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Groups with cyclic Sylow subgroups and finiteness conditions for certain complexes

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Introduction

Let π denote a finite group of order n whose Sylow subgroups are all cyclic and let $N=\sum x \in \mathbf{Z}\pi$, $x \in \pi$, denote the norm element. The augmentation $\mathbf{Z}\pi \to \mathbf{Z}$ induces a map $j:\mathbf{Z}\pi/N \to \mathbf{Z}/n$ which we use to consider $\mathbf{Z}/n$ as a $\mathbf{Z}\pi/N$-module. We show (Theorem 1.3) that

$$\text{proj.}\dim_{\mathbf{Z}\pi/N}(\mathbf{Z}/n)<\infty.$$

Thus there is a *transfer map*

$$j^*: K_0(\mathbf{Z}/n) \to K_0(\mathbf{Z}\pi/N)$$

between projective class groups. It turns out that $\operatorname{im}(j^*) \subset \operatorname{im}(pr_*: K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N))$ and, since $\operatorname{im}(K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N)) \cong (K_0\mathbf{Z}\pi)/\operatorname{im} S$ where $S: K_1(\mathbf{Z}/n) \to K_0(\mathbf{Z}\pi)$ denotes the *Swan homomorphism* (cf. Section 2), we can think of the transfer map to map $K_0(\mathbf{Z}/n)$ into $(K_0\mathbf{Z}\pi)/\operatorname{im} S$. If we compose this map with the obvious homomorphism

$$u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to K_0(\mathbf{Z}/n)$$

we obtain a "transfer" homomorphism

$$T: u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to (K_0\mathbf{Z}\pi)/\operatorname{im} S$$

($u(\mathbf{Z}[1/n])$ denotes the group of units of $\mathbf{Z}[1/n]$). The homomorphism T is in general non-trivial, even if π is cyclic (in which case $\operatorname{im} S=0$). However, we

Dedicated to Beno Eckmann on the occasion of his sixtieth birthday.

show that $T=0$ if n is a prime power or if $n=2p$, p an odd prime (cf. Theorem 2.5).

In the second half of the paper we make use of the homomorphism T to compute the *Wall obstruction* $wX \in K_0\mathbf{Z}\pi_1X$ for certain complexes. We will consider spaces X for which π_1X operates nilpotently on $H_*\tilde{X}$ (i.e. X is homologically nilpotent in the sense of Brown–Kahn [3]). If such a space is dominated by a finite complex and has a finite fundamental group of order n, then the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X})/\text{card } H_{\text{ev}}(X, \tilde{X})$$

is well defined and is a unit in $\mathbf{Z}[1/n]$; $\rho(X)$ is related to the finiteness obstruction wX in the following way, (cf. Theorem 3.3).

THEOREM I. *Let X be a finitely dominated homologically nilpotent space with non-zero finite fundamental group of square free order. Then*

$$T\rho(X) = \bar{w}X$$

where $\bar{w}X$ *denotes the image of* wX *in* $(K_0\mathbf{Z}\pi_1X)/\text{im } S$.

In particular, if the space X in Theorem I is supposed to be *nilpotent*, then π_1X–being nilpotent and of square free order–is necessarily cyclic and therefore im $S=0$ by a result of Swan [14]. The formula reduces then to

$$T\rho(X) = wX$$

yielding new information concerning the Wall obstruction for nilpotent spaces.

Under suitable conditions on X the rational number $\rho(X)$ depends only upon H_*X: Suppose that π_1X is cyclic of square free order n operating trivially on $H_*^{\pi}(X; I\mathbf{Z}\pi)$. Then we show that

$$\rho(X) = \prod_{p/n} p^{e_p(X)}$$

the product being taken over all prime divisors of n, and $e_p(X)$ denoting the value at -1 of the derivative of the *Poincaré polynomial* of X with respect to $\mathbf{Z}/p$-coefficients, a quantity depending only upon H_*X.

As an illustration we show that for X an H-space of rank ≥ 2 one has $e_p(X)=0$ for all primes p, and hence $\rho(X)=1$. The following vanishing theorem for the Wall obstruction for H-spaces then follows.

THEOREM II. *Let X be a finitely dominated H-complex with finite fundamental group of square free order. Then $wX=0$ and X is therefore of the homotopy type of a finite complex.*

1. Groups with cyclic Sylow subgroups and $\mathbf{Z}\pi/N$-modules

Let π denote a finite group whose p-Sylow subgroups are cyclic of order p^k for a fixed prime p. Such a group π is p-periodic in the sense of Cartan–Eilenberg [4]. If q denotes the smallest p-period of π, then $H^q(\pi;\mathbf{Z}_{(p)})\cong\mathbf{Z}/p^k$, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p. Furthermore, if $H^i(\pi;\mathbf{Z}_{(p)})\cong\mathbf{Z}/p^k$ for some $i>0$, then i is necessarily a multiple of q (see Swan [15]). It has been observed by Lundmark [8] that

$$H^i(\pi;\mathbf{Z}_{(p)})=0 \quad \text{for} \quad 0<i<q.$$

Namely, suppose i is an integer with $0<i<q$ and let π_p denote a p-Sylow subgroup of π. Then from the decomposition

$$H^i(\pi_p;\mathbf{Z})\cong \operatorname{im}\iota(\pi_p,\pi)\oplus\ker t(\pi,\pi_p)$$

(cf. [4]) and the fact that the map induced by inclusion $\iota(\pi_p,\pi):H^i(\pi;\mathbf{Z})\to H^i(\pi_p;\mathbf{Z})$ is monic on the p-primary subgroup, we infer, because π_p is cyclic, that $H^i(\pi;\mathbf{Z}_{(p)})\cong\mathbf{Z}/p^k$ or $H^i(\pi;\mathbf{Z}_{(p)})=0$. The former case is impossible since i is not a multiple of q and hence $H^i(\pi;\mathbf{Z}_{(p)})=0$ for $0<i<q$.

Let π be an arbitrary finite group of order n and $N=\sum x\in\mathbf{Z}\pi$, $x\in\pi$. Then

$$\begin{array}{ccc}\mathbf{Z}\pi & \longrightarrow & \mathbf{Z}\pi/N\\ \downarrow & & \downarrow\\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/n\end{array}$$

is a pullback square of rings (with obvious maps). Hence there is a short exact sequence of $\mathbf{Z}\pi/N$-modules

$$0\to I\mathbf{Z}\pi\to\mathbf{Z}\pi/N\to\mathbf{Z}/n\to 0$$

where $I\mathbf{Z}\pi$ denotes the augmentation ideal. Notice that a $\mathbf{Z}\pi/N$-module may be considered as a π-module via the projection $\mathbf{Z}\pi\to\mathbf{Z}\pi/N$.

DEFINITION 1.1. A $\mathbf{Z}\pi/N$-module M is said to be *trivial*, if it is trivial as a π-module; M is called *nilpotent*, if M possesses a finite filtration with associated graded module a trivial $\mathbf{Z}\pi/N$-module.

If M is a $\mathbf{Z}\pi/N$-module, then we will write IM for $(I\mathbf{Z}\pi)M$ and I^kM for $I(I^{k-1}M)$, $k \geq 2$. Obviously, M is then nilpotent if and only if $I^kM = 0$ for some k, (if and only if M is nilpotent as a π-module, respectively). Furthermore, M is a trivial $\mathbf{Z}\pi/N$-module if and only if $IM = 0$; hence a trivial $\mathbf{Z}\pi/N$-module is the same as a $\mathbf{Z}/n$-module. It is plain that the underlying abelian group of a nilpotent $\mathbf{Z}\pi/N$-module is an n-torsion group.

LEMMA 1.2. *Let π denote a finite group whose p-Sylow subgroups are cyclic of order p^k, p a fixed prime. Then, for $\mathbf{Z}/p^k$ considered as a trivial $\mathbf{Z}\pi/N$-module*

$$\text{proj. dim}_{\mathbf{Z}\pi/N}\,(\mathbf{Z}/p^k) \leq q$$

where q denotes the minimal p-period of π.

Proof. By [14] there exists a periodic resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbf{Z}_{(p)} \to 0$$

with P_i projective $\mathbf{Z}_{(p)}\pi$-modules, $P_i = P_{i+q}$ and $P_q \to P_{q-1}$ factoring through $\mathbf{Z}_{(p)}$. Let $\Lambda = \mathbf{Z}\pi/N$ and $\Lambda_p = \Lambda \otimes \mathbf{Z}_{(p)}$. From the short exact sequence $\mathbf{Z}_{(p)} \xrightarrow{N} \mathbf{Z}_{(p)}\pi \to \Lambda_p$ we deduce $H_i(\pi; \Lambda_p) \cong H_{i-1}(\pi; \mathbf{Z}_{(p)})$ for $i \geq 2$, and an exact sequence

$$0 \to H_1(\pi; \Lambda_p) \to \mathbf{Z}_{(p)} \xrightarrow{n} \mathbf{Z}_{(p)} \to H_0(\pi; \Lambda_p) \to 0.$$

Since $H_i(\pi; \mathbf{Z}_{(p)}) = 0$ for $0 < i < q-1$ we conclude that

$$H_i(\pi; \Lambda_p) = \begin{cases} \mathbf{Z}/p^k & \text{if} \quad i = 0, q \\ 0 & \text{if} \quad 0 < i < q. \end{cases}$$

These groups are the homology groups of the complex $\cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to 0$ in dimension $\leq q$, where $Q_i = \Lambda_p \otimes_\pi P_i$. Notice that Q_i is torsionfree as an abelian group, since it is Λ_p-projective. We know that $d_q: Q_q \to Q_{q-1}$ factors through $\Lambda_p \otimes_\pi \mathbf{Z}_{(p)} \cong \mathbf{Z}/p^k$ and therefore, since $\text{im}\,(d_q)$ is a torsionfree abelian group, we infer $d_q = 0$. Thus

$$0 \to Q_{q-1} \to Q_{q-2} \to \cdots \to Q_0 \twoheadrightarrow \mathbf{Z}/p^k$$

is a projective resolution of the trivial Λ_p-module $\mathbf{Z}/p^k$. As a result proj. $\dim_{\Lambda_p} (\mathbf{Z}/p^k) \leq q-1$. Of course proj. $\dim_\Lambda (\Lambda_p) = 1$, as one can see by tensoring a free abelian presentation of $\mathbf{Z}_{(p)}$ with Λ. As a consequence

$$\text{proj.}\dim_\Lambda (\mathbf{Z}/p^k) \leq \text{proj.}\dim_{\Lambda_p} (\mathbf{Z}/p^k) + \text{proj.}\dim_\Lambda (\Lambda_p) \leq q$$

which completes the proof of the lemma.

An immediate consequence is the following theorem which was mentioned in the introduction.

THEOREM 1.3. *Suppose π is a finite group of order n with cyclic Sylow subgroups. Then $\mathbf{Z}/n$ considered as a trivial $\mathbf{Z}\pi/N$-module has finite projective dimension.*

Proof. Write $\mathbf{Z}/n = \oplus \mathbf{Z}/p^{k(p)}$, the sum taken over all prime divisors of n. Then

$$\text{proj.}\dim_{\mathbf{Z}\pi/N} (\mathbf{Z}/n) = \max (\text{proj.}\dim_{\mathbf{Z}\pi/N} (\mathbf{Z}/p^{k(p)}) \mid p/n) < \infty$$

Remark. From the short exact sequence $I\mathbf{Z}\pi \to \mathbf{Z}\pi/N \to \mathbf{Z}/n$ we see that proj. $\dim_{\mathbf{Z}\pi/N} (I\mathbf{Z}\pi) = \text{proj.}\dim_{\mathbf{Z}\pi/N} (\mathbf{Z}/n) - 1$. Hence, if π has cyclic Sylow subgroups, we get from Theorem 1.3

$$\text{proj.}\dim_{\mathbf{Z}\pi/N} (I\mathbf{Z}\pi) < \infty$$

This generalizes a well known fact on the augmentation ideal of a finite cyclic group, in which case $I\mathbf{Z}\pi$ is free of rank 1 over $\mathbf{Z}\pi/N$.

We will apply later Lemma 1.2 and Theorem 1.3 in case π has square free order; for such a π the Sylow subgroups are of course cyclic of prime order.

LEMMA 1.4. *Let π be a finite group of square free order n and let M denote a nilpotent $\mathbf{Z}\pi/N$-module. Then*

(i) proj. $\dim_{\mathbf{Z}\pi/N} (M) < \infty$;

if, in addition, M is finitely generated, then

(ii) *M is of type FP and* card (M) *is a unit in* $\mathbf{Z}[1/n]$.

Proof. We first assume that M is a trivial $\mathbf{Z}\pi/N$-module. Then M is a direct sum of modules of the form $\mathbf{Z}/p$, p dividing n. From Lemma 1.2 we see then that proj. dim $M<\infty$. If M is a general nilpotent $\mathbf{Z}\pi/N$-module, we choose a finite filtration of M such that $gr(M)$ is a trivial $\mathbf{Z}\pi/N$-module. Clearly proj. dim $gr(M)\geq$ proj. dim M and *i*) follows. If M is finitely generated then, $\mathbf{Z}\pi/N$ being noetherian, we can find a projective resolution of M of finite length, which is also of finite type; by definition, M is therefore of type *FP*. Finally, a finitely generated nilpotent $\mathbf{Z}\pi/N$-module has as underlying abelian group a finitely generated n-torsion group. Hence card (M) is a unit in $\mathbf{Z}[1/n]$.

2. The transfer homomorphism $T: u(\mathbf{Z}[1/n]) \to (K_0\mathbf{Z}\pi)/\mathrm{im}\, S$

Let π denote a finite group of order n with cyclic Sylow subgroups. Then according to Theorem 1.3, proj. $\dim_{\mathbf{Z}\pi/N}(\mathbf{Z}/n)<\infty$, and therefore the canonical projection $j:\mathbf{Z}\pi/N\to\mathbf{Z}/n$ gives rise to a transfer map (cf. Bass [1, Chapter IX, 1.7])

$$j^*: K_0(\mathbf{Z}/n)\to K_0(\mathbf{Z}\pi/N).$$

The map j^* is defined on a generator $[\mathbf{Z}/p^k]$ of $K_0(\mathbf{Z}/n)$ by choosing a $\mathbf{Z}\pi/N$-projective resolution of finite type

$$0\to P_m\to P_{m-1}\to\cdots\to P_0\to\mathbf{Z}/p^k\to 0$$

of the trivial $\mathbf{Z}\pi/N$-module $\mathbf{Z}/p^k$, and setting

$$j^*[\mathbf{Z}/p^k]=\sum(-1)^i[P_i]\in K_0(\mathbf{Z}\pi/N).$$

Let $j_*: K_0(\mathbf{Z}\pi/N)\to K_0(\mathbf{Z}/n)$ denote the map induced by the projection $j:\mathbf{Z}\pi/N\to\mathbf{Z}/n$.

LEMMA 2.1. *$j_*j^*: K_0(\mathbf{Z}/n)\to K_0(\mathbf{Z}/n)$ is the 0-homomorphism.*

Proof. Let q denote the minimal p-periode of π and let $[\mathbf{Z}/p^k]\in K_0(\mathbf{Z}/n)$ denote a generator. Choose a $\mathbf{Z}\pi/N$-projective resolution of finite type of $\mathbf{Z}/p^k$

which has length q (cf. Lemma 1.2)

$$0 \to L_q \to L_{q-1} \to \cdots \to L_0 \to \mathbf{Z}/p^k \to 0.$$

Then

$$\begin{aligned} j_* j^*[\mathbf{Z}/p^k] &= j_*(\textstyle\sum (-1)^i [L_i]) \\ &= \textstyle\sum (-1)^i [\mathbf{Z}/n \otimes_\pi L_i] \\ &= \sum_{r/n} (\textstyle\sum (-1)^i [\mathbf{Z}/n(r) \otimes_\pi L_i]) \end{aligned}$$

where $n(r)$ stands for the highest power of the prime r, which divides n. For $r \neq p$ we have

$$\operatorname{Tor}^*_{\mathbf{Z}\pi/N}(\mathbf{Z}/n(r), \mathbf{Z}/p^k) = 0$$

and therefore the complex

$$0 \to \mathbf{Z}/n(r) \otimes_\pi L_q \to \cdots \to \mathbf{Z}/n(r) \otimes_\pi L_0 \to 0$$

is exact. Hence $\sum (-1)^i [\mathbf{Z}/n(r) \otimes_\pi L_i] = 0$ for $r \neq p$, and therefore $j_* j^*[\mathbf{Z}/p^k] = \sum (-1)^i [\mathbf{Z}/p^k \otimes_\pi L_i]$. To compute $\sum (-1)^i [\mathbf{Z}/p^k \otimes_\pi L_i]$ and the homology of $\{\mathbf{Z}/p^k \otimes_\pi L_i\}$ we can as well use the $\mathbf{Z}\pi/N \otimes \mathbf{Z}_{(p)}$-projective resolution $\{Q_i\}$ of $\mathbf{Z}/p^k$, which was considered in the proof of Lemma 1.2. Hence

$$\sum (-1)^i [\mathbf{Z}/p^k \otimes_\pi L_i] = \sum_{i=0}^{q-1} (-1)^i [\mathbf{Z}/p^k \otimes_\pi Q_i]$$

and plainly for $0 \leq i \leq q-1$ one has

$$\operatorname{Tor}^i_{\mathbf{Z}\pi/N}(\mathbf{Z}/p^k, \mathbf{Z}/p^k) = H_i(\pi; \mathbf{Z}/p^k) = \begin{cases} 0 & \text{for } 0 < i < q-1 \\ \mathbf{Z}/p^k & \text{for } i = 0, q-1 \end{cases}$$

Therefore $j_* j^*[\mathbf{Z}/p^k] = [\mathbf{Z}/p^k] + (-1)^{q-1}[\mathbf{Z}/p^k] = 0$ because the p-period q of π is an even number [15].

If π denotes an arbitrary group of order n then associated with the square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{pr} & \mathbf{Z}\pi/N \\ \downarrow & & \downarrow{\scriptstyle j} \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/n \end{array}$$

there is an exact sequence (cf. Milnor [9]) which reduces to

$$u(\mathbf{Z}/n) \xrightarrow{S} K_0(\mathbf{Z}\pi) \xrightarrow{pr_*} K_0(\mathbf{Z}\pi/N) \xrightarrow{j_*} \tilde{K}_0(\mathbf{Z}/n) \to 0 \tag{2.2}$$

We call S the *Swan homomorphism* (cf. [14]). S can be described in the following way: for k a unit mod n, $S(k) = [(k, N)]$ where (k, N) denotes the projective ideal in $\mathbf{Z}\pi$ generated by k and N.

Consider now the case of a π with cyclic Sylow subgroups. Then $j_*j^* = 0$ by Lemma 2.1 and, by the exactness of (2.2), the transfer j^* gives therefore rise to a homomorphism

$$t : K_0(\mathbf{Z}/n) \to (K_0\mathbf{Z}\pi)/\mathrm{im}\, S$$

such that $\overline{pr}_* t = j^*$, $\overline{pr}_* : (K_0\mathbf{Z}\pi)/\mathrm{im}\, S \to K_0(\mathbf{Z}\pi/N)$ denoting the map induced by pr_*.

If $n = p_1^{k_1} \cdots p_m^{k_m}$ then $K_0(\mathbf{Z}/n)$ is a free abelian group, freely generated by $\{[\mathbf{Z}/p_i^{k_i}], 1 \le i \le m\}$. Hence there is a unique group homomorphism

$$\varphi : u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to K_0(\mathbf{Z}/n)$$

such that $\varphi(\pm p_i) = [\mathbf{Z}/p_i^{k_i}]$. If we compose φ with t we get a map $T = t\varphi$ which we will also call a *transfer*, since it is induced by j^*. For π a group with cyclic Sylow subgroups we get therefore a commutative diagram

$$\begin{tikzcd} u(\mathbf{Z}[1/n]) \arrow[r, "T"] \arrow[d, "\varphi"] & (K_0\mathbf{Z}\pi)/\mathrm{im}\, S \arrow[d, tail, "\overline{pr}_*"] \\ K_0(\mathbf{Z}/n) \arrow[r, "j^*"] \arrow[ur, "t"] & K_0(\mathbf{Z}\pi/N) \end{tikzcd} \tag{2.3}$$

We will sometimes consider $K_0(\mathbf{Z}\pi/N)$ to be the range of T; this should not give rise to any confusion, since $\overline{pr}_*$ is injective.

It is well known that if R is a ring and M an R-module of type FP, then M defines an element $[M] \in K_0R$ (depending only upon the isomorphism class of M) by choosing any finite projective resolution of finite type

$$0 \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

and setting $[M]=\sum(-1)^i[P_i]\in K_0R$; if $0\to M'\to M\to M''\to 0$ is an exact sequence of modules of type *FP*, then $[M]=[M']+[M'']$ (cf. [1] and [11]).

LEMMA 2.4. *Let π denote a finite group of square free order n and let M denote a finitely generated nilpotent $\mathbf{Z}\pi/N$-module. Then*

$$T(\text{card } M)=[M]\in K_0(\mathbf{Z}\pi/N).$$

Proof. Notice that M is of type *FP* over $\mathbf{Z}\pi/N$ and card $M\in u(\mathbf{Z}[1/n])$ by Lemma 1.4. Hence $T(\text{card } M)$ and $[M]$ are well defined elements of $K_0(\mathbf{Z}\pi/N)$. If M is a trivial $\mathbf{Z}\pi/N$-module, then $T(\text{card } M)=j^*\varphi(\text{card } M)=[M]$ where the second equation follows from the definition of φ, j^* and $[M]$ respectively. For the general case we choose a finite filtration of M with $gr(M)$ a trivial $\mathbf{Z}\pi/N$-module. Clearly card $M=$ card $gr(M)$ and $[M]=[gr(M)]$; therefore $T(\text{card } M)=[M]$.

For the applications in the next section we will be particularly interested in groups π for which im $S=0$. The following theorem gives some information on T for such cases.

THEOREM 2.5. *Let π denote a finite group of order n with cyclic Sylow subgroups. Then*

(i) *$T\equiv 0$ in case n is a prime power or $n=2p$, p an odd prime.*
(ii) *$T(p_1\cdots p_m)=0$ if π is cyclic of order $p_1^{k_1}\cdots p_m^{k_m}$.*
(iii) *$T(3)=T(5)\neq 0$ if $n=15$, and $T(3)$ has order 2.*

Furthermore, in all three cases listed above one has im $S=0$, *and T can therefore be considered as a map $T: u(\mathbf{Z}[1/n])\to K_0\mathbf{Z}\pi$.*

We will break the proof up into a couple of lemmas.

LEMMA 2.6. *Let π be a cyclic group of order n. Then $j^*: K_0(\mathbf{Z}/n)\to K(\mathbf{Z}\pi/N)$ factors through $\tilde{K}_0(\mathbf{Z}/n)$.*

Proof. We may assume $n>1$. Let x denote a generator of π. Then $I\mathbf{Z}\pi$ is freely generated by $(1-x)$ over $\mathbf{Z}\pi/N$ and hence there is an exact sequence $0\to\mathbf{Z}\pi/N\to\mathbf{Z}\pi/N\to\mathbf{Z}/n\to 0$, from which we infer that $j^*[\mathbf{Z}/n]=[\mathbf{Z}\pi/N]-[\mathbf{Z}\pi/N]=0$. Thus j^* factors through $\tilde{K}_0(\mathbf{Z}/n)=K_0(\mathbf{Z}/n)/\langle[\mathbf{Z}/n]\rangle$.

LEMMA 2.7. *Let p denote an odd prime and $\pi=\mathbf{Z}/2p$ or the dihedral group D_{2p}. Then*

$$j^*=0: K_0(\mathbf{Z}/2p)\to K_0(\mathbf{Z}\pi/N)$$

Proof. We will first consider the case $\pi = \mathbf{Z}/2p$. Since j^* factors through $\tilde{K}_0(\mathbf{Z}/2p)$ which is cyclic, generated by the equivalence class of $[\mathbf{Z}/2]$, it suffices to prove that $j^*[\mathbf{Z}/2]=0$. Let $\pi = \langle x, y \mid x^2 = y^p = 1, xy = yx \rangle$, $R = \mathbf{Z}[\omega]$ with $\omega = \exp(2\pi i/p)$ and $R[\mathbf{Z}/2] \cong \mathbf{Z}\pi/(1+y+\cdots y^{p-1})$ the obvious isomorphism (mapping ω to y). Consider the pullback square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\lambda} & R[\mathbf{Z}/2] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \longrightarrow & \mathbf{F}_p[\mathbf{Z}/2] \end{array}$$

with obvious maps. Since $u(R[\mathbf{Z}/2]) \to u(\mathbf{F}_p[\mathbf{Z}/2])$ is surjective (cf. Reiner–Ullom [12, §7]) we get from the associated Milnor–Mayer–Vietoris sequence a monomorphism

$$\lambda_* : K_0\mathbf{Z}\pi \to K_0R[\mathbf{Z}/2].$$

Let $P \subset \mathbf{Z}\pi$ be the ideal generated by $(1-y)$ and 2. Then $\mathbf{Z}\pi/P \cong \mathbf{F}_2[\mathbf{Z}/2]$ is certainly cohomologically trivial and hence P is projective (cf. Rim [13]). Since $\mathbf{F}_2[\mathbf{Z}/2]/N = \mathbf{Z}/2$ we see that $j^*[\mathbf{Z}/2] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_\pi P]$. It suffices therefore to show that $[P] = [\mathbf{Z}\pi] \in K_0\mathbf{Z}\pi$. But $\lambda_*[P] = [(1-\omega, 2)] = [R[\mathbf{Z}/2]]$ since $R/(1-\omega)R \cong \mathbf{Z}/p$ and p odd. Hence $[P] = [\mathbf{Z}\pi]$ because λ_* is injective, from where we conclude that $j^*[\mathbf{Z}/2] = 0$. In case $\pi = D_{2p}$ we proceed in a similar way. Notice that $K_0(\mathbf{Z}/2p)$ is freely generated by $[\mathbf{Z}/2]$ and $[\mathbf{Z}/p]$. From Corollary 3.5 we infer that $j^*[\mathbf{Z}/p] = 0$ and we are therefore left showing that $j^*[\mathbf{Z}/2] = 0$. Let $D_{2p} = \langle x, y \mid x^2 = y^p, yxy = x \rangle$. Notice that $P = (1-y)\mathbf{Z}\pi + 2\mathbf{Z}\pi$ is a twosided ideal with $\mathbf{Z}\pi/P \cong \mathbf{F}_2[\mathbf{Z}/2]$, which is cohomologically trivial. Hence P is a projective π-module and clearly $j^*[\mathbf{Z}/2] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_\pi P]$. In order to see that $[P] = [\mathbf{Z}\pi]$ we consider the square of rings

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\lambda} & R_t[\mathbf{Z}/2] \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \longrightarrow & \mathbf{F}_p[\mathbf{Z}/2] \end{array}$$

with $R_t[\mathbf{Z}/2] = \mathbf{Z}\pi/(1+y+\cdots y^{p-1})$ a twisted group ring. By [12, §7] $u(R_t[\mathbf{Z}/2]) \to u(\mathbf{F}_p[\mathbf{Z}/2])$ is surjective and hence

$$\lambda_* : K_0\mathbf{Z}\pi \to K_0R_t[\mathbf{Z}/2]$$

is injective. Since $\mathbf{F}_2[\mathbf{Z}/2]\otimes_\pi R_t[\mathbf{Z}/2]\cong\mathbf{F}_2[\mathbf{Z}/2]\otimes_\pi \mathbf{F}_p[\mathbf{Z}/2]=0$ we infer that $\lambda_*[P]=[R_t[\mathbf{Z}/2]]$ and whence $[P]=[\mathbf{Z}\pi]$ from the injectivity of λ_*. Therefore $j^*[\mathbf{Z}/2]=0$, which completes the proof.

LEMMA 2.8. *Let* $\pi=\mathbf{Z}/15$. *Then* $T(3)$ *is the element of order* 2 *in* $K_0\mathbf{Z}[\mathbf{Z}/15]$.

Proof. Let $\pi=\langle x, y \mid x^3=y^5=1, xy=yx\rangle$ and let $P=(x+2)\mathbf{Z}\pi+(y-1)\mathbf{Z}\pi$. Then $\mathbf{Z}\pi/P=M$ is cyclic of order 9 with y operating trivially and x operating by multiplication with 7 mod 9. One checks easily that M is cohomologically trivial using the criterion of [13]. Hence P is projective. Since $M/NM\cong\mathbf{Z}/3$ as trivial $\mathbf{Z}\pi/N$-module we infer that

$$j^*[\mathbf{Z}/3]=[\mathbf{Z}\pi/N]-[\mathbf{Z}\pi/N\otimes_\pi P]\in K_0(\mathbf{Z}\pi/N).$$

Notice that $\operatorname{im} S=0$ since π is cyclic. Hence we can think of $T(3)$ to be the element $[\mathbf{Z}\pi]-[P]\in K_0(\mathbf{Z}\pi)$. Recall that $K_0\mathbf{Z}\pi\cong\mathbf{Z}\oplus\mathbf{Z}/2$ by Kervaire–Murthy [7]. Since P is projective of rank 1, it remains therefore to prove that $[P]^{\tilde{}}\neq 0$ in $\tilde{K}_0(\mathbf{Z}\pi)\cong\mathbf{Z}/2$. For this we consider the pullback square (with obvious maps)

$$\begin{array}{ccc}
\mathbf{Z}\pi & \longrightarrow & R[\mathbf{Z}/3] \\
\downarrow & & \downarrow \\
\mathbf{Z}[\mathbf{Z}/3] & \longrightarrow & \mathbf{F}_5[\mathbf{Z}/3]
\end{array}$$

where $R=\mathbf{Z}[\exp(2\pi i/5)]$. The associated Milnor–Mayer–Vietoris sequence yields a map

$$\partial: u(\mathbf{F}_5[\mathbf{Z}/3])\to \tilde{K}_0\mathbf{Z}[\mathbf{Z}/15]$$

By [7] ∂ factors through $u(\mathbf{F}_5(\omega))\cong\mathbf{Z}/24$ where $\mathbf{F}_5(\omega)$ is the field $\mathbf{F}_5[\mathbf{Z}/3]/(1+x+x^2)$, ω is the residue class of the generator $x\in\mathbf{Z}/3$. Furthermore, ∂ is surjective (cf. [7]). Notice that $R[\mathbf{Z}/3]\otimes_\pi M=0$ and therefore $P'=R[\mathbf{Z}/3]\otimes_\pi P\cong R[\mathbf{Z}/3]$. Furthermore $P''=\mathbf{Z}[\mathbf{Z}/3]\otimes_\pi P\subset\mathbf{Z}[\mathbf{Z}/3]$ is the principal ideal generated by $(x+2)$. Hence (cf. [9])

$$P=\{((x+2)a, b)\in\mathbf{Z}[\mathbf{Z}/3]\times R[\mathbf{Z}/3] \mid \overline{(x+2)}\bar{a}=\bar{b}\in\mathbf{F}_5[\mathbf{Z}/3]\}.$$

Notice that $\overline{x+2}\in u(\mathbf{F}_5[\mathbf{Z}/3])$ corresponds to $(3, \omega+2)\in u(\mathbf{F}_5)\times u(\mathbf{F}_5(\omega))$ under the obvious isomorphism $\mathbf{F}_5[\mathbf{Z}/3]\cong\mathbf{F}_5\times\mathbf{F}_5(\omega)$. It follows therefore that $\partial(\overline{x+2})=\partial(\omega+2)=[P]^{\tilde{}}$ and, since $(\omega+2)$ has order 24 in $u(\mathbf{F}_5(\omega))$, we conclude that $[P]^{\tilde{}}$ must have order 2. This completes the proof of the lemma.

We can now complete the proof of Theorem 2.5: First, if n is a prime power, π (having cyclic Sylow subgroups) is necessarily cyclic and therefore $j^*: K_0(\mathbf{Z}/n) \to K_0(\mathbf{Z}\pi/N)$ factors through $\tilde{K}_0(\mathbf{Z}/n)$ by Lemma 2.6. But $\tilde{K}_0(\mathbf{Z}/n) = 0$ for n a prime power. Hence $T \equiv 0$ in this case. If $n = 2p$, p an odd prime, then $\pi = \mathbf{Z}/2p$ or D_{2p} and it follows from Lemma 2.7 that $T \equiv 0$. Thus (i) holds. Assume now that π is cyclic of order $\prod p_i^{k_i} = n$. Then $T(\prod p_i) = t\varphi(\prod p_i) = t[\mathbf{Z}/n] = 0$, since $j^*[\mathbf{Z}/n] = 0$ by Lemma 2.6. Therefore (ii) holds. For (iii) notice that in case $n = 15$, π is necessarily cyclic. Hence, by applying Lemma 2.6, $j^*[\mathbf{Z}/3] = -j^*[\mathbf{Z}/5]$ or $T(3) = -T(5)$. Moreover $T(3)$ has order 2 by Lemma 2.8 and therefore $T(3) = T(5) \neq 0$. Finally im $S = 0$ for π cyclic or $\pi = D_{2p}$, p an odd prime (cf. Ullom [16]) and hence im $S = 0$ for the groups considered in (i)–(iii) of Theorem 2.5.

3. Applications to homologically nilpotent spaces

Following [3] we call a connected space X *homologically nilpotent*, if $\pi_1 X$ operates nilpotently on $H_i\tilde{X}$ for all i. In particular, if X is a nilpotent space, then X is homologically nilpotent and conversely, a homologically nilpotent space X is nilpotent if and only if $\pi_1 X$ is a nilpotent group (cf. [10]). Let $H_*^\pi(X; I\mathbf{Z}\pi)$ denote the homology of the (left) $\mathbf{Z}\pi/N$-complex $I\mathbf{Z}\pi \otimes_\pi C_*\tilde{X}$, where $\pi = \pi_1 X$; similarly for $H_*^\pi(X; \mathbf{Z}\pi/N)$.

LEMMA 3.1. *If X is homologically nilpotent with $\pi_1 X$ of finite order n, then $H_i^\pi(X; I\mathbf{Z}\pi)$ and $H_i^\pi(X; \mathbf{Z}\pi/N)$ are nilpotent $\mathbf{Z}\pi/N$-modules for all i. If, in addition, n is square free and X finitely dominated, then $H_i^\pi(X; I\mathbf{Z}\pi)$ and $H_i^\pi(X; \mathbf{Z}\pi/N)$ are of type FP over $\mathbf{Z}\pi/N$.*

Proof. Consider the long exact homology sequence associated with the exact sequence of chain complexes $0 \to I\mathbf{Z}\pi \otimes_\pi C_*\tilde{X} \to C_*\tilde{X} \to C_*X \to 0$. Standard results on nilpotent actions (cf. [6, Chapter I.4]) imply then that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a nilpotent π-module. Since the π-action on $H_i^\pi(X; I\mathbf{Z}\pi)$ factors through $\mathbf{Z}\pi/N$, we conclude that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a nilpotent $\mathbf{Z}\pi/N$-module. If X is dominated by a finite complex and $\pi_1 X$ finite, then certainly $H_i^\pi(X; I\mathbf{Z}\pi)$ is a finitely generated $\mathbf{Z}\pi/N$-module. Hence, by Lemma 1.4, $H_i^\pi(X; I\mathbf{Z}\pi)$ is of type *FP*. The proof for $H_i^\pi(X; \mathbf{Z}\pi/N)$ is similar, using $0 \to C_*X \to C_*\tilde{X} \to \mathbf{Z}\pi/N \otimes_\pi C_*\tilde{X} \to 0$.

LEMMA 3.2. *Let X be a finitely dominated homologically nilpotent space with finite fundamental group of order n and let $H_*(X, \tilde{X})$ denote the homology of the mapping cylinder of $\tilde{X} \to X$* mod $\tilde{X}$. *Then the groups $H_i^\pi(X, I\mathbf{Z}\pi)$ and $H_{i+1}(X, \tilde{X})$ have the same finite cardinality $c(i)$ for $i \geq 0$, and $c(i)$ is a unit in $\mathbf{Z}[1/n]$.*

Proof. Since $H_i^\pi(X; I\mathbf{Z}\pi)$ is a finitely generated nilpotent $\mathbf{Z}\pi/N$-module, it has by Lemma 1.4 a cardinality $c(i)$ which is a unit in $\mathbf{Z}[1/n]$. From long exact homology sequences it is obvious that $c(i)$ is also given by

$$c(i) = (\text{card coker } (H_{i+1}\tilde{X} \to H_{i+1}X)) \cdot (\text{card ker } (H_i\tilde{X} \to H_iX))$$

which equals card $H_{i+1}(X, \tilde{X})$.

It follows that for X as in Lemma 3.2, the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X})/\text{card } H_{\text{ev}}(X, \tilde{X})$$

is a well defined unit in $\mathbf{Z}[1/n]$. This unit $\rho(X)$ is related to the finiteness obstruction $wX \in K_0\mathbf{Z}\pi_1X$ of Wall (cf. [17], [10]) in the following way.

THEOREM 3.3. *Let X be a homologically nilpotent space with non-trivial fundamental group π of square free order n. Suppose further that X is dominated by a finite complex and let $\bar{w}X$ denote the image of wX in $(K_0\mathbf{Z}\pi_1X)/\text{im } S$. Then*

$$T\rho(X) = \bar{w}X$$

If X is in addition nilpotent (i.e. π_1X is cyclic) then

(i) $T\rho(X) = wX$, and

(ii) $wX = 0$ in case $\rho(X)$ is a power (positive or negative) of n.

Before proving Theorem 3.3 we will establish a different way of computing $\rho(X)$.

LEMMA 3.4. *Let X be as in* (3.2) *and let $H_*^\pi(X; \mathbf{Z}\pi/N)$ denote the homology of $\mathbf{Z}\pi/N \otimes_\pi C_*(\tilde{X})$. Then*

$$\rho(X) = \text{card } H_{\text{ev}}^\pi(X; \mathbf{Z}\pi/N)/\text{card } H_{\text{odd}}^\pi(X; \mathbf{Z}\pi/N)$$

Proof. We may of course assume that $n = \text{card } \pi_1X > 1$. Since $H_*(\tilde{X}; \mathbf{Q})$ is semisimple and nilpotent as π-module, we have $H_*(\tilde{X}; \mathbf{Q}) \cong H_*(X; \mathbf{Q})$ and therefore the Euler characteristic of X vanishes. Thus, for all primes p, $\chi_p(X) = \sum (-1)^i \dim H_i(X; \mathbf{Z}/p) = 0$. Since n is square free, we infer then that

$$\text{card } H_{\text{ev}}(X; \mathbf{Z}/n) = \text{card } H_{\text{odd}}(X; \mathbf{Z}/n).$$

The long exact homology sequence associated with the exact sequence $I\mathbf{Z}\pi \to \mathbf{Z}\pi/N \to \mathbf{Z}/n$ then yields

$$\operatorname{card} H^{\pi}_{\mathrm{ev}}(X; \mathbf{Z}\pi/N)/\operatorname{card} H^{\pi}_{\mathrm{odd}}(X; \mathbf{Z}\pi/N) = \operatorname{card} H^{\pi}_{\mathrm{ev}}(X; I\mathbf{Z}\pi)/\operatorname{card} H^{\pi}_{\mathrm{odd}}(X; I\mathbf{Z}\pi)$$

from where the result follows, using Lemma 3.2.

Proof of Theorem 3.3. Let $pr_*: K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N)$ be the map induced by the projection. If $\bar{C}_*$ denotes a chain complex of type *FP*, homotopy equivalent to the singular complex of $\tilde{X}$, then $pr_* wX = \sum (-1)^i [\mathbf{Z}\pi/N \otimes_\pi \bar{C}_i]$. Since $H^{\pi}_i(X; \mathbf{Z}\pi/N)$ is of type *FP* (cf. Lemma 3.1) we infer that (cf. [11])

$$\sum (-1)^i [\mathbf{Z}\pi/N \otimes_\pi \bar{C}_i] = \sum (-1)^i [H^{\pi}_i(X; \mathbf{Z}\pi/N)]$$

and therefore, since $[H^{\pi}_i(X; \mathbf{Z}\pi/N)] = T(\operatorname{card} H^{\pi}_i(X; \mathbf{Z}\pi/N))$ by Lemma 2.4, we get

$$pr_* wX = \bar{w}X = T(\operatorname{card} H^{\pi}_{\mathrm{ev}}(X; \mathbf{Z}\pi/N)/\operatorname{card} H^{\pi}_{\mathrm{odd}}(X; \mathbf{Z}\pi/N)) = T\rho(X).$$

In case X is in addition nilpotent, $\pi_1 X$ is necessarily cyclic and therefore im $S = 0$. Hence $T\rho(X) = wX$ in this case. Furthermore, if $\rho(X) = n^k$, we conclude from Theorem 2.5(ii) that $wX = T(n^k) = 0$. This completes the proof of Theorem 3.3.

As a first application we will prove the following algebraic result, which is used to prove Lemma 2.7.

COROLLARY 3.5. *Let π denote the dihedral group D_{2p}, p an odd prime. Then $j^*[\mathbf{Z}/p] = 0 \in K_0(\mathbf{Z}D_{2p}/N)$.*

Proof. By Theorem A [14] there is a finite simplicial complex $\tilde{X}$ of the homotopy type of S^3 on which D_{2p} acts freely and simplicially. Let $X = \tilde{X}/D_{2p}$. Then

$$H_i(X, \tilde{X}) = \begin{cases} 0, & \text{if } i = 0, 2 \\ \mathbf{Z}/2, & \text{if } i = 1 \\ \mathbf{Z}/2p, & \text{if } i = 3 \end{cases}$$

Thus $\rho(X) = 4p$. Since X is a finite complex with trivial action of $\pi_1 X$ on $H_* \tilde{X}$, we infer that $wX = 0$ and therefore $T(4p) = 0$. We have already observed in course of the proof of Lemma 2.7 that $j^*[\mathbf{Z}/2] = 0$. Hence

$$j^*[\mathbf{Z}/p] = j^*[(\mathbf{Z}/2) \oplus (\mathbf{Z}/2) \oplus (\mathbf{Z}/p)] = T(4p) = 0.$$

Recall that two nilpotent spaces X and Y of finite type are of the same *genus* (cf. [6]) if their p-localizations X_p, Y_p are homotopy equivalent for all primes p. In [10] one finds an example of two finitely dominated nilpotent spaces X and Y of the same genus with fundamental groups of order 8 such that $wX \neq 0$ but $wY = 0$. For fundamental groups of square free orders, such an example is impossible. Namely one has

COROLLARY 3.6. *Let X and Y be two finitely dominated nilpotent spaces of the same genus, with non trivial fundamental groups of square free order n. Then $\rho(X) = \rho(Y)$ and wX, wY have the same finite orders.*

Proof. Notice that $\pi_1 X$ and $\pi_1 Y$ are abelian and whence $\pi_1 X \cong \pi_1 Y$, since X and Y are of the same genus (cf. [6]). Furthermore, there are for all primes p commutative diagrams

$$\begin{tikzcd} \tilde{X}_p \arrow[r] \arrow[d, "pr"'] & \tilde{Y}_p \arrow[d, "pr"] \\ X_p \arrow[r] & Y_p \end{tikzcd}$$

with the horizontal maps being homotopy equivalences. Thus

$$H_i(X, \tilde{X}; \mathbf{Z}_{(p)}) \cong H_i(X_p, \tilde{X}_p; \mathbf{Z}_{(p)}) \cong H_i(Y_p, \tilde{Y}_p; \mathbf{Z}_{(p)}) \cong H_i(Y, \tilde{Y}; \mathbf{Z}_{(p)})$$

and therefore $H_i(X, \tilde{X}) \cong H_i(Y, \tilde{Y})$ since the groups $H_i(X, \tilde{X})$ and $H_i(Y, \tilde{Y})$ are finite. Hence $\rho X = \rho Y$ and, since $\pi_1 X \cong \pi_1 Y$, it follows from Theorem 3.3(i) that wX and wY have the same orders; the orders must be finite, because X and Y must have vanishing Euler characteristic (cf. proof of Lemma 3.4).

Another application of Theorem 3.3 is the following vanishing theorem for the Wall obstruction.

COROLLARY 3.7. *Let X be a finitely dominated homologically nilpotent space with fundamental group of order p or $2p$, p a prime. Then $wX = 0$ and X is therefore of the homotopy type of a finite complex.*

Proof. First consider the case $p = 2$. By a result of Fröhlich (cf. [5, Theorem 6(i)]) we infer that $\tilde{K}_0 \pi_1 X = 0$ and, since the Euler characteristic of X is necessarily 0 (cf. proof of Lemma 3.4), it follows that $wX = 0$. Second, let p denote an odd prime. Then we may apply Theorem 3.3 and obtain $T\rho(X) = \tilde{w}X$. Since $T \equiv 0$ and im $S = 0$ for the $\pi_1 X$ in question (cf. Theorem 2.5), we infer that $wX = 0$.

It is sometimes possible to compute $\rho(X)$ directly from H_*X, giving rise to a particular simple formula for wX. We will treat one such case in the next section and plan to treat other cases in a forthcoming paper.

4. The Wall obstruction for H-spaces

We want to prove the Theorem II mentioned in the introduction.

Suppose X is a space with $\oplus H_i(X; \mathbf{Z})$ finitely generated and let p denote a fixed prime. Then we write

$$\chi_p(X, t) = \sum (-1)^i \beta_i t^i, \qquad \beta_i = \dim H_i(X; \mathbf{Z}/p)$$

for the *Poincaré polynomial* of X with respect to $\mathbf{Z}/p$. Define

$$e_p(X) = \frac{d}{dt} \chi_p(X, t) \bigg|_{t=-1}$$

and, in case $\pi_1 X$ has finite order n, define

$$e(X) = \prod_{p/n} p^{e_p(X)}$$

THEOREM 4.1. *Let X be a finitely dominated nilpotent space with fundamental group π of square free order. Suppose that $H_i^\pi(X; I\mathbf{Z}\pi)$ is a trivial π-module for all i. Then*

$$\rho(X) = e(X)$$

Proof. We may assume that $\pi \neq \{1\}$. Let x denote a generator of the necessarily cyclic group π. Then there is a exact sequence of $\mathbf{Z}\pi/N$-modules

$$0 \to I\mathbf{Z}\pi \xrightarrow{1-x} I\mathbf{Z}\pi \to \mathbf{Z}/n \to 0$$

where n denotes the order of π. Since $(1-x)$ induces 0 in $H_*^\pi(X; I\mathbf{Z}\pi)$, the associated long exact homology sequence breaks up into short exact sequences

$$0 \to H_i^\pi(X; I\mathbf{Z}\pi) \to H_i(X; \mathbf{Z}/n) \to H_{i-1}^\pi(X; I\mathbf{Z}\pi) \to 0$$

for $i \geq 0$. Since $H_i^\pi(X; I\mathbf{Z}\pi)$ is a trivial $\mathbf{Z}\pi/N$-module and n is square free, $H_i^\pi(X; I\mathbf{Z}\pi)\otimes\mathbf{Z}_{(p)}$ is a $\mathbf{Z}/p$-vector space. Define

$$\beta_i = \dim H_i(X; \mathbf{Z}/p), \qquad \gamma_i = \dim H_i^\pi(X; I\mathbf{Z}\pi)\otimes\mathbf{Z}_{(p)}.$$

Then

$$\gamma_i = \beta_i - \gamma_{i-1} = \beta_i - \beta_{i-1} + \cdots + (-1)^i\beta_0$$

and

$$\begin{aligned}\sum(-1)^j\gamma_j &= \sum(\beta_0 - \beta_1 + \cdots + (-1)^j\beta_j)\\ &= (m+1)\beta_0 - m\beta_1 + \cdots + (-1)^m\beta_m\end{aligned}$$

where m denotes the largest integer k with $\beta_k \neq 0$. Hence

$$\begin{aligned}\sum(-1)^j\gamma_j &= (-1)^m\frac{d}{dt}\left(t^{m+1}\chi_p\left(X, \frac{1}{t}\right)\right)\Big|_{t=-1}\\ &= (m+1)\chi_p(X,-1) + \chi_p'(X,-1).\end{aligned}$$

Since X is homologically nilpotent with non-trivial finite fundamental group, the Euler characteristic of X is 0 (cf. proof of Lemma 3.4) and hence $\chi_p(X,-1) = 0$. The above equation reduces therefore to

$$\sum(-1)^j\gamma_j = \chi_p'(X,-1) = e_p(X).$$

Hence card $H_{\mathrm{ev}}^\pi(X; I\mathbf{Z}\pi)$/card $H_{\mathrm{odd}}^\pi(X; I\mathbf{Z}\pi) = \prod_{p/n} p^{e_p(X)} = e(X)$ and therefore $\rho(X) = e(X)$.

In order to prove that for an H-space X the π-operation on $H_*(X; I\mathbf{Z}\pi)$ is trivial, we will need the following lemma.

LEMMA 4.2. *Let X be an H-space and $a \in \pi_1 X$. Then the induced covering transformation $a_*: \tilde{X} \to \tilde{X}$ is equivariantly homotopic to the identity.*

Proof. Without loss of generality we may assume that the H-structure $\mu: X\times X \to X$ has the base point as a strict identity. Equip $\tilde{X}$ with the canonical H-structure $\tilde{\mu}$. Then $pr^{-1}(*)$ is a central subgroup of $(\tilde{X}, \tilde{\mu})$, naturally isomorphic to $\pi_1 X$. We may thus think of a as an element of $pr^{-1}(*)$ which acts on $\tilde{X}$ by left multiplication. Choosing a path from $a \in pr^{-1}(*)$ to $*$ we get a homotopy of the map a_* to Id, which is equivariant with respect to the $\pi_1 X = pr^{-1}(*)$-action, because $pr^{-1}(*)$ is central.

COROLLARY 4.3. *If X is an H-space, then $H_*^{\pi}(X; I\mathbf{Z}\pi)$ is a trivial $\mathbf{Z}\pi/N$-module.*

This is clear since by 4.2, the operation of $a \in \pi_1 X$ on $C_*\tilde{X}$ is chain homotopic to Id as map of $\pi_1 X$-complexes, and therefore the induced action of a on $I\mathbf{Z}\pi \otimes_\pi C_*\tilde{X}$ is chain homotopic to Id.

Proof of Theorem II of the Introduction. If X is of rank 1, then X is equivalent to one of the spaces S^1, S^3, S^7, $\mathbf{R}P^3$ or $\mathbf{R}P^7$ (cf. Browder [2]). Hence $wX = 0$ in these cases. If rank $(X) \geq 2$, then $\chi_p(X, t)$ contains a factor $(1-t^{n_1})(1-t^{n_2})$ with n_1 and n_2 odd. Therefore

$$e_p(X) = \chi_p'(X, -1) = 0$$

for all primes p. In particular we obtain $e(X) = 1$ and, since $H_i^{\pi}(X; I\mathbf{Z}\pi)$ is a trivial π-module for all i we infer from Theorem 4.1 that $\rho(X) = e(X)$. Hence $wX = T\rho(X) = 0$.

5. Appendix

If X denotes a homologically nilpotent space with finite fundamental group, then there is a simple criterion for deciding whether X is dominated by a finite complex.

THEOREM 5.1. *Let X be a homologically nilpotent complex with finite fundamental group. Then the following are equivalent.*

(i) *X is dominated by a finite complex*

(ii) *$H_i(\tilde{X}; \mathbf{Z})$ and $H_i(X; \mathbf{Z})$ are finitely generated abelian groups for all i and zero for i sufficiently large.*

Proof. Certainly (i) implies (ii). If (ii) is given, then from [3, Corollary 3.4] we infer that X has the homotopy type of a finite dimensional complex. Since $\mathbf{Z}\pi_1 X$ is noetherian and $H_i(\tilde{X}; \mathbf{Z})$ finitely generated for all i, Theorems B and F of [17] imply that X is dominated by a finite complex.

A more general result of this type in case X is nilpotent was proved in [11].

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