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Groups with cyclic Sylow subgroups and finiteness conditions for certain complexes

G. MISLIN

Introduction

Let π denote a finite group of order n whose Sylow subgroups are all cyclic and let $N = \sum x \in \mathbf{Z}\pi$, $x \in \pi$, denote the norm element. The augmentation $\mathbf{Z}\pi \rightarrow \mathbf{Z}$ induces a map $j: \mathbf{Z}\pi/N \rightarrow \mathbf{Z}/n$ which we use to consider \mathbf{Z}/n as a $\mathbf{Z}\pi/N$ -module. We show (Theorem 1.3) that

$$\text{proj. dim}_{\mathbf{Z}\pi/N} (\mathbf{Z}/n) < \infty.$$

Thus there is a *transfer map*

$$j^*: K_0(\mathbf{Z}/n) \rightarrow K_0(\mathbf{Z}\pi/N)$$

between projective class groups. It turns out that $\text{im}(j^*) \subset \text{im}(pr_*: K_0(\mathbf{Z}\pi) \rightarrow K_0(\mathbf{Z}\pi/N))$ and, since $\text{im}(K_0(\mathbf{Z}\pi) \rightarrow K_0(\mathbf{Z}\pi/N)) \cong (K_0\mathbf{Z}\pi)/\text{im } S$ where $S: K_1(\mathbf{Z}/n) \rightarrow K_0(\mathbf{Z}\pi)$ denotes the *Swan homomorphism* (cf. Section 2), we can think of the transfer map to map $K_0(\mathbf{Z}/n)$ into $(K_0\mathbf{Z}\pi)/\text{im } S$. If we compose this map with the obvious homomorphism

$$u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \rightarrow K_0(\mathbf{Z}/n)$$

we obtain a “transfer” homomorphism

$$T: u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \rightarrow (K_0\mathbf{Z}\pi)/\text{im } S$$

($u(\mathbf{Z}[1/n])$ denotes the group of units of $\mathbf{Z}[1/n]$). The homomorphism T is in general non-trivial, even if π is cyclic (in which case $\text{im } S = 0$). However, we

show that $T = 0$ if n is a prime power or if $n = 2p$, p an odd prime (cf. Theorem 2.5).

In the second half of the paper we make use of the homomorphism T to compute the Wall obstruction $wX \in K_0\mathbb{Z}\pi_1 X$ for certain complexes. We will consider spaces X for which $\pi_1 X$ operates nilpotently on $H_*\tilde{X}$ (i.e. X is homologically nilpotent in the sense of Brown–Kahn [3]). If such a space is dominated by a finite complex and has a finite fundamental group of order n , then the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X}) / \text{card } H_{\text{ev}}(X, \tilde{X})$$

is well defined and is a unit in $\mathbb{Z}[1/n]$; $\rho(X)$ is related to the finiteness obstruction wX in the following way, (cf. Theorem 3.3).

THEOREM I. *Let X be a finitely dominated homologically nilpotent space with non-zero finite fundamental group of square free order. Then*

$$T\rho(X) = \bar{w}X$$

where $\bar{w}X$ denotes the image of wX in $(K_0\mathbb{Z}\pi_1 X)/\text{im } S$.

In particular, if the space X in Theorem I is supposed to be *nilpotent*, then $\pi_1 X$ —being nilpotent and of square free order—is necessarily cyclic and therefore $\text{im } S = 0$ by a result of Swan [14]. The formula reduces then to

$$T\rho(X) = wX$$

yielding new information concerning the Wall obstruction for nilpotent spaces.

Under suitable conditions on X the rational number $\rho(X)$ depends only upon H_*X : Suppose that $\pi_1 X$ is cyclic of square free order n operating trivially on $H_*^\pi(X; I\mathbb{Z}\pi)$. Then we show that

$$\rho(X) = \prod_{p|n} p^{e_p(X)}$$

the product being taken over all prime divisors of n , and $e_p(X)$ denoting the value at -1 of the derivative of the *Poincaré polynomial* of X with respect to \mathbb{Z}/p -coefficients, a quantity depending only upon H_*X .

As an illustration we show that for X an H -space of rank ≥ 2 one has $e_p(X) = 0$ for all primes p , and hence $\rho(X) = 1$. The following vanishing theorem for the Wall obstruction for H -spaces then follows.

THEOREM II. *Let X be a finitely dominated H -complex with finite fundamental group of square free order. Then $wX=0$ and X is therefore of the homotopy type of a finite complex.*

1. Groups with cyclic Sylow subgroups and $\mathbb{Z}\pi/N$ -modules

Let π denote a finite group whose p -Sylow subgroups are cyclic of order p^k for a fixed prime p . Such a group π is p -periodic in the sense of Cartan–Eilenberg [4]. If q denotes the smallest p -period of π , then $H^q(\pi; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p^k$, where $\mathbb{Z}_{(p)}$ denotes the integers localized at p . Furthermore, if $H^i(\pi; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p^k$ for some $i > 0$, then i is necessarily a multiple of q (see Swan [15]). It has been observed by Lundmark [8] that

$$H^i(\pi; \mathbb{Z}_{(p)}) = 0 \quad \text{for } 0 < i < q.$$

Namely, suppose i is an integer with $0 < i < q$ and let π_p denote a p -Sylow subgroup of π . Then from the decomposition

$$H^i(\pi_p; \mathbb{Z}) \cong \text{im } \iota(\pi_p, \pi) \oplus \ker t(\pi, \pi_p)$$

(cf. [4]) and the fact that the map induced by inclusion $\iota(\pi_p, \pi): H^i(\pi; \mathbb{Z}) \rightarrow H^i(\pi_p; \mathbb{Z})$ is monic on the p -primary subgroup, we infer, because π_p is cyclic, that $H^i(\pi; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p^k$ or $H^i(\pi; \mathbb{Z}_{(p)}) = 0$. The former case is impossible since i is not a multiple of q and hence $H^i(\pi; \mathbb{Z}_{(p)}) = 0$ for $0 < i < q$.

Let π be an arbitrary finite group of order n and $N = \sum x \in \mathbb{Z}\pi$, $x \in \pi$. Then

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}\pi/N \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/n \end{array}$$

is a pullback square of rings (with obvious maps). Hence there is a short exact sequence of $\mathbb{Z}\pi/N$ -modules

$$0 \rightarrow I\mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/N \rightarrow \mathbb{Z}/n \rightarrow 0$$

where $I\mathbb{Z}\pi$ denotes the augmentation ideal. Notice that a $\mathbb{Z}\pi/N$ -module may be considered as a π -module via the projection $\mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/N$.

DEFINITION 1.1. A $\mathbf{Z}\pi/N$ -module M is said to be *trivial*, if it is trivial as a π -module; M is called *nilpotent*, if M possesses a finite filtration with associated graded module a trivial $\mathbf{Z}\pi/N$ -module.

If M is a $\mathbf{Z}\pi/N$ -module, then we will write IM for $(I\mathbf{Z}\pi)M$ and $I^k M$ for $I(I^{k-1}M)$, $k \geq 2$. Obviously, M is then nilpotent if and only if $I^k M = 0$ for some k , (if and only if M is nilpotent as a π -module, respectively). Furthermore, M is a trivial $\mathbf{Z}\pi/N$ -module if and only if $IM = 0$; hence a trivial $\mathbf{Z}\pi/N$ -module is the same as a \mathbf{Z}/n -module. It is plain that the underlying abelian group of a nilpotent $\mathbf{Z}\pi/N$ -module is an n -torsion group.

LEMMA 1.2. Let π denote a finite group whose p -Sylow subgroups are cyclic of order p^k , p a fixed prime. Then, for \mathbf{Z}/p^k considered as a trivial $\mathbf{Z}\pi/N$ -module

$$\text{proj. dim}_{\mathbf{Z}\pi/N} (\mathbf{Z}/p^k) \leq q$$

where q denotes the minimal p -period of π .

Proof. By [14] there exists a periodic resolution

$$\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbf{Z}_{(p)} \rightarrow 0$$

with P_i projective $\mathbf{Z}_{(p)}\pi$ -modules, $P_i = P_{i+q}$ and $P_q \rightarrow P_{q-1}$ factoring through $\mathbf{Z}_{(p)}$. Let $\Lambda = \mathbf{Z}\pi/N$ and $\Lambda_p = \Lambda \otimes \mathbf{Z}_{(p)}$. From the short exact sequence $\mathbf{Z}_{(p)} \xrightarrow{N} \mathbf{Z}_{(p)}\pi \rightarrow \Lambda_p$ we deduce $H_i(\pi; \Lambda_p) \cong H_{i-1}(\pi; \mathbf{Z}_{(p)})$ for $i \geq 2$, and an exact sequence

$$0 \rightarrow H_1(\pi; \Lambda_p) \rightarrow \mathbf{Z}_{(p)} \xrightarrow{n} \mathbf{Z}_{(p)} \rightarrow H_0(\pi; \Lambda_p) \rightarrow 0.$$

Since $H_i(\pi; \mathbf{Z}_{(p)}) = 0$ for $0 < i < q-1$ we conclude that

$$H_i(\pi; \Lambda_p) = \begin{cases} \mathbf{Z}/p^k & \text{if } i = 0, q \\ 0 & \text{if } 0 < i < q. \end{cases}$$

These groups are the homology groups of the complex $\cdots \rightarrow Q_i \rightarrow Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow 0$ in dimension $\leq q$, where $Q_i = \Lambda_p \otimes_{\pi} P_i$. Notice that Q_i is torsionfree as an abelian group, since it is Λ_p -projective. We know that $d_q: Q_q \rightarrow Q_{q-1}$ factors through $\Lambda_p \otimes_{\pi} \mathbf{Z}_{(p)} \cong \mathbf{Z}/p^k$ and therefore, since $\text{im}(d_q)$ is a torsionfree abelian

group, we infer $d_q = 0$. Thus

$$0 \rightarrow Q_{q-1} \rightarrow Q_{q-2} \rightarrow \cdots \rightarrow Q_0 \twoheadrightarrow \mathbf{Z}/p^k$$

is a projective resolution of the trivial Λ_p -module \mathbf{Z}/p^k . As a result $\text{proj. dim}_{\Lambda_p}(\mathbf{Z}/p^k) \leq q-1$. Of course $\text{proj. dim}_{\Lambda}(\Lambda_p) = 1$, as one can see by tensoring a free abelian presentation of $\mathbf{Z}_{(p)}$ with Λ . As a consequence

$$\text{proj. dim}_{\Lambda}(\mathbf{Z}/p^k) \leq \text{proj. dim}_{\Lambda_p}(\mathbf{Z}/p^k) + \text{proj. dim}_{\Lambda}(\Lambda_p) \leq q$$

which completes the proof of the lemma.

An immediate consequence is the following theorem which was mentioned in the introduction.

THEOREM 1.3. *Suppose π is a finite group of order n with cyclic Sylow subgroups. Then \mathbf{Z}/n considered as a trivial $\mathbf{Z}\pi/N$ -module has finite projective dimension.*

Proof. Write $\mathbf{Z}/n = \bigoplus \mathbf{Z}/p^{k(p)}$, the sum taken over all prime divisors of n . Then

$$\text{proj. dim}_{\mathbf{Z}\pi/N}(\mathbf{Z}/n) = \max(\text{proj. dim}_{\mathbf{Z}\pi/N}(\mathbf{Z}/p^{k(p)}) \mid p|n) < \infty$$

Remark. From the short exact sequence $I\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/N \rightarrow \mathbf{Z}/n$ we see that $\text{proj. dim}_{\mathbf{Z}\pi/N}(I\mathbf{Z}\pi) = \text{proj. dim}_{\mathbf{Z}\pi/N}(\mathbf{Z}/n) - 1$. Hence, if π has cyclic Sylow subgroups, we get from Theorem 1.3

$$\text{proj. dim}_{\mathbf{Z}\pi/N}(I\mathbf{Z}\pi) < \infty$$

This generalizes a well known fact on the augmentation ideal of a finite cyclic group, in which case $I\mathbf{Z}\pi$ is free of rank 1 over $\mathbf{Z}\pi/N$.

We will apply later Lemma 1.2 and Theorem 1.3 in case π has square free order; for such a π the Sylow subgroups are of course cyclic of prime order.

LEMMA 1.4. *Let π be a finite group of square free order n and let M denote a nilpotent $\mathbf{Z}\pi/N$ -module. Then*

$$(i) \text{ proj. dim}_{\mathbf{Z}\pi/N}(M) < \infty;$$

if, in addition, M is finitely generated, then

(ii) M is of type FP and $\text{card}(M)$ is a unit in $\mathbb{Z}[1/n]$.

Proof. We first assume that M is a trivial $\mathbb{Z}\pi/N$ -module. Then M is a direct sum of modules of the form \mathbb{Z}/p , p dividing n . From Lemma 1.2 we see then that $\text{proj. dim } M < \infty$. If M is a general nilpotent $\mathbb{Z}\pi/N$ -module, we choose a finite filtration of M such that $\text{gr}(M)$ is a trivial $\mathbb{Z}\pi/N$ -module. Clearly $\text{proj. dim } \text{gr}(M) \geq \text{proj. dim } M$ and $i)$ follows. If M is finitely generated then, $\mathbb{Z}\pi/N$ being noetherian, we can find a projective resolution of M of finite length, which is also of finite type; by definition, M is therefore of type FP . Finally, a finitely generated nilpotent $\mathbb{Z}\pi/N$ -module has as underlying abelian group a finitely generated n -torsion group. Hence $\text{card}(M)$ is a unit in $\mathbb{Z}[1/n]$.

2. The transfer homomorphism $T: u(\mathbb{Z}[1/n]) \rightarrow (K_0\mathbb{Z}\pi)/\text{im } S$

Let π denote a finite group of order n with cyclic Sylow subgroups. Then according to Theorem 1.3, $\text{proj. dim}_{\mathbb{Z}\pi/N}(\mathbb{Z}/n) < \infty$, and therefore the canonical projection $j: \mathbb{Z}\pi/N \rightarrow \mathbb{Z}/n$ gives rise to a transfer map (cf. Bass [1, Chapter IX, 1.7])

$$j^*: K_0(\mathbb{Z}/n) \rightarrow K_0(\mathbb{Z}\pi/N).$$

The map j^* is defined on a generator $[\mathbb{Z}/p^k]$ of $K_0(\mathbb{Z}/n)$ by choosing a $\mathbb{Z}\pi/N$ -projective resolution of finite type

$$0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z}/p^k \rightarrow 0$$

of the trivial $\mathbb{Z}\pi/N$ -module \mathbb{Z}/p^k , and setting

$$j^*[\mathbb{Z}/p^k] = \sum (-1)^i [P_i] \in K_0(\mathbb{Z}\pi/N).$$

Let $j_*: K_0(\mathbb{Z}\pi/N) \rightarrow K_0(\mathbb{Z}/n)$ denote the map induced by the projection $j: \mathbb{Z}\pi/N \rightarrow \mathbb{Z}/n$.

LEMMA 2.1. $j_*j^*: K_0(\mathbb{Z}/n) \rightarrow K_0(\mathbb{Z}/n)$ is the 0-homomorphism.

Proof. Let q denote the minimal p -periode of π and let $[\mathbb{Z}/p^k] \in K_0(\mathbb{Z}/n)$ denote a generator. Choose a $\mathbb{Z}\pi/N$ -projective resolution of finite type of \mathbb{Z}/p^k

which has length q (cf. Lemma 1.2)

$$0 \rightarrow L_q \rightarrow L_{q-1} \rightarrow \cdots \rightarrow L_0 \rightarrow \mathbf{Z}/p^k \rightarrow 0.$$

Then

$$\begin{aligned} j_* j^*[\mathbf{Z}/p^k] &= j_* \left(\sum (-1)^i [L_i] \right) \\ &= \sum (-1)^i [\mathbf{Z}/n \otimes_{\pi} L_i] \\ &= \sum_{r|n} \left(\sum (-1)^i [\mathbf{Z}/n(r) \otimes_{\pi} L_i] \right) \end{aligned}$$

where $n(r)$ stands for the highest power of the prime r , which divides n . For $r \neq p$ we have

$$\mathrm{Tor}_{\mathbf{Z}\pi/N}^*(\mathbf{Z}/n(r), \mathbf{Z}/p^k) = 0$$

and therefore the complex

$$0 \rightarrow \mathbf{Z}/n(r) \otimes_{\pi} L_q \rightarrow \cdots \rightarrow \mathbf{Z}/n(r) \otimes_{\pi} L_0 \rightarrow 0$$

is exact. Hence $\sum (-1)^i [\mathbf{Z}/n(r) \otimes_{\pi} L_i] = 0$ for $r \neq p$, and therefore $j_* j^*[\mathbf{Z}/p^k] = \sum (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} L_i]$. To compute $\sum (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} L_i]$ and the homology of $\{\mathbf{Z}/p^k \otimes_{\pi} L_i\}$ we can as well use the $\mathbf{Z}\pi/N \otimes \mathbf{Z}_{(p)}$ -projective resolution $\{Q_i\}$ of \mathbf{Z}/p^k , which was considered in the proof of Lemma 1.2. Hence

$$\sum (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} L_i] = \sum_{i=0}^{q-1} (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} Q_i]$$

and plainly for $0 \leq i \leq q-1$ one has

$$\mathrm{Tor}_{\mathbf{Z}\pi/N}^i(\mathbf{Z}/p^k, \mathbf{Z}/p^k) = H_i(\pi; \mathbf{Z}/p^k) = \begin{cases} 0 & \text{for } 0 < i < q-1 \\ \mathbf{Z}/p^k & \text{for } i = 0, q-1 \end{cases}$$

Therefore $j_* j^*[\mathbf{Z}/p^k] = [\mathbf{Z}/p^k] + (-1)^{q-1} [\mathbf{Z}/p^k] = 0$ because the p -period q of π is an even number [15].

If π denotes an arbitrary group of order n then associated with the square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{pr} & \mathbf{Z}\pi/N \\ \downarrow & & \downarrow j \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/n \end{array}$$

there is an exact sequence (cf. Milnor [9]) which reduces to

$$u(\mathbf{Z}/n) \xrightarrow{S} K_0(\mathbf{Z}\pi) \xrightarrow{pr_*} K_0(\mathbf{Z}\pi/N) \xrightarrow{j_*} \tilde{K}_0(\mathbf{Z}/n) \rightarrow 0 \quad (2.2)$$

We call S the *Swan homomorphism* (cf. [14]). S can be described in the following way: for k a unit mod n , $S(k) = [(k, N)]$ where (k, N) denotes the projective ideal in $\mathbf{Z}\pi$ generated by k and N .

Consider now the case of a π with cyclic Sylow subgroups. Then $j_*j^* = 0$ by Lemma 2.1 and, by the exactness of (2.2), the transfer j^* gives therefore rise to a homomorphism

$$t: K_0(\mathbf{Z}/n) \rightarrow (K_0\mathbf{Z}\pi)/\text{im } S$$

such that $\bar{pr}_*t = j^*$, $\bar{pr}_*: (K_0\mathbf{Z}\pi)/\text{im } S \rightarrow K_0(\mathbf{Z}\pi/N)$ denoting the map induced by pr_* .

If $n = p_1^{k_1} \cdots p_m^{k_m}$ then $K_0(\mathbf{Z}/n)$ is a free abelian group, freely generated by $\{[\mathbf{Z}/p_i^{k_i}], 1 \leq i \leq m\}$. Hence there is a unique group homomorphism

$$\varphi: u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \rightarrow K_0(\mathbf{Z}/n)$$

such that $\varphi(\pm p_i) = [\mathbf{Z}/p_i^{k_i}]$. If we compose φ with t we get a map $T = t\varphi$ which we will also call a *transfer*, since it is induced by j^* . For π a group with cyclic Sylow subgroups we get therefore a commutative diagram

$$\begin{array}{ccc} u(\mathbf{Z}[1/n]) & \xrightarrow{T} & (K_0\mathbf{Z}\pi)/\text{im } S \\ \downarrow \varphi & \nearrow t & \downarrow \bar{pr}_* \\ K_0(\mathbf{Z}/n) & \xrightarrow{j^*} & K_0(\mathbf{Z}\pi/N) \end{array} \quad (2.3)$$

We will sometimes consider $K_0(\mathbf{Z}\pi/N)$ to be the range of T ; this should not give rise to any confusion, since \bar{pr}_* is injective.

It is well known that if R is a ring and M an R -module of type FP , then M defines an element $[M] \in K_0R$ (depending only upon the isomorphism class of M) by choosing any finite projective resolution of finite type

$$0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

and setting $[M] = \sum (-1)^i [P_i] \in K_0 R$; if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of modules of type FP , then $[M] = [M'] + [M'']$ (cf. [1] and [11]).

LEMMA 2.4. *Let π denote a finite group of square free order n and let M denote a finitely generated nilpotent $\mathbf{Z}\pi/N$ -module. Then*

$$T(\text{card } M) = [M] \in K_0(\mathbf{Z}\pi/N).$$

Proof. Notice that M is of type FP over $\mathbf{Z}\pi/N$ and $\text{card } M \in u(\mathbf{Z}[1/n])$ by Lemma 1.4. Hence $T(\text{card } M)$ and $[M]$ are well defined elements of $K_0(\mathbf{Z}\pi/N)$. If M is a trivial $\mathbf{Z}\pi/N$ -module, then $T(\text{card } M) = j^* \varphi(\text{card } M) = [M]$ where the second equation follows from the definition of φ , j^* and $[M]$ respectively. For the general case we choose a finite filtration of M with $\text{gr}(M)$ a trivial $\mathbf{Z}\pi/N$ -module. Clearly $\text{card } M = \text{card } \text{gr}(M)$ and $[M] = [\text{gr}(M)]$; therefore $T(\text{card } M) = [M]$.

For the applications in the next section we will be particularly interested in groups π for which $\text{im } S = 0$. The following theorem gives some information on T for such cases.

THEOREM 2.5. *Let π denote a finite group of order n with cyclic Sylow subgroups. Then*

- (i) $T \equiv 0$ in case n is a prime power or $n = 2p$, p an odd prime.
- (ii) $T(p_1 \cdots p_m) = 0$ if π is cyclic of order $p_1^{k_1} \cdots p_m^{k_m}$.
- (iii) $T(3) = T(5) \neq 0$ if $n = 15$, and $T(3)$ has order 2.

Furthermore, in all three cases listed above one has $\text{im } S = 0$, and T can therefore be considered as a map $T: u(\mathbf{Z}[1/n]) \rightarrow K_0 \mathbf{Z}\pi$.

We will break the proof up into a couple of lemmas.

LEMMA 2.6. *Let π be a cyclic group of order n . Then $j^*: K_0(\mathbf{Z}/n) \rightarrow K(\mathbf{Z}\pi/N)$ factors through $\tilde{K}_0(\mathbf{Z}/n)$.*

Proof. We may assume $n > 1$. Let x denote a generator of π . Then $I\mathbf{Z}\pi$ is freely generated by $(1-x)$ over $\mathbf{Z}\pi/N$ and hence there is an exact sequence $0 \rightarrow \mathbf{Z}\pi/N \rightarrow \mathbf{Z}\pi/N \rightarrow \mathbf{Z}/n \rightarrow 0$, from which we infer that $j^*[\mathbf{Z}/n] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N] = 0$. Thus j^* factors through $\tilde{K}_0(\mathbf{Z}/n) = K_0(\mathbf{Z}/n)/\langle [\mathbf{Z}/n] \rangle$.

LEMMA 2.7. *Let p denote an odd prime and $\pi = \mathbf{Z}/2p$ or the dihedral group D_{2p} . Then*

$$j^* = 0: K_0(\mathbf{Z}/2p) \rightarrow K_0(\mathbf{Z}\pi/N)$$

Proof. We will first consider the case $\pi = \mathbf{Z}/2p$. Since j^* factors through $\tilde{K}_0(\mathbf{Z}/2p)$ which is cyclic, generated by the equivalence class of $[\mathbf{Z}/2]$, it suffices to prove that $j^*[\mathbf{Z}/2] = 0$. Let $\pi = \langle x, y \mid x^2 = y^p = 1, xy = yx \rangle$, $R = \mathbf{Z}[\omega]$ with $\omega = \exp(2\pi i/p)$ and $R[\mathbf{Z}/2] \cong \mathbf{Z}\pi/(1 + y + \cdots + y^{p-1})$ the obvious isomorphism (mapping ω to y). Consider the pullback square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\lambda} & R[\mathbf{Z}/2] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \longrightarrow & \mathbf{F}_p[\mathbf{Z}/2] \end{array}$$

with obvious maps. Since $u(R[\mathbf{Z}/2]) \rightarrow u(\mathbf{F}_p[\mathbf{Z}/2])$ is surjective (cf. Reiner–Ullom [12, §7]) we get from the associated Milnor–Mayer–Vietoris sequence a monomorphism

$$\lambda_*: K_0\mathbf{Z}\pi \rightarrow K_0R[\mathbf{Z}/2].$$

Let $P \subset \mathbf{Z}\pi$ be the ideal generated by $(1 - y)$ and 2 . Then $\mathbf{Z}\pi/P \cong \mathbf{F}_2[\mathbf{Z}/2]$ is certainly cohomologically trivial and hence P is projective (cf. Rim [13]). Since $\mathbf{F}_2[\mathbf{Z}/2]/N = \mathbf{Z}/2$ we see that $j^*[\mathbf{Z}/2] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_\pi P]$. It suffices therefore to show that $[P] = [\mathbf{Z}\pi] \in K_0\mathbf{Z}\pi$. But $\lambda_*[P] = [(1 - \omega, 2)] = [R[\mathbf{Z}/2]]$ since $R/(1 - \omega)R \cong \mathbf{Z}/p$ and p odd. Hence $[P] = [\mathbf{Z}\pi]$ because λ_* is injective, from where we conclude that $j^*[\mathbf{Z}/2] = 0$. In case $\pi = D_{2p}$ we proceed in a similar way. Notice that $K_0(\mathbf{Z}/2p)$ is freely generated by $[\mathbf{Z}/2]$ and $[\mathbf{Z}/p]$. From Corollary 3.5 we infer that $j^*[\mathbf{Z}/p] = 0$ and we are therefore left showing that $j^*[\mathbf{Z}/2] = 0$. Let $D_{2p} = \langle x, y \mid x^2 = y^p, yxy = x \rangle$. Notice that $P = (1 - y)\mathbf{Z}\pi + 2\mathbf{Z}\pi$ is a two-sided ideal with $\mathbf{Z}\pi/P \cong \mathbf{F}_2[\mathbf{Z}/2]$, which is cohomologically trivial. Hence P is a projective π -module and clearly $j^*[\mathbf{Z}/2] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_\pi P]$. In order to see that $[P] = [\mathbf{Z}\pi]$ we consider the square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\lambda} & R_t[\mathbf{Z}/2] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \longrightarrow & \mathbf{F}_p[\mathbf{Z}/2] \end{array}$$

with $R_t[\mathbf{Z}/2] = \mathbf{Z}\pi/(1 + y + \cdots + y^{p-1})$ a twisted group ring. By [12, §7] $u(R_t[\mathbf{Z}/2]) \rightarrow u(\mathbf{F}_p[\mathbf{Z}/2])$ is surjective and hence

$$\lambda_*: K_0\mathbf{Z}\pi \rightarrow K_0R_t[\mathbf{Z}/2]$$

is injective. Since $\mathbf{F}_2[\mathbf{Z}/2] \otimes_{\pi} R_t[\mathbf{Z}/2] \cong \mathbf{F}_2[\mathbf{Z}/2] \otimes_{\pi} \mathbf{F}_p[\mathbf{Z}/2] = 0$ we infer that $\lambda_*[P] = [R_t[\mathbf{Z}/2]]$ and whence $[P] = [\mathbf{Z}\pi]$ from the injectivity of λ_* . Therefore $j^*[\mathbf{Z}/2] = 0$, which completes the proof.

LEMMA 2.8. *Let $\pi = \mathbf{Z}/15$. Then $T(3)$ is the element of order 2 in $K_0\mathbf{Z}[\mathbf{Z}/15]$.*

Proof. Let $\pi = \langle x, y \mid x^3 = y^5 = 1, xy = yx \rangle$ and let $P = (x+2)\mathbf{Z}\pi + (y-1)\mathbf{Z}\pi$. Then $\mathbf{Z}\pi/P = M$ is cyclic of order 9 with y operating trivially and x operating by multiplication with 7 mod 9. One checks easily that M is cohomologically trivial using the criterion of [13]. Hence P is projective. Since $M/NM \cong \mathbf{Z}/3$ as trivial $\mathbf{Z}\pi/N$ -module we infer that

$$j^*[\mathbf{Z}/3] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_{\pi} P] \in K_0(\mathbf{Z}\pi/N).$$

Notice that $\text{im } S = 0$ since π is cyclic. Hence we can think of $T(3)$ to be the element $[\mathbf{Z}\pi] - [P] \in K_0(\mathbf{Z}\pi)$. Recall that $K_0\mathbf{Z}\pi \cong \mathbf{Z} \oplus \mathbf{Z}/2$ by Kervaire–Murthy [7]. Since P is projective of rank 1, it remains therefore to prove that $[P] \neq 0$ in $\tilde{K}_0(\mathbf{Z}\pi) \cong \mathbf{Z}/2$. For this we consider the pullback square (with obvious maps)

$$\begin{array}{ccc} \mathbf{Z}\pi & \longrightarrow & R[\mathbf{Z}/3] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/3] & \longrightarrow & \mathbf{F}_5[\mathbf{Z}/3] \end{array}$$

where $R = \mathbf{Z}[\exp(2\pi i/5)]$. The associated Milnor–Mayer–Vietoris sequence yields a map

$$\partial: u(\mathbf{F}_5[\mathbf{Z}/3]) \rightarrow \tilde{K}_0\mathbf{Z}[\mathbf{Z}/15]$$

By [7] ∂ factors through $u(\mathbf{F}_5(\omega)) \cong \mathbf{Z}/24$ where $\mathbf{F}_5(\omega)$ is the field $\mathbf{F}_5[\mathbf{Z}/3]/(1+x+x^2)$, ω is the residue class of the generator $x \in \mathbf{Z}/3$. Furthermore, ∂ is surjective (cf. [7]). Notice that $R[\mathbf{Z}/3] \otimes_{\pi} M = 0$ and therefore $P' = R[\mathbf{Z}/3] \otimes_{\pi} P \cong R[\mathbf{Z}/3]$. Furthermore $P'' = \mathbf{Z}[\mathbf{Z}/3] \otimes_{\pi} P \subset \mathbf{Z}[\mathbf{Z}/3]$ is the principal ideal generated by $(x+2)$. Hence (cf. [9])

$$P = \{((x+2)a, b) \in \mathbf{Z}[\mathbf{Z}/3] \times R[\mathbf{Z}/3] \mid \overline{(x+2)}\bar{a} = \bar{b} \in \mathbf{F}_5[\mathbf{Z}/3]\}.$$

Notice that $\overline{x+2} \in u(\mathbf{F}_5[\mathbf{Z}/3])$ corresponds to $(3, \omega+2) \in u(\mathbf{F}_5) \times u(\mathbf{F}_5(\omega))$ under the obvious isomorphism $\mathbf{F}_5[\mathbf{Z}/3] \cong \mathbf{F}_5 \times \mathbf{F}_5(\omega)$. It follows therefore that $\partial(\overline{x+2}) = \partial(\omega+2) = [P]$ and, since $(\omega+2)$ has order 24 in $u(\mathbf{F}_5(\omega))$, we conclude that $[P]$ must have order 2. This completes the proof of the lemma.

We can now complete the proof of Theorem 2.5: First, if n is a prime power, π (having cyclic Sylow subgroups) is necessarily cyclic and therefore $j^*: K_0(\mathbf{Z}/n) \rightarrow K_0(\mathbf{Z}\pi/N)$ factors through $\tilde{K}_0(\mathbf{Z}/n)$ by Lemma 2.6. But $\tilde{K}_0(\mathbf{Z}/n) = 0$ for n a prime power. Hence $T \equiv 0$ in this case. If $n = 2p$, p an odd prime, then $\pi = \mathbf{Z}/2p$ or D_{2p} and it follows from Lemma 2.7 that $T \equiv 0$. Thus (i) holds. Assume now that π is cyclic of order $\prod p_i^{k_i} = n$. Then $T(\prod p_i) = t\varphi(\prod p_i) = t[\mathbf{Z}/n] = 0$, since $j^*[\mathbf{Z}/n] = 0$ by Lemma 2.6. Therefore (ii) holds. For (iii) notice that in case $n = 15$, π is necessarily cyclic. Hence, by applying Lemma 2.6, $j^*[\mathbf{Z}/3] = -j^*[\mathbf{Z}/5]$ or $T(3) = -T(5)$. Moreover $T(3)$ has order 2 by Lemma 2.8 and therefore $T(3) = T(5) \neq 0$. Finally $\text{im } S = 0$ for π cyclic or $\pi = D_{2p}$, p an odd prime (cf. Ullom [16]) and hence $\text{im } S = 0$ for the groups considered in (i)–(iii) of Theorem 2.5.

3. Applications to homologically nilpotent spaces

Following [3] we call a connected space X *homologically nilpotent*, if $\pi_1 X$ operates nilpotently on $H_i \tilde{X}$ for all i . In particular, if X is a nilpotent space, then X is homologically nilpotent and conversely, a homologically nilpotent space X is nilpotent if and only if $\pi_1 X$ is a nilpotent group (cf. [10]). Let $H_*^\pi(X; I\mathbf{Z}\pi)$ denote the homology of the (left) $\mathbf{Z}\pi/N$ -complex $I\mathbf{Z}\pi \otimes_\pi C_* \tilde{X}$, where $\pi = \pi_1 X$; similarly for $H_*^\pi(X; \mathbf{Z}\pi/N)$.

LEMMA 3.1. *If X is homologically nilpotent with $\pi_1 X$ of finite order n , then $H_i^\pi(X; I\mathbf{Z}\pi)$ and $H_i^\pi(X; \mathbf{Z}\pi/N)$ are nilpotent $\mathbf{Z}\pi/N$ -modules for all i . If, in addition, n is square free and X finitely dominated, then $H_i^\pi(X; I\mathbf{Z}\pi)$ and $H_i^\pi(X; \mathbf{Z}\pi/N)$ are of type FP over $\mathbf{Z}\pi/N$.*

Proof. Consider the long exact homology sequence associated with the exact sequence of chain complexes $0 \rightarrow I\mathbf{Z}\pi \otimes_\pi C_* \tilde{X} \rightarrow C_* \tilde{X} \rightarrow C_* X \rightarrow 0$. Standard results on nilpotent actions (cf. [6, Chapter I.4]) imply then that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a nilpotent π -module. Since the π -action on $H_i^\pi(X; I\mathbf{Z}\pi)$ factors through $\mathbf{Z}\pi/N$, we conclude that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a nilpotent $\mathbf{Z}\pi/N$ -module. If X is dominated by a finite complex and $\pi_1 X$ finite, then certainly $H_i^\pi(X; I\mathbf{Z}\pi)$ is a finitely generated $\mathbf{Z}\pi/N$ -module. Hence, by Lemma 1.4, $H_i^\pi(X; I\mathbf{Z}\pi)$ is of type FP. The proof for $H_i^\pi(X; \mathbf{Z}\pi/N)$ is similar, using $0 \rightarrow C_* X \rightarrow C_* \tilde{X} \rightarrow \mathbf{Z}\pi/N \otimes_\pi C_* \tilde{X} \rightarrow 0$.

LEMMA 3.2. *Let X be a finitely dominated homologically nilpotent space with finite fundamental group of order n and let $H_*(X, \tilde{X})$ denote the homology of the mapping cylinder of $\tilde{X} \rightarrow X \bmod \tilde{X}$. Then the groups $H_i^\pi(X, I\mathbf{Z}\pi)$ and $H_{i+1}(X, \tilde{X})$ have the same finite cardinality $c(i)$ for $i \geq 0$, and $c(i)$ is a unit in $\mathbf{Z}[1/n]$.*

Proof. Since $H_i^\pi(X; I\mathbb{Z}\pi)$ is a finitely generated nilpotent $\mathbb{Z}\pi/N$ -module, it has by Lemma 1.4 a cardinality $c(i)$ which is a unit in $\mathbb{Z}[1/n]$. From long exact homology sequences it is obvious that $c(i)$ is also given by

$$c(i) = (\text{card coker } (H_{i+1}\tilde{X} \rightarrow H_{i+1}X)) \cdot (\text{card ker } (H_i\tilde{X} \rightarrow H_iX))$$

which equals $\text{card } H_{i+1}(X, \tilde{X})$.

It follows that for X as in Lemma 3.2, the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X}) / \text{card } H_{\text{ev}}(X, \tilde{X})$$

is a well defined unit in $\mathbb{Z}[1/n]$. This unit $\rho(X)$ is related to the finiteness obstruction $wX \in K_0\mathbb{Z}\pi_1X$ of Wall (cf. [17], [10]) in the following way.

THEOREM 3.3. *Let X be a homologically nilpotent space with non-trivial fundamental group π of square free order n . Suppose further that X is dominated by a finite complex and let $\bar{w}X$ denote the image of wX in $(K_0\mathbb{Z}\pi_1X)/\text{im } S$. Then*

$$T\rho(X) = \bar{w}X$$

If X is in addition nilpotent (i.e. π_1X is cyclic) then

- (i) $T\rho(X) = wX$, and
- (ii) $wX = 0$ in case $\rho(X)$ is a power (positive or negative) of n .

Before proving Theorem 3.3 we will establish a different way of computing $\rho(X)$.

LEMMA 3.4. *Let X be as in (3.2) and let $H_*^\pi(X; \mathbb{Z}\pi/N)$ denote the homology of $\mathbb{Z}\pi/N \otimes_\pi C_*(\tilde{X})$. Then*

$$\rho(X) = \text{card } H_{\text{ev}}^\pi(X; \mathbb{Z}\pi/N) / \text{card } H_{\text{odd}}^\pi(X; \mathbb{Z}\pi/N)$$

Proof. We may of course assume that $n = \text{card } \pi_1X > 1$. Since $H_*(\tilde{X}; \mathbb{Q})$ is semisimple and nilpotent as π -module, we have $H_*(\tilde{X}; \mathbb{Q}) \cong H_*(X; \mathbb{Q})$ and therefore the Euler characteristic of X vanishes. Thus, for all primes p , $\chi_p(X) = \sum (-1)^i \dim H_i(X; \mathbb{Z}/p) = 0$. Since n is square free, we infer then that

$$\text{card } H_{\text{ev}}(X; \mathbb{Z}/n) = \text{card } H_{\text{odd}}(X; \mathbb{Z}/n).$$

The long exact homology sequence associated with the exact sequence $I\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/N \rightarrow \mathbf{Z}/n$ then yields

$$\begin{aligned} \text{card } H_{\text{ev}}^{\pi}(X; \mathbf{Z}\pi/N) / \text{card } H_{\text{odd}}^{\pi}(X; \mathbf{Z}\pi/N) \\ = \text{card } H_{\text{ev}}^{\pi}(X; I\mathbf{Z}\pi) / \text{card } H_{\text{odd}}^{\pi}(X; I\mathbf{Z}\pi) \end{aligned}$$

from where the result follows, using Lemma 3.2.

Proof of Theorem 3.3. Let $pr_*: K_0(\mathbf{Z}\pi) \rightarrow K_0(\mathbf{Z}\pi/N)$ be the map induced by the projection. If \bar{C}_* denotes a chain complex of type *FP*, homotopy equivalent to the singular complex of \tilde{X} , then $pr_*wX = \sum (-1)^i [\mathbf{Z}\pi/N \otimes_{\pi} \bar{C}_i]$. Since $H_i^{\pi}(X; \mathbf{Z}\pi/N)$ is of type *FP* (cf. Lemma 3.1) we infer that (cf. [11])

$$\sum (-1)^i [\mathbf{Z}\pi/N \otimes_{\pi} \bar{C}_i] = \sum (-1)^i [H_i^{\pi}(X; \mathbf{Z}\pi/N)]$$

and therefore, since $[H_i^{\pi}(X; \mathbf{Z}\pi/N)] = T(\text{card } H_i^{\pi}(X; \mathbf{Z}\pi/N))$ by Lemma 2.4, we get

$$pr_*wX = \bar{w}X = T(\text{card } H_{\text{ev}}^{\pi}(X; \mathbf{Z}\pi/N) / \text{card } H_{\text{odd}}^{\pi}(X; \mathbf{Z}\pi/N)) = T\rho(X).$$

In case X is in addition nilpotent, $\pi_1 X$ is necessarily cyclic and therefore $\text{im } S = 0$. Hence $T\rho(X) = wX$ in this case. Furthermore, if $\rho(X) = n^k$, we conclude from Theorem 2.5(ii) that $wX = T(n^k) = 0$. This completes the proof of Theorem 3.3.

As a first application we will prove the following algebraic result, which is used to prove Lemma 2.7.

COROLLARY 3.5. *Let π denote the dihedral group D_{2p} , p an odd prime. Then $j^*[\mathbf{Z}/p] = 0 \in K_0(\mathbf{Z}D_{2p}/N)$.*

Proof. By Theorem A [14] there is a finite simplicial complex \tilde{X} of the homotopy type of S^3 on which D_{2p} acts freely and simplicially. Let $X = \tilde{X}/D_{2p}$. Then

$$H_i(X, \tilde{X}) = \begin{cases} 0, & \text{if } i = 0, 2 \\ \mathbf{Z}/2, & \text{if } i = 1 \\ \mathbf{Z}/2p, & \text{if } i = 3 \end{cases}$$

Thus $\rho(X) = 4p$. Since X is a finite complex with trivial action of $\pi_1 X$ on $H_*\tilde{X}$, we infer that $wX = 0$ and therefore $T(4p) = 0$. We have already observed in course of the proof of Lemma 2.7 that $j^*[\mathbf{Z}/2] = 0$. Hence

$$j^*[\mathbf{Z}/p] = j^*[(\mathbf{Z}/2) \oplus (\mathbf{Z}/2) \oplus (\mathbf{Z}/p)] = T(4p) = 0.$$

Recall that two nilpotent spaces X and Y of finite type are of the same *genus* (cf. [6]) if their p -localizations X_p, Y_p are homotopy equivalent for all primes p . In [10] one finds an example of two finitely dominated nilpotent spaces X and Y of the same genus with fundamental groups of order 8 such that $wX \neq 0$ but $wY = 0$. For fundamental groups of square free orders, such an example is impossible. Namely one has

COROLLARY 3.6. *Let X and Y be two finitely dominated nilpotent spaces of the same genus, with non trivial fundamental groups of square free order n . Then $\rho(X) = \rho(Y)$ and wX, wY have the same finite orders.*

Proof. Notice that $\pi_1 X$ and $\pi_1 Y$ are abelian and whence $\pi_1 X \cong \pi_1 Y$, since X and Y are of the same genus (cf. [6]). Furthermore, there are for all primes p commutative diagrams

$$\begin{array}{ccc} \tilde{X}_p & \longrightarrow & \tilde{Y}_p \\ pr \downarrow & & \downarrow pr \\ X_p & \longrightarrow & Y_p \end{array}$$

with the horizontal maps being homotopy equivalences. Thus

$$H_i(X, \tilde{X}; \mathbf{Z}_{(p)}) \cong H_i(X_p, \tilde{X}_p; \mathbf{Z}_{(p)}) \cong H_i(Y_p, \tilde{Y}_p; \mathbf{Z}_{(p)}) \cong H_i(Y, \tilde{Y}; \mathbf{Z}_{(p)})$$

and therefore $H_i(X, \tilde{X}) \cong H_i(Y, \tilde{Y})$ since the groups $H_i(X, \tilde{X})$ and $H_i(Y, \tilde{Y})$ are finite. Hence $\rho X = \rho Y$ and, since $\pi_1 X \cong \pi_1 Y$, it follows from Theorem 3.3(i) that wX and wY have the same orders; the orders must be finite, because X and Y must have vanishing Euler characteristic (cf. proof of Lemma 3.4).

Another application of Theorem 3.3 is the following vanishing theorem for the Wall obstruction.

COROLLARY 3.7. *Let X be a finitely dominated homologically nilpotent space with fundamental group of order p or $2p$, p a prime. Then $wX = 0$ and X is therefore of the homotopy type of a finite complex.*

Proof. First consider the case $p = 2$. By a result of Fröhlich (cf. [5, Theorem 6(i)]) we infer that $\tilde{K}_0 \pi_1 X = 0$ and, since the Euler characteristic of X is necessarily 0 (cf. proof of Lemma 3.4), it follows that $wX = 0$. Second, let p denote an odd prime. Then we may apply Theorem 3.3 and obtain $T\rho(X) = \bar{w}X$. Since $T \equiv 0$ and $\text{im } S = 0$ for the $\pi_1 X$ in question (cf. Theorem 2.5), we infer that $wX = 0$.

It is sometimes possible to compute $\rho(X)$ directly from H_*X , giving rise to a particular simple formula for wX . We will treat one such case in the next section and plan to treat other cases in a forthcoming paper.

4. The Wall obstruction for H -spaces

We want to prove the Theorem II mentioned in the introduction.

Suppose X is a space with $\oplus H_i(X; \mathbf{Z})$ finitely generated and let p denote a fixed prime. Then we write

$$\chi_p(X, t) = \sum (-1)^i \beta_i t^i, \quad \beta_i = \dim H_i(X; \mathbf{Z}/p)$$

for the *Poincaré polynomial* of X with respect to \mathbf{Z}/p . Define

$$e_p(X) = \left. \frac{d}{dt} \chi_p(X, t) \right|_{t=-1}$$

and, in case $\pi_1 X$ has finite order n , define

$$e(X) = \prod_{p|n} p^{e_p(X)}$$

THEOREM 4.1. *Let X be a finitely dominated nilpotent space with fundamental group π of square free order. Suppose that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a trivial π -module for all i . Then*

$$\rho(X) = e(X)$$

Proof. We may assume that $\pi \neq \{1\}$. Let x denote a generator of the necessarily cyclic group π . Then there is an exact sequence of $\mathbf{Z}\pi/N$ -modules

$$0 \rightarrow I\mathbf{Z}\pi \xrightarrow{1-x} I\mathbf{Z}\pi \rightarrow \mathbf{Z}/n \rightarrow 0$$

where n denotes the order of π . Since $(1-x)$ induces 0 in $H_*^\pi(X; I\mathbf{Z}\pi)$, the associated long exact homology sequence breaks up into short exact sequences

$$0 \rightarrow H_i^\pi(X; I\mathbf{Z}\pi) \rightarrow H_i(X; \mathbf{Z}/n) \rightarrow H_{i-1}^\pi(X; I\mathbf{Z}\pi) \rightarrow 0$$

for $i \geq 0$. Since $H_i^\pi(X; I\mathbf{Z}\pi)$ is a trivial $\mathbf{Z}\pi/N$ -module and n is square free, $H_i^\pi(X; I\mathbf{Z}\pi) \otimes \mathbf{Z}_{(p)}$ is a \mathbf{Z}/p -vector space. Define

$$\beta_i = \dim H_i(X; \mathbf{Z}/p), \quad \gamma_i = \dim H_i^\pi(X; I\mathbf{Z}\pi) \otimes \mathbf{Z}_{(p)}.$$

Then

$$\gamma_i = \beta_i - \gamma_{i-1} = \beta_i - \beta_{i-1} + \cdots + (-1)^i \beta_0$$

and

$$\begin{aligned} \sum (-1)^j \gamma_j &= \sum (\beta_0 - \beta_1 + \cdots + (-1)^j \beta_j) \\ &= (m+1)\beta_0 - m\beta_1 + \cdots + (-1)^m \beta_m \end{aligned}$$

where m denotes the largest integer k with $\beta_k \neq 0$. Hence

$$\begin{aligned} \sum (-1)^j \gamma_j &= (-1)^m \frac{d}{dt} \left(t^{m+1} \chi_p \left(X, \frac{1}{t} \right) \right) \Big|_{t=-1} \\ &= (m+1) \chi_p(X, -1) + \chi'_p(X, -1). \end{aligned}$$

Since X is homologically nilpotent with non-trivial finite fundamental group, the Euler characteristic of X is 0 (cf. proof of Lemma 3.4) and hence $\chi_p(X, -1) = 0$. The above equation reduces therefore to

$$\sum (-1)^j \gamma_j = \chi'_p(X, -1) = e_p(X).$$

Hence $\text{card } H_{\text{ev}}^\pi(X; I\mathbf{Z}\pi) / \text{card } H_{\text{odd}}^\pi(X; I\mathbf{Z}\pi) = \prod_{p|n} p^{e_p(X)} = e(X)$ and therefore $\rho(X) = e(X)$.

In order to prove that for an H -space X the π -operation on $H_*(X; I\mathbf{Z}\pi)$ is trivial, we will need the following lemma.

LEMMA 4.2. *Let X be an H -space and $a \in \pi_1 X$. Then the induced covering transformation $a_*: \tilde{X} \rightarrow \tilde{X}$ is equivariantly homotopic to the identity.*

Proof. Without loss of generality we may assume that the H -structure $\mu: X \times X \rightarrow X$ has the base point as a strict identity. Equip \tilde{X} with the canonical H -structure $\tilde{\mu}$. Then $pr^{-1}(*)$ is a central subgroup of $(\tilde{X}, \tilde{\mu})$, naturally isomorphic to $\pi_1 X$. We may thus think of a as an element of $pr^{-1}(*)$ which acts on \tilde{X} by left multiplication. Choosing a path from $a \in pr^{-1}(*)$ to $*$ we get a homotopy of the

map a_* to Id , which is equivariant with respect to the $\pi_1 X = pr^{-1}(*)$ -action, because $pr^{-1}(*)$ is central.

COROLLARY 4.3. *If X is an H -space, then $H_*^\pi(X; I\mathbb{Z}\pi)$ is a trivial $\mathbb{Z}\pi/N$ -module.*

This is clear since by 4.2, the operation of $a \in \pi_1 X$ on $C_*\tilde{X}$ is chain homotopic to Id as map of $\pi_1 X$ -complexes, and therefore the induced action of a on $I\mathbb{Z}\pi \otimes_\pi C_*\tilde{X}$ is chain homotopic to Id .

Proof of Theorem II of the Introduction. If X is of rank 1, then X is equivalent to one of the spaces S^1 , S^3 , S^7 , $\mathbb{R}P^3$ or $\mathbb{R}P^7$ (cf. Browder [2]). Hence $wX = 0$ in these cases. If $\text{rank}(X) \geq 2$, then $\chi_p(X, t)$ contains a factor $(1 - t^{n_1})(1 - t^{n_2})$ with n_1 and n_2 odd. Therefore

$$e_p(X) = \chi'_p(X, -1) = 0$$

for all primes p . In particular we obtain $e(X) = 1$ and, since $H_i^\pi(X; I\mathbb{Z}\pi)$ is a trivial π -module for all i we infer from Theorem 4.1 that $\rho(X) = e(X)$. Hence $wX = T\rho(X) = 0$.

5. Appendix

If X denotes a homologically nilpotent space with finite fundamental group, then there is a simple criterion for deciding whether X is dominated by a finite complex.

THEOREM 5.1. *Let X be a homologically nilpotent complex with finite fundamental group. Then the following are equivalent.*

- (i) X is dominated by a finite complex
- (ii) $H_i(\tilde{X}; \mathbb{Z})$ and $H_i(X; \mathbb{Z})$ are finitely generated abelian groups for all i and zero for i sufficiently large.

Proof. Certainly (i) implies (ii). If (ii) is given, then from [3, Corollary 3.4] we infer that X has the homotopy type of a finite dimensional complex. Since $\mathbb{Z}\pi_1 X$ is noetherian and $H_i(\tilde{X}; \mathbb{Z})$ finitely generated for all i , Theorems B and F of [17] imply that X is dominated by a finite complex.

A more general result of this type in case X is nilpotent was proved in [11].

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