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Flat manifolds with non-zero Euler characteristics

By JOHN SMILLIE

A flat structure on a smooth vector bundle ξ can be given either by a connection on ξ with zero curvature or by a reduction of the structure group of ξ from $GL(\mathbb{R}^n)$ to $GL(\mathbb{R}^n)$ with the discrete topology. Using either definition we see that if ξ and ξ' admit flat structures and f is a map then $\xi + \xi'$ and $f^*\xi$ admit flat structures. A flat structure on a manifold M is a flat structure on $T(M)$. We say a manifold is flat if it admits a flat structure.

Let M_g denote the surface of genus g for $g \geq 1$. Milnor (1) proves that an oriented bundle ξ over M_g admits a flat structure if and only if

$$|e(\xi)[M_g]| \leq -\frac{1}{2}\chi(M_g) \quad (1)$$

where $e(\xi)$ is the Euler class of ξ and $\chi(M)$ is the Euler characteristic of M . If M_g is a flat manifold this formula implies $\chi(M_g) = 0$. According to Hirsch-Thurston (2) it is an old conjecture that the Euler characteristic of any flat manifold is zero. The question is raised by Milnor (1) and Kamber-Tondeur (3, p. 47). In this paper we prove the following.

THEOREM. *There are flat manifolds M^{2n} with non-zero Euler characteristic for all $n > 1$.*

Starting with surfaces we construct manifolds M^{2n} by taking products and connected sums. To prove the existence of a flat structure on $T(M^{2n})$ we establish that $T(M^{2n})$ is isomorphic to a second bundle ζ . ζ admits a flat structure because it is the sum of pullbacks of 2-plane bundles which admit flat structures. A bundle isomorphism allows us to transfer the flat structure from ζ to $T(M)$.

ALMOST TRIVIAL BUNDLES. We will call a bundle ξ^n almost trivial if $\xi^n + \varepsilon^1 = \varepsilon^{n+1}$ where ε^n denotes the trivial bundle of dimension n .^{*} Sums and pullbacks of almost trivial bundles are almost trivial. If $T(M)$ is almost trivial then M is said to be almost parallelizable.

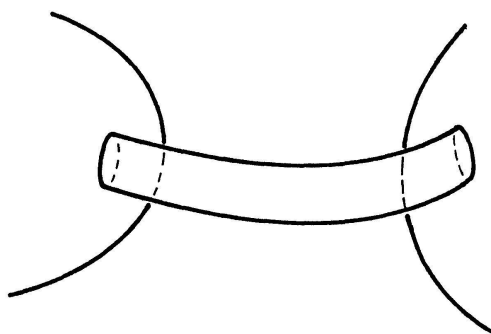
^{*} If the base space of ξ^n is n -dimensional then ξ^n is almost trivial if and only if ξ^n is stably trivial.

PROPOSITION 2. *An oriented 2-plane bundle ξ over a surface M_g is almost trivial if $e(\xi)[M_g]$ is even.*

Proof. $\xi + \varepsilon^1$ admits a two-frame over M_g if a certain obstruction class $\mathcal{O}_2(\xi + \varepsilon^1)$ vanishes. $\mathcal{O}_2$ can be identified with $w_2(\xi + \varepsilon^1) = w_2(\xi)$ which vanishes when $e(\xi)[M_g]$ is even. Assuming $\mathcal{O}_2 = 0$, $\xi + \varepsilon^1 = \eta + \varepsilon^2$ but $w_1(\eta) = w_1(\xi) = 0$ so $\eta = \varepsilon^1$.

PROPOSITION 3. *If n dimensional manifolds N and M are almost parallelizable then $N \# M$ is almost parallelizable.*

Proof. We can immerse N and M in $\mathbb{R}^{n+1}$. We can connect them by a tube. Smoothing the corners we see that we have constructed an immersion of $N \# M$ in $\mathbb{R}^{n+1}$ hence $N \# M$ is almost parallelizable.



LEMMA. *If ξ is an almost trivial bundle over M then there is a map $g: M \rightarrow S^n$ so that $\xi = g^*T(S^n)$.*

Proof. Choose a bundle map $f: \xi + \varepsilon^1 \rightarrow \mathbb{R}^{n+1}$. We have a non-zero section $s: M \rightarrow \xi + \varepsilon^1$ associated to ε^1 . Using a Gram-Schmidt procedure we may assume that $fs(x)$ is of unit length and perpendicular to $f(\xi_x)$.

$$\begin{array}{ccc} \xi & \xrightarrow{fsp \times f|_{\xi}} & S^n \times \mathbb{R}^{n+1} \\ \downarrow p & & \downarrow \pi \\ M & \xrightarrow{fs} & S^n \end{array}$$

Set $g = fs$. ξ is the pullback by g of the sub-bundle $T(S^n) \subset S^n \times \mathbb{R}^{n+1}$ consisting of pairs (x, y) where y is perpendicular to x .

PROPOSITION 4. *Let η_1 and η_2 be almost trivial oriented $2n$ -plane bundles over an oriented $2n$ -manifold. If $e(\eta_1) = e(\eta_2)$ then η_1 and η_2 are isomorphic.*

Proof. The lemma yields maps g_i such that $\eta_i = g_i^*T(S^{2n})$. $e(\eta_i) = 2g_i^*u$ where $u \in H^{2n}(S^{2n}; \mathbb{Z})$ is dual to the orientation class. The homotopy class of $g_i: M^{2n} \rightarrow S^{2n}$ is determined by the cohomology class g_i^*u hence g_1 and g_2 are homotopic. The isomorphism type of $g_i^*T(S^{2n})$ depends only on the homotopy type of g_1 hence η_1 and η_2 are isomorphic.

PROOF OF THEOREM. We construct a flat 4-manifold N with $\chi(N)=4$ and a flat 6-manifold Q with $\chi(Q)=8$. Products of these two manifolds with themselves give examples of flat manifolds with non-zero Euler characteristics of any even dimension greater than 2.

Let P be a parallelizable 4-manifold such as $S^1 \times S^3$. Let $N = (M_3 \times M_3) \# \underbrace{P \# \cdots \# P}_{6\text{-copies}}$ where M_3 is the surface of genus 3. Clearly $M_3 \times M_3$ is almost parallelizable. N is almost parallelizable by proposition 3. We have the formulas

$$\chi(M_g) = 2 - 2g$$

$$\chi(M^{2n} \# N^{2n}) = \chi(M^{2n}) + \chi(N^{2n}) - 2.$$

We calculate that $\chi(N)=4$.

Let ξ be the oriented bundle over M_3 with $e(\xi)[M_3]=2$. Proposition 2 implies that ξ is almost trivial. $\xi \times \xi$ is a bundle over $M_3 \times M_3$. Let $f: N \rightarrow M_3$ be a degree one map that, for example, sends the six P summands to a point. $f^*\xi \times \xi$ is almost trivial and $e(f^*\xi \times \xi)[N]=4$. $T(N)$ is almost trivial and $e(T(N))[N]=\chi(N)=4$. It follows from proposition 4 that $T(N)$ and $f^*\xi \times \xi$ are isomorphic. Milnor's formula implies that ξ admits a flat structure hence $f^*\xi \times \xi$ admits a flat structure. Choosing an isomorphism we transfer the flat structure from $f^*\xi \times \xi$ to $T(N)$.

Let Q be the 6-manifold $\{(M_3 \times M_3) \# \underbrace{P \# \cdots \# P}_{9\text{-copies}}\} \times M_3$. We calculate that

$\chi(Q)=8$. We have a flat bundle $\eta = \{h^*(\xi \times \xi)\} \times \xi$ over Q where h is the appropriate degree one map. $e(\eta)[Q]=8$. $T(Q)$ and η are almost trivial and have the same Euler class. By proposition 4 they are isomorphic hence $T(Q)$ admits a flat structure.

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