Relative vanishing theorems I: applications to ample divisors.

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Introduction

We present in this paper various vanishing theorems connected with positivity, and we also give one application to ample divisors to prove a generalization of one of the main results of [11].

More precisely, in paragraph 1 we present a generalization of Kodaira's vanishing theorem along the lines of [2] and [3] and an Enriques–Severi–Zariski type corollary. In an appendix, a sharpened version of the vanishing theorem of Grauert–Riemenschneider, [3], is also given.

In paragraph 2 we give one application to showing that certain holomorphic maps from an ample divisor extend to the ambient space.

We would like to thank D. C. Spencer for making available his notes on Griffiths' paper, [4].

§I

1.1. In this section we present a very general form of Kodaira's precise vanishing theorem. It is a combination of Grauert–Riemenschneider [3] and of Andreotti–Vesentini [2, cf. also 8]. It has as a corollary a general form of a theorem of Enriques–Severi–Zariski, (see Serre, [9]).

We will first recall the relevant facts about the generalized canonical sheaf of Grauert–Riemenschneider, [3].

1.2. Let X be a reduced and irreducible analytic space and let $p:\tilde{X}\to X$ be a desingularization. Then $\mathcal{K}_X=p_*\mathcal{O}(K_{\tilde{X}})$ is the generalized canonical sheaf of Grauert and Riemenschneider where $K_{\tilde{X}}$ is the canonical bundle of $\tilde{X}$. It is easy to show[1] $\mathcal{K}_X$ doesn't depend on $\tilde{X}$. One simply notes first that given a second desingularization $p':X'\to X$, there exists a manifold Z and maps q and q' such that

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "q'"'] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "p'"] & X \end{tikzcd}$$

[1] By Hironaka's theorem.

commutes, and q and q' are modifications and, second, $q_*\mathcal{O}(K_Z)\cong\mathcal{O}(K_{\tilde X})$ and $q_*\mathcal{O}(K_Z)\cong\mathcal{O}(K_{X'})$. Also $\mathcal{R}^i_{p_*}\mathcal{O}(K_{\tilde X})=0$ for $i>0$. Grauert and Riemenschneider prove this latter fact only for Kaehler $\tilde X$; we prove it for general $\tilde X$ in an appendix to this section.

1.3 DEFINITION. Let $\mathbf{L}$, $\mathbf{L}=(\pi: L\to X)$ be an holomorphic line bundle on a reduced analytic space X and let $p: X\to S$ be a proper map onto a reduced analytic space S. $\mathbf{L}$ is said to be semipositive (semiample, ample) relative to p if given any point $s\in S$ there exists a neighborhood $U(s)$ such that there is an Hermitian metric on L over $p^{-1}(U(s))$ whose curvature form is positive semi-definite (positive semi-definite and positive definite at one point of each irreducible component of $p^{-1}(U(s))$, positive definite everywhere).[(2)]

We can prove now the Kodaira's type vanishing theorem and its Enriques–Severi–Zariski–Serre type corollary:

1.4. PROPOSITION. *Let $p: X\to S$ be a proper map from a normal irreducible analytic space X onto a reduced analytic space S. Assume given any point s of S, there exists a neighborhood $U(s)$ such that $p^{-1}(U(s))$ has a Kaehler desingularization. Let $\mathbf{L}$ be an holomorphic line bundle on X that is semiample relative to p. Let $\mathcal{F}$ be a coherent sheaf on S; then $\mathcal{R}^j_{p_*}(X, p^*\mathcal{F}\otimes\mathcal{O}(\mathbf{L})\otimes\mathcal{K}_X)=0$ for $j>0$.*

Proof. Following [11, pg. 60] we immediately reduce to showing $\mathcal{R}^j_{p_*}(X, \mathcal{O}(\mathbf{L})\otimes\mathcal{K}_X)=0$ for $j>0$ where S is Stein and there exists a Kaehler manifold $\tilde X$ such that $q:\tilde X\to X$ is a desingularization. Now one sees $\mathcal{R}^j_{(p\circ q)_*}(\tilde X, \mathcal{O}(K_{\tilde X}\otimes p^*\mathbf{L}))=0$ for $j>0$, and $\mathcal{R}^j_{q_*}(\tilde X, \mathcal{O}(K_{\tilde X}\otimes p^*\mathbf{L}))=0$ for $j>0$. (Indeed the second part follows from [11, Lemma II-A]. To prove the first, along the same lines as the above mentioned lemma, for every $s\in S$, replace S by an open Stein neighborhood U of s, since the statement is local with respect to S. It suffices to show that given a class $\eta\in H^j(\tilde U, \mathcal{O}(K_{\tilde X}\otimes p^*\mathbf{L}))$, where $\tilde U=(p\circ q)^{-1}(U)$, then the image of η, by restriction, in $H^j(\tilde V, \mathcal{O}(K_{\tilde X}\otimes p^*\mathbf{L}))$ is zero, where V is some neighborhood of s contained in U and $\tilde V=(p\circ q)^{-1}(V)$. By assumption, if U is small enough, one has an Hermitian norm $\|\ \|$ on $\mathbf{L}$ such that

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[2] If the Hermitian metric is positive definite everywhere, the definition of ample relatively to p coincides with the one given in [6]. It will be also understood from now on that to give an Hermitian metric on E, E being the total space of a vector bundle $\mathbf{E}=(\pi: E\to X)$ of rank r on the analytic space X, means the following: let $\mathcal{U}=\{V_i\}$ be a covering of X trivializing for $\mathbf{E}$ and $\{g_{ij}\}$ be the transition matrices attached to $\mathcal{U}$; an hermitian metric is then a collection $\{h_i\}$ of $r\times r$ differentiable positive definite matrices, h_i defined in V_i, that has to transform under the law $h_j={}^t\bar g_{ij}h_i g_{ij}$ in $V_i\cap V_j$ in the following sense: if $\mathcal{U}$ is chosen consisting of small enough open subsets, then for every i V_i can be realized as an analytic subset of an open subset G of $\mathbf{C}^n$. We require then that for every i there exists a positive definite, differentiable $r\times r$ matrix $\hat h_i$ on G, such that $h_i=\hat h_i|V_i$. If $r=1$, the curvature form of the metric $\{h_i\}$ is then $-\partial\bar\partial\log\hat h_i|V_i$.

its curvature form induces a Kaehler metric on $\tilde{U}$. If φ is a plurisubharmonic function on some $V \subset U$, the Kaehler metric obtained from the curvature form $e^{-\varphi}\,\|\;\|$ is complete and the representative of η in $H^j(\tilde{V}, (K_{\tilde{X}} \otimes p^*\mathbf{L}|_{\tilde{V}})$ has finite L^2 norm with respect to $e^{-\varphi}\,\|\;\|$ and the associated complete Kaehler metric. One will be done then by applying Lemma A of [10]). Now noting that $q_*\mathcal{O}(K_{\tilde{X}} \otimes p^*\mathbf{L}) \approx \mathcal{K}_X \otimes \mathcal{O}(\mathbf{L})$ one is done by means of the Grothendieck spectral sequence for the composition of the two functors p_* and q_*. Q.E.D.

1.5. COROLLARY. *Let p, X,* $\mathbf{L}$*, Y, and* $\mathcal{F}$ *be as above. If* $\dim_{\mathbf{C}} X - \dim_{\mathbf{C}} S > 1$ *then* $\mathcal{R}^j_{p_*}(X, p^*\mathcal{F} \otimes \mathcal{O}(\mathbf{L}^{-1})) = 0$ *for* $j = 0$ *and* 1. *If X is also a manifold then* $\mathcal{R}^j_{p_*}(X, p^*\mathcal{F} \otimes \mathbf{L}^{-1}) = 0$ *for* $j < \dim_{\mathbf{C}} X - \dim_{\mathbf{C}} S$.

Proof. Again one follows page 60 of [11] and reduces to showing that for $\mathbf{L}$ semiample one has $H^j(X, \mathcal{O}(\mathbf{L}^{-1})) = 0$ for appropriate j and S Stein. By Serre duality when X is a manifold this reduces to showing that the cohomology group with compact supports $H^j_c(X, \mathcal{O}(\mathbf{L} \otimes K_X)) = 0$ for $j > \dim_{\mathbf{C}} S$. By the Leray spectral sequence we must only show $H^{j-r}_c(S, \mathcal{R}^r_{p_*}(X, \mathbf{L} \otimes K_X)) = 0$ for $r \geq j - \dim_{\mathbf{C}} S > 0$. This is clear by (1.4).

Now in the case X is normal one reduces to showing $H^j(X, \mathcal{O}(\mathbf{L}^{-1})) = 0$ for $j = 0$ and 1 when S a Stein space and $\mathbf{L}$ is semiample. Let $q : \tilde{X} \to X$ be the hypothesized Kaehler desingularization of X. One has $H^j(\tilde{X}, q^*\mathbf{L}^{-1}) = 0$ for $j = 0$ and 1 by the last paragraph. Now note $q_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and thus $q_*(q^*\mathcal{O}(\mathbf{L}^{-1}) \approx \mathcal{O}(\mathbf{L}^{-1})$ for a normal space X. Thus $H^0(X, \mathbf{L}^{-1}) \approx H^0(\tilde{X}, q^*\mathbf{L}^{-1}) = 0$. By the edge exact sequence $0 \to H^1(X, \mathbf{L}^{-1}) \to H^1(\tilde{X}, q^*\mathbf{L}^{-1})$ and the theorem is proved. Q.E.D.

1.5. It should be noted that by using the reduction to S a Stein space one can prove a more refined vanishing theorem if one uses the profondeur of $\mathcal{R}^0_{p_*}(X, \mathbf{L} \otimes K_X)$. Also in the case of normal X one can prove a more refined result if one knows that some of the $\mathcal{R}^j_{q_*}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $q : \tilde{X} \to X$ a Kaehler desingularization, e.g. if X has only rational singularities (i.e. $\mathcal{R}^j_{q_*}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $j > 0$). The vanishing result for X is then the same as for the manifold case. Without some such conditions the proposition is clearly false. Let us give an example with S a point. Let $\mathbf{L}$ be an ample line bundle on a product $R_1 \times R_2$ of Riemann surfaces of genus $g > 0$. Let $\tilde{X}$ be the projectivization of $\mathbf{C} \oplus \mathbf{L}$ where $\mathbf{C}$ is the trivial bundle. One has $p : \tilde{X} \to R_1 \times R_2$ where $\tilde{X}$ is $\mathbf{L}$ plus a copy of $R_1 \times R_2$ at infinity. Blow down the copy of $R_1 \times R_2$ at infinity to get the normal space X with $q : \tilde{X} \to X$ a desingularization. Now let $\mathbf{E}$ be any ample line bundle on X. Clearly $\mathcal{R}^1_{q_*}(\tilde{X}, \mathcal{O}(\mathbf{E}^{-1})) \approx \mathcal{R}^1_{q_*}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \otimes \mathcal{O}(\mathbf{E}^{-1})$ has a nontrivial stalk supported at the singular point. Thus by the Leray spectral sequence and the above result for manifolds one has $H^2(X, \mathbf{E}^{-1}) \neq 0$. Q.E.D.

Appendix to §I

A.1. The following lemma allows the properties of the generalized canonical sheaf of Grauert and Riemenschneider [3] to be developed in full generality.

A.2. LEMMA. *Let $p: X \to Y$ be a proper, generically finite to one holomorphic map from a complex connected manifold X onto a reduced analytic space Y. Let* L *be an holomorphic line bundle on X that is semi-positive relative to p. Then, if K_X denotes the canonical bundle of X:*

$$\mathcal{R}^j_{p_*}(\mathbf{L} \otimes K_X) = 0 \quad \text{for} \quad j > 0.$$

Proof: Since the theorem is local on Y one can assume Y is Stein and $\mathbf{L}$ is semi-ample. Now let $p = s \circ r$ be the Remmert–Stein factorization of p where $s: S \to Y$ is a finite to one holomorphic proper map, S is normal and $r: X \to S$ is a proper holomorphic surjection with connected fibres. Since $\mathcal{R}^q_{s_*}(\mathcal{F}) = 0$ for all $q > 0$ and any coherent sheaf $\mathcal{F}$ on one immediately sees one can replace Y by S and p by r; i.e. one can assume Y is a normal Stein space and $p: X \to Y$ is a proper bimeromorphic morphism with connected fibers. By Hironaka's basic result [5, Cor. 2] one can find an analytic space X' and a proper projective bimeromorphic surjection $q: X' \to Y$ and a holomorphic proper surjection $\lambda: X' \to X$ such that $p \circ \lambda = q$. Now by shrinking Y one can assume X' is embedded in $Y \times \mathbf{C}P^N$ for some N. Now there exists a proper modification $\mu: Z \to Y \times \mathbf{C}P^N$ with Z a complex manifold obtained from $Y \times \mathbf{C}P^N$ by a locally finite sequence of monoidal transformations with nonsingular centers such that there is a submanifold $X'' \subset Z$ with $\mu|_{X''}: X'' \to X'$ a proper bimeromorphic surjection.(3) By shrinking Y the sequence of monoidal transformations is finite. It is easy to check that a monoidal transformation with smooth center of a holomorphically convex Kaehler manifold is a holomorphically convex Kaehler manifold. Thus Z and hence X'' is a Kaehler manifold. Thus we have the commutative diagram with $\varphi = \lambda \circ \mu$:

$$\begin{tikzcd}
X'' \arrow[r, "\mu"] \arrow[dr, "\phi"'] & X' \arrow[r, "q"] \arrow[d, "\lambda"] & Y \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

Now $\tilde{\mathbf{L}} = \mu^*(\lambda^*\mathbf{L})$ is clearly semi-ample. Thus a direct application of [11, Lemma II-A] and the proof of [10, Lemma A] shows $\mathcal{R}^t_{\gamma_*}(K_{X''} \otimes \tilde{L}) = 0$ for $t > 0$ and γ is either φ or $p \circ \varphi$. Further since X'' and X are manifolds and φ is a proper

[3] This method is implicit in the discussion of [5, Cor. 3].

bimeromorphic surjection one has $\varphi_*(K_{X''}\otimes\tilde{\mathbf{L}})=K_X\otimes\mathbf{L}$. Now one has the Grothendieck–Leray spectral sequence

$$\bigoplus_{\alpha+\beta=\nu}\mathcal{R}^{\alpha}_{p*}(\mathcal{R}^{\beta}_{\varphi*}(\tilde{\mathbf{L}}\otimes K_{X''}))\Rightarrow\mathcal{R}^{\nu}_{p\circ\varphi*}(\tilde{\mathbf{L}}\otimes K_{X''}).$$

By the last few lines this gives $\mathcal{R}^{\nu}_{p*}(\mathbf{L}\otimes K_X)=0$ if $\nu>0$. Q.E.D.

§II

2.1. In this section we prove generalizations of a number of results of Sommese [11]. Let us recall that a divisor A in the complex analytic space[4] X is ample if the line bundle $[A]$ defined by A is ample.

We use the Andreotti–Frankel version [1] of Lefschetz theorem:

2.2. LEFSCHETZ THEOREM: *Let A be an ample divisor of a connected projective variety X. If $X-A$ is a manifold, then:*

$$H^i(A,\mathbf{Z})\approx H^i(X,\mathbf{Z})\quad\text{for}\quad i<\dim_{\mathbf{C}}A$$

and $0\to H^i(X,\mathbf{Z})\to H^i(A,\mathbf{Z})$ *and the cokernel has no torsion for* $i=\dim_{\mathbf{C}}A$.

The following lemma is a generalization of [11, Lemma I-B].

2,3. LEMMA. *Let A be a Cartier divisor in an analytic space X and let $\mathcal{F}$ be a locally free sheaf on X. If $H^i(A,\ \mathcal{F}|_A\otimes\mathcal{O}([A]|_A^{-N}))=0$ for all $N>0$ and if $H^i(X,\ \mathcal{F}\otimes\mathcal{O}([A]^{-M}))=0$ for $M\gg0$, then $H^i(X,\mathcal{F}\otimes\mathcal{O}([A]^{-M}))=0$ for all $M>0$.*

Proof. Consider:

$$0\to\mathcal{F}\otimes\mathcal{O}([A]^{-M-1})\to\mathcal{F}\otimes\mathcal{O}([A]^{-M})\to\mathcal{F}|_A\otimes\mathcal{O}([A]|_A^{-M})\to0$$

The hypotheses translates into:

$$H^i(X,\mathcal{F}\otimes\mathcal{O}([A]^{-M-1}))\to H^i(X,\mathcal{F}\otimes\mathcal{O}([A]^{-M}))\to0$$

for all $M>0$, and the first group is zero for $M\gg0$. Q.E.D.

2.4. PROPOSITION: *Let $p:A\to Y$ be an holomorphic surjection of a compact normal connected analytic space onto a projective variety. Assume that A is a Cartier divisor in a reduced analytic space X and that $[A]|_A$ is ample. Assume*

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[4] Every complex analytic space throughout this section is supposed reduced.

Pic $(X) \to$ Pic $(A) \to 0$. *Then if* $\dim_{\mathbf{C}} A \geq 2 + \dim_{\mathbf{C}} Y$, *p extends to an holomorphic map of a neighborhood U of A. If in addition X is compact,* $[A]$ *is ample, and* $X - A$ *is a manifold, then p extends holomorphically to X.*

Proof. Following the proof of [11, Proposition III] one shows there is a line bundle $\mathscr{L}$ in X such that the map ϕ associated to $H^0(A, \mathscr{L}|_A)$ is p composed with an embedding of Y into some $\mathbf{C}P^N$. To get extension of p one first shows $H^0(X, \mathscr{L}) \to H^0(A, \mathscr{L}|_A) \to 0$ and that the map associated to $H^0(X, \mathscr{L})$ has image ϕ and is holomorphic.

Note that if one shows $H^0(U, \mathscr{L}|_U) \to H^0(A, \mathscr{L}|_A) \to 0$ for some neighborhood U of A, then one gets the desired result for compact X. This is because $X - A$ is Stein and any holomorphic section of $\mathscr{L}$ on $U - A$ extends to $X - A$ by Hartog's theorem. The rest follows from the proof of [11, Proposition III].

So we have reduced to the case of a neighborhood U of A. By (1.5), [4], and (2.3) one has $H^0(U, \mathscr{L}) \to H^0(A, \mathscr{L}|_A) \to 0$ for some U. Thus the map $\phi : A \to \mathbf{C}P^N$ extends to an map $\tilde{\phi} : U \to \mathbf{C}P^N$ for some U. We must merely show $\tilde{\phi}(U) = \phi(A)$. Assume otherwise, then given a point x of $\phi(A)$, one can find a neighborhood V of x and a holomorphic function f in V zero on $V \cap \phi(A)$ but not on $V \cap \tilde{\phi}(U)$. f must give rise to some nontrivial section of $([A]|_{\phi^{-1}(V \cap \phi(A))})^{-r}$ for some r. But this has no sections by (1.5). Q.E.D.

2.5. It is an immediate consequence of (2.2) that $\mathrm{Pic}(X) \to \mathrm{Pic}(A)$ has a cokernel without torsion when $\dim_{\mathbf{C}} A \geq 2$ and A is a manifold that is an ample divisor in a projective manifold X. This allows to extend some of the results of [11]. §V had the blanket assumption that $\dim_{\mathbf{C}} X \geq 3$; this can now be changed to $\dim_{\mathbf{C}} X \geq 2$. One has:

2.6. *Let $\mathscr{A}$ be a contiguity class generated by the given manifold X. If X is such that* $H^i(X, [q]_X) = 0$ *for* $0 < i < \dim_{\mathbf{C}} X$ *and all q, then for each* $Y \in \mathscr{A}$ *one has* $H^i(X, [q]_Y) = 0$ *for* $0 < i < \dim_{\mathbf{C}} Y$ *and all q.* Also the Corollary [11, p. 72] becomes: *If* $n \geq 2$, *then* $\mathbf{C}P^n$ *can only be an ample divisor in* $\mathbf{C}P^{n+1}$, *and a nonsingular quadric,* $\mathscr{Q}^n$, *can only be an ample divisor in* $\mathbf{C}P^{n+1}$ *or* $\mathscr{Q}^{n+1}$. (The proof of this corollary can be simplified by the use of a result of Kobayashi and Ochiai [7].)

2.7. The following question has been asked in [11], p. 62: Let A be a projective manifold such that the cotangent bundle of A is the direct sum of two nontrivial sub-bundles. If A is an ample divisor in some projective manifold, then must A be $\mathbf{CP}^1 \times \mathbf{CP}^1$? The answer is no, in general, and the claim that it was true if A were two dimensional and very ample is false.

Indeed let R be any Riemann surface of genus g. Let L be a very ample line bundle of degree $>g-1$. Then $L\oplus L$ has a trivial sub-bundle $\mathbf{C}$ and an ample quotient bundle F of degree $>2g-2$. Thus F is very ample. Consider the sequence obtained by direct summing copies of F,

$$0\to\mathbf{C}\to L\oplus L\oplus\underbrace{F\oplus F\oplus F\cdots}_{n\text{ copies}}\to\underbrace{F\oplus\cdots}_{n+1\text{ copies}}\to 0$$

Then the projectivization of the dual of the last vector bundle is an ample divisor. It is clearly biholomorphic to $R\times\mathbf{C}P^n$ and one can check it is very ample.

(Takao Fujita in a personal communication has informed A. J. Sommese that he also has shown $R\times\mathbf{C}P^n$ can be an ample divisor, he has also made some progress on the blowing down problem in appendix I, of [11], and he has produced a fiber bundle that is an ample divisor and does not have $\mathbf{C}P^N$ as a fiber.)

Added in Proof. A particular case of (1.4) has been proved by G. R. Kempf, Inv. Math. *37*, (1976), p. 236.

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