $\mathcal{R}_{1/2} = \mathcal{S}_{1/2}$ and $\mathcal{R}_0 = \mathcal{K}$, the class of univalent functions $f \in \mathcal{A}$ with $f(\Delta)$ convex, Theorems D, E are contained in the following more general result (also appearing to be new).

THEOREM D′. *Let $f \in \mathcal{R}_\alpha$, $\alpha \le \frac{1}{2}$. Then for every $z_0 \in \Delta$ we have*

$$z \frac{f(z) - f(z_0)}{z - z_0} \in \mathcal{R}_{\alpha + 1/2}.$$

Prestarlike functions (in a different parametrization) have already been considered in [5] and Theorems A′, B′, D′ correspond to the conjectures (i), (ii), (iii) of that paper. All of these results and Theorem C′ are contained in our Main Theorem.

MAIN THEOREM. *For $\alpha \le \beta \le 1$ let $p(z) \in \mathcal{S}_{1+\alpha-\beta}$ be analytic in $\bar{\Delta}$. With $g \in \mathcal{R}_\beta$ let* **T** *be the continuous linear operator*

$$f \mapsto (\mathbf{T}f)(z) := \left[g(yz) \frac{p(y)}{y} *_y f(y) \right] \Bigg|_{y=1}$$

acting on $\mathcal{A}$. Then $\mathbf{T} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$.

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1 $*$ denotes the Hadamard product or convolution of two analytic functions in Δ. If it is necessary to indicate the acting variable we shall write $*_z$, $*_y$ etc.

Note that the choices

$$\begin{aligned}&p(y)=y, \qquad \alpha=\beta,\\ &p(y)=y, \qquad g(y)=y/(1-y),\\ &p(y)=y/(1-z_0y)\in\mathcal{S}_{1/2}, \qquad g(y)=y/(1-y), \qquad \beta=\alpha+\tfrac{1}{2}.\end{aligned}$$

give Theorems A′, B′, D′ respectively. Theorem C′ is an obvious consequence of the following Corollary to the Main Theorem (merely a reformulation of the case $\beta=1$).

COROLLARY 1. *For $\alpha\leq 1$ let $f\in\mathcal{R}_\alpha$, $p\in\mathcal{S}_\alpha$. Let $F(z)$ be analytic and with positive real part in Δ. Then*

$$\operatorname{Re}\frac{f(z)*(p(z)F(z))}{f(z)*p(z)}>0, \qquad z\in\Delta.$$

Remarks. (1) The cases $\alpha=0, \frac{1}{2}$ of Corollary 1 are in [3].

(2) In the notation of T. Sheil-Small ([10], [11]) Corollary 1 is equivalent to the fact that

$$\Lambda F:=\frac{f*pF}{f*p}$$

is a *convexity preserving generalized convolution operator.* For the very interesting properties of these operators we refer to his work.

As a further application of Corollary 1 we wish to point out the particular relevance of prestarlike functions to certain classes of univalent functions in $\mathcal{A}$. Let $\alpha>0$, $\beta\in\mathbb{R}$. A function $F\in\mathcal{A}$ represented by

$$F(z)=\left[\int_0^z h(t)g^\alpha(t)t^{i\beta-1}\,dt\right]^{1/(\alpha+i\beta)}$$

with $g\in\mathcal{S}_0$ and h analytic with positive real part in Δ is called *Bazilevič function of type* (α,β). These functions are known to be univalent in Δ (see [1]).

THEOREM. *Let $F(z)$ be a Bazilevič function of type (α,β) and let $f\in\mathcal{R}_{1-\alpha}$.*

Then

$$z\left[\left(\frac{F(z)}{z}\right)^{\alpha+i\beta} * \frac{f(z)}{z}\right]^{1/(\alpha+i\beta)}$$

is a Bazilevič function of type (α, β).

For a method to deduce the Theorem from Corollary 1 we refer to [4] where the case $\alpha = 1$ has been considered.

An intrinsic definition of prestarlike functions is given by

LEMMA 1. *$f \in \mathscr{A}$ is prestarlike of order $\alpha \leq 1$ if and only if*

$$f(z) *_z z\frac{1-xz}{(1-z)^{3-2\alpha}} \neq 0, \qquad 0<|z|<1, \qquad |x|=1. \tag{1.1}$$

We shall show that the Main Theorem is a consequence of

LEMMA 2. *Let $f \in \mathscr{A}$ be prestarlike of order α, $\alpha \leq 1$, and let $p \in \mathscr{S}_\alpha$. Then*

$$f(z) *_z \frac{1-xz}{1-yz} p(z) \neq 0, \qquad 0<|z|<1, \qquad |x|\leq 1, \qquad |y|\leq 1. \tag{1.2}$$

We have two alternate proofs for Lemma 2. The first one, which we present in Section 2, makes use of Suffridge's Theorem B. The second one is direct and elementary but less concise. We shall give a short outline of it in Section 3.

2. Proof of the Main Theorem

In the sequel we write $\dot{\Delta}$ for $\Delta\backslash\{0\}$.

Proof of Lemma 1. Let $f \in \mathscr{A}$ satisfy (1.1) and put

$$g(z) = \frac{z}{(1-z)^{2(1-\alpha)}} * f(z).$$

The case $x=1$ gives $g(z)\neq 0$, $z\in\dot{\Delta}$, and for $x\neq 1$ we can rewrite (1.1):

$$G(z):=\frac{\dfrac{z}{(1-z)^{3-2\alpha}}*f(z)}{\dfrac{z}{(1-z)^{2-2\alpha}}*f(z)}\neq\frac{x}{x-1},\qquad |x|=1,\qquad x\neq 1,\qquad z\in\Delta.$$

This is equivalent to

$$\operatorname{Re} G(z)>\tfrac{1}{2},\qquad z\in\Delta. \tag{2.1}$$

For $\alpha=1$ the conclusion follows. In the case $\alpha<1$ one uses the identity

$$\frac{z}{(1-z)^{3-2\alpha}}=\frac{z}{(1-z)^{2-2\alpha}}*\left[\frac{1-2\alpha}{2-2\alpha}\frac{z}{1-z}+\frac{1}{2-2\alpha}\frac{z}{(1-z)^2}\right] \tag{2.2}$$

to obtain

$$\operatorname{Re}\left[\frac{1-2\alpha}{2-2\alpha}+\frac{1}{2-2\alpha}\frac{zg'(z)}{g(z)}\right]>\frac{1}{2} \tag{2.3}$$

and thus $g\in\mathcal{S}_\alpha$. All steps are invertible and the Lemma is settled.

Remarks. (3) By the principle of subordination we deduce from (2.1) for $z\in\Delta$

$$G(yz)\neq\frac{x}{x-y},\qquad x\in\bar{\Delta},\qquad y\in\bar{\Delta}\backslash\{0\},$$

and hence for $f\in\mathcal{R}_\alpha$

$$f(z)*z\frac{1-xz}{(1-yz)^{3-2\alpha}}\neq 0,\qquad z\in\dot{\Delta},\qquad x,y\in\bar{\Delta}. \tag{2.4}$$

(4) From (1.1) it is easily seen that Theorem B ($\mathcal{R}_\alpha\subset\mathcal{R}_\beta$ for $\alpha\leq\beta<1$ extends to $\beta=1$ and thus to Theorem B'.

(5) It should be noted that the subclass of functions $f\in\mathcal{R}_\alpha$ satisfying $f'(0)=1$ is a compact family (in the sense of compact convergence in Δ).

In the sequel we shall use Hadamard products with respect to two different variables. The following trivial fact should be kept in mind: if $f(z)$, $g(w)$ and

$F(z, w)$ are analytic in $z \in \Delta$, $w \in \Delta$ and $(z, w) \in \Delta^2$ respectively then

$$f(z) *_z [F(z, w) *_w g(w))] = [f(z) *_z F(z, w)] *_w g(w).$$

Proof of Lemma 2. Let

$$\mathcal{S}_\alpha(0) := \{f(z) = z\},$$
$$\mathcal{S}_\alpha(n) := \left\{ f \in \mathcal{A} \,\middle|\, f(z) = z \prod_{k=1}^{n} (1 - z_k z)^{-\alpha_k}, |z_k| < 1, \alpha_k \geq 0, \sum_{k=1}^{n} \alpha_k = 2(1-\alpha) \right\}.$$

It is well known that $\bigcup_{n \in N} \mathcal{S}_\alpha(n)$ is dense in $\mathcal{S}_\alpha \cap \{f \in \mathcal{A} \mid f'(0) = 1\}$. Therefore it will be enough to establish Lemma 2 for the members of $\mathcal{S}_\alpha(n)$, $n = 0, 1, \ldots$, which will be done by induction.

For $p \in \mathcal{S}_\alpha(0)$ (1.2) follows from Theorem B$'$: $\mathcal{R}_\alpha \subset \mathcal{R}_1$, and from (2.4). Now assume (1.2) is valid for arbitrary $p \in \mathcal{S}_\alpha(n-1)$, $\alpha \leq 1$. We wish to prove (1.2) for

$$p(z) = z \prod_{k=1}^{n} (1 - z_k z)^{-\alpha_k}, \qquad |z| < 1, \qquad \alpha_k \geq 0, \qquad \sum_{k=1}^{n} \alpha_k = 2(1-\alpha).$$

Let

$$a(\tau) = \frac{\tau + z_n}{1 + \bar{z}_n \tau}, \qquad b(\tau) = \frac{\tau - \bar{z}_n}{1 - z_n \tau}$$

be automorphisms of $\bar{\Delta}$. The function

$$\tilde{p}(w) := w \prod_{k=1}^{n-1} (1 + a(-z_k) w)^{-\alpha_k}$$

is a member of $\mathcal{S}_{\alpha + \alpha_n/2}(n-1)$ and satisfies

$$z \tilde{p}(b(z))(1 - z_n z)^{2(\alpha - 1)} = c b(z) p(z), \qquad z \in \Delta, \tag{2.5}$$

where

$$c = -\frac{1}{\bar{z}_n} \tilde{p}(-\bar{z}_n) \neq 0$$

is a constant.

For fixed $w \in \dot{\Delta}$, $\eta \in \bar{\Delta}$, $\zeta \in \bar{\Delta}$ and $f \in \mathscr{R}_\alpha$ we have from (2.4)

$$zw \frac{1-a(\eta w)z}{(1-a(\zeta w)z)^{3-2\alpha}} *_z f(z) \neq 0, \qquad z \in \dot{\Delta}. \tag{2.6}$$

We use the relation

$$1-a(\tau)z = \frac{1-z_n z}{1+\bar{z}_n \tau}(1-b(z)\tau) \tag{2.7}$$

to rearrange (2.6). After multiplication with a nonvanishing factor we get

$$H(z, w) *_z f(z) \neq 0, \qquad z \in \dot{\Delta}, \qquad w \in \dot{\Delta},$$

with

$$\begin{aligned} H(z, w) &= zw(1-z_n z)^{2(\alpha-1)} \frac{1-\eta w b(z)}{(1-\zeta w b(z))^{3-2\alpha}} \\ &= w \frac{1-\eta w}{(1-\zeta w)^{3-2\alpha}} *_w \frac{zw(1-z_n z)^{2(\alpha-1)}}{1-wb(z)}. \end{aligned}$$

Hence for fixed $z \in \dot{\Delta}$

$$w \frac{1-\eta w}{(1-\zeta w)^{3-2\alpha}} *_w \tilde{f}(w) \neq 0, \qquad w \in \dot{\Delta}, \qquad \eta \in \bar{\Delta}, \qquad \zeta \in \bar{\Delta},$$

where

$$\tilde{f}(w) = \frac{zw(1-z_n z)^{2(\alpha-1)}}{1-wb(z)} *_z f(z). \tag{2.8}$$

(2.4) with $x = y = z_n$ shows

$$\tilde{f}'(0) = \frac{z}{(1-z_n z)^{2(1-\alpha)}} *_z f(z) \neq 0$$

and we conclude $\tilde{f} \in \mathscr{R}_\alpha$ by Lemma 1. From Theorem B′ we get $\tilde{f} \in \mathscr{R}_{\alpha+\alpha_n/2}$ and thus

$$\frac{1-\eta w}{1-\zeta w}\tilde{p}(w) *_w \tilde{f}(w) \neq 0, \qquad w \in \dot{\Delta}, \qquad \eta \in \bar{\Delta}, \qquad \zeta \in \bar{\Delta},$$

by the assumption of induction. (2.8) leads to

$$z\frac{1-\eta wb(z)}{1-\zeta wb(z)}\frac{\tilde{p}(wb(z))}{b(z)}(1-z_n z)^{2(\alpha-1)} *_z f(z) \neq 0$$

and applying (2.7) one obtains

$$z\frac{1-a(\eta w)z}{1-a(\zeta w)z}\frac{\tilde{p}(wb(z))}{b(z)}(1-z_n z)^{2(\alpha-1)} *_z f(z) \neq 0, \qquad z \in \dot{\Delta}. \tag{2.9}$$

(2.9) remains valid when $w \to 1$ and from (2.5) we get

$$\frac{1-a(\eta)z}{1-a(\zeta)z}p(z) *_z f(z) \neq 0, \qquad z \in \dot{\Delta}, \qquad \eta \in \bar{\Delta}, \qquad \zeta \in \bar{\Delta},$$

which is our assertion.

Proof of the Main Theorem. (1) It follows from [3, Lemma 2.4] that Lemma 2 implies Corollary 1: For $f \in \mathcal{R}_\alpha$, $p \in \mathcal{S}_\alpha$ and F analytic with positive real part in Δ we have

$$\operatorname{Re}\frac{f*(pF)}{f*p} > 0, \qquad z \in \Delta. \tag{2.10}$$

This can be used to prove Theorem A′ which is well known for $\alpha = 1$. In fact, assume f, $g \in \mathcal{R}_\alpha$, $\alpha < 1$. Then

$$p = \frac{z}{(1-z)^{2(1-\alpha)}} * g \in \mathcal{S}_\alpha$$

and from (2.10) we deduce

$$\operatorname{Re}\frac{z(f*p)'}{f*p} = \alpha + \operatorname{Re}\frac{f*\left\{p\left[\frac{zp'}{p}-\alpha\right]\right\}}{f*p} > \alpha, \qquad z \in \Delta.$$

Hence $f*p \in \mathcal{S}_\alpha$ and $f*g \in \mathcal{R}_\alpha$.

(2) Let $\alpha \le \beta \le 1$ and $p(z) \in \mathcal{S}_{1+\alpha-\beta}$ be analytic in $\bar{\Delta}$. For every $y \in \Delta$ we then have

$$p_y(z) := \frac{p(z)}{(1-yz)^{2(1-\beta)}} \in \mathcal{S}_\alpha$$

and Lemma 2 gives for $f \in \mathcal{R}_\alpha$

$$y \frac{1-xyz}{(1-yz)^{3-2\beta}} p(z) *_z f(z) \neq 0, \qquad z \in \dot{\Delta}, \qquad x \in \bar{\Delta}, \qquad y \in \dot{\Delta}.$$

But

$$y \frac{1-xyz}{(1-yz)^{3-2\beta}} p(z) = y \frac{1-xy}{(1-y)^{3-2\beta}} *_y \frac{y}{1-yz} p(z)$$

and hence for every fixed $z \in \Delta$

$$y \frac{1-xy}{(1-y)^{3-2\beta}} *_y h_z(y) \neq 0, \qquad y \in \dot{\Delta}, \qquad x \in \bar{\Delta}, \tag{2.11}$$

where

$$h_z(y) = \frac{y}{1-yz} p(z) *_z f(z).$$

Since $h_z'(0) \neq 0$ (2.11) shows $h_z(y) \in \mathcal{R}_\beta$. Now let $g \in \mathcal{R}_\beta$. By Part (1) we conclude

$$H_z(y) := g(y) *_y h_z(y) = g(yz) \frac{p(z)}{z} *_z f(z) \in \mathcal{R}_\beta$$

for $z \in \Delta$. Since

$$\lim_{z \to 1} H_z'(0) = \lim_{z \to 1} [g'(0) p(z) * f(z)] \neq 0$$

the compactness property mentioned in Remark 5 shows

$$\lim_{z \to 1} H_z(y) = (\mathbf{T}f)(y) \in \mathcal{R}_\beta.$$

3. An alternate proof of Lemma 2

We extend the definition of starlike and prestarlike functions of order α to $1<\alpha<\frac{3}{2}$: $f\in\mathcal{A}$ is starlike of order α, $1<\alpha<\frac{3}{2}$, if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)}<\alpha, \qquad z\in\Delta.$$

$f\in\mathcal{A}$ is prestarlike of order α, $1<\alpha<\frac{3}{2}$, if and only if $z/(1-z)^{2(1-\alpha)}*f(z)$ is starlike of order α.

By means of elementary methods one can prove

LEMMA 3. *Let* $\alpha<2$, $\alpha\neq 1$. *If* $g\in\mathcal{A}$ *satisfies* $\alpha g(z)-zg'(z)\in\mathcal{S}_{\alpha/2}$ *then* $g\in\mathcal{S}_{(\alpha+1)/2}$.

An application of Lemma 1 and the identity (2.2) gives

$$\mathcal{R}_\alpha\subset\mathcal{R}_{\alpha+1/2}, \qquad \alpha<1. \tag{3.1}$$

If one applies the ideas of the first proof of Lemma 2 but uses (3.1) instead of Theorem B′ one obtains for $f\in\mathcal{R}_{(3-\alpha-k)/2}$ with

$$0<\alpha\leq 1, \qquad k\in\mathbb{N}:$$

$$\frac{1-xz}{1-yz}\frac{z}{(1-\epsilon z)^\alpha P_{k-1}(z)}*_z f(z)\neq 0 \tag{3.2}$$

for arbitrary $z\in\dot{\Delta}$, $x\in\bar{\Delta}$, $y\in\bar{\Delta}$, $\epsilon\in\bar{\Delta}$ and $P_{k-1}\in\pi_{k-1}$. π_n, $n=0,1,\ldots$, denotes the set of polynomials of degree $\leq n$ which are nonvanishing in Δ.

By [3, Lemma 2.4] we deduce

$$\operatorname{Re}\frac{f*\dfrac{zF(z)}{(1-\epsilon z)^\alpha P_{k-1}(z)}}{f*\dfrac{z}{(1-\epsilon z)^\alpha P_{k-1}(z)}}>0, \qquad z\in\Delta, \tag{3.3}$$

for an arbitrary F analytic and with positive real part in Δ. Since the denominator of the function in (3.3) is nonvanishing in Δ the same holds for the numerator and hence

$$f(z) * g(z) \neq 0, \qquad z \in \dot{\Delta}, \tag{3.4}$$

for every $g \in \mathscr{A}$ with the following property: there exist $\epsilon \in \bar{\Delta}$, $P_{k-1}(z) \in \pi_{k-1}$ such that

$$\operatorname{Re}\left[(1-\epsilon z)^{\alpha} P_{k-1}(z) \frac{g(z)}{z}\right] > 0, \qquad z \in \Delta. \tag{3.5}$$

The following result is in [6].

LEMMA 4. *Let $q(z) \in \mathscr{S}_{(2-k)/2}$, $k \in \mathbb{N}$, and let $t_0 \in \mathbb{R}$. Then there exists a polynomial $P_{k-1} \in \pi_{k-1}$ such that*

$$\operatorname{Re}\left[(1-e^{it_0}z) P_{k-1}(z) \frac{q(z)}{z}\right] > 0, \qquad z \in \Delta.$$

Now assume $p \in \mathscr{S}_{(3-\alpha-k)/2}$ and let $|x|=1$, $y \in \bar{\Delta}$. We then have

$$q(z) = \frac{p(z)}{(1-yz)^{1-\alpha}} \in \mathscr{S}_{(2-k)/2},$$

to which we apply Lemma 4. With $x = e^{it_0}$ we obtain $P_{k-1} \in \pi_{k-1}$ such that

$$\operatorname{Re}\left[(1-yz)^{\alpha} P_{k-1}(z) \frac{1-xz}{1-yz} \frac{p(z)}{z}\right] > 0, \qquad z \in \Delta.$$

Hence (3.5) is satisfied for

$$g(z) = \frac{1-xz}{1-yz} p(z)$$

and (3.4) settles Lemma 2 for $|x|=1$. That this can be extended to $|x| \leq 1$ is a simple exercise.

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Mathematisches Institut
der Universität Würzburg
Am Hubland
D-8700 Würzburg
Federal Republic of Germany

Current Address:
P.O. Box 13
Kabul 1
Afghanistan

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