**Zeitschrift:** Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

**Band:** 52 (1977)

**Artikel:** Groups with cyclic Sylow subgroups and finiteness conditions for

certain complexes.

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**DOI:** https://doi.org/10.5169/seals-40005

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# Groups with cyclic Sylow subgroups and finiteness conditions for certain complexes

G. MISLIN

### Introduction

Let  $\pi$  denote a finite group of order n whose Sylow subgroups are all cyclic and let  $N = \sum x \in \mathbb{Z}\pi$ ,  $x \in \pi$ , denote the norm element. The augmentation  $\mathbb{Z}\pi \to \mathbb{Z}$  induces a map  $j: \mathbb{Z}\pi/N \to \mathbb{Z}/n$  which we use to consider  $\mathbb{Z}/n$  as a  $\mathbb{Z}\pi/N$ -module. We show (Theorem 1.3) that

proj. 
$$\dim_{\mathbf{Z}_{\pi/N}}(\mathbf{Z}/n) < \infty$$
.

Thus there is a transfer map

$$j^*: K_0(\mathbb{Z}/n) \to K_0(\mathbb{Z}\pi/N)$$

between projective class groups. It that im  $(i^*) \subset$ turns out im  $(pr_*: K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N))$ and, since  $\operatorname{im} (K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N)) \cong$  $(K_0 \mathbb{Z}\pi)/\text{im } S \text{ where } S: K_1(\mathbb{Z}/n) \to K_0(\mathbb{Z}\pi) \text{ denotes the Swan homomorphism (cf. }$ Section 2), we can think of the transfer map to map  $K_0(\mathbf{Z}/n)$  into  $(K_0\mathbf{Z}\pi)/\text{im } S$ . If we compose this map with the obvious homomorphism

$$u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to K_0(\mathbf{Z}/n)$$

we obtain a "transfer" homomorphism

$$T: u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to (K_0\mathbf{Z}\pi)/\text{im } S$$

 $(u(\mathbf{Z}[1/n]))$  denotes the group of units of  $\mathbf{Z}[1/n]$ ). The homomorphism T is in general non-trivial, even if  $\pi$  is cyclic (in which case im S=0). However, we

show that T = 0 if n is a prime power or if n = 2p, p an odd prime (cf. Theorem 2.5).

In the second half of the paper we make use of the homomorphism T to compute the Wall obstruction  $wX \in K_0\mathbb{Z}\pi_1X$  for certain complexes. We will consider spaces X for which  $\pi_1X$  operates nilpotently on  $H_*\tilde{X}$  (i.e. X is homologically nilpotent in the sense of Brown-Kahn [3]). If such a space is dominated by a finite complex and has a finite fundamental group of order n, then the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X})/\text{card } H_{\text{ev}}(X, \tilde{X})$$

is well defined and is a unit in  $\mathbb{Z}[1/n]$ ;  $\rho(X)$  is related to the finiteness obstruction wX in the following way, (cf. Theorem 3.3).

THEOREM I. Let X be a finitely dominated homologically nilpotent space with non-zero finite fundamental group of square free order. Then

$$T\rho(X) = \bar{w}X$$

where  $\bar{w}X$  denotes the image of wX in  $(K_0\mathbf{Z}\pi_1X)/\text{im }S$ .

In particular, if the space X in Theorem I is supposed to be *nilpotent*, then  $\pi_1 X$ -being nilpotent and of square free order-is necessarily cyclic and therefore im S = 0 by a result of Swan [14]. The formula reduces then to

$$T\rho(X) = wX$$

yielding new information concerning the Wall obstruction for nilpotent spaces.

Under suitable conditions on X the rational number  $\rho(X)$  depends only upon  $H_*X$ : Suppose that  $\pi_1X$  is cyclic of square free order n operating trivially on  $H_*^{\pi}(X; I\mathbf{Z}\pi)$ . Then we show that

$$\rho(X) = \prod_{p/n} p^{e_p(X)}$$

the product being taken over all prime divisors of n, and  $e_p(X)$  denoting the value at -1 of the derivative of the *Poincaré polynomial* of X with respect to  $\mathbb{Z}/p$ -coefficients, a quantity depending only upon  $H_*X$ .

As an illustration we show that for X an H-space of rank  $\geq 2$  one has  $e_p(X) = 0$  for all primes p, and hence  $\rho(X) = 1$ . The following vanishing theorem for the Wall obstruction for H-spaces then follows.

THEOREM II. Let X be a finitely dominated H-complex with finite fundamental group of square free order. Then wX = 0 and X is therefore of the homotopy type of a finite complex.

## 1. Groups with cyclic Sylow subgroups and $Z\pi/N$ -modules

Let  $\pi$  denote a finite group whose p-Sylow subgroups are cyclic of order  $p^k$  for a fixed prime p. Such a group  $\pi$  is p-periodic in the sense of Cartan-Eilenberg [4]. If q denotes the smallest p-period of  $\pi$ , then  $H^q(\pi; \mathbf{Z}_{(p)}) \cong \mathbf{Z}/p^k$ , where  $\mathbf{Z}_{(p)}$  denotes the integers localized at p. Furthermore, if  $H^i(\pi; \mathbf{Z}_{(p)}) \cong \mathbf{Z}/p^k$  for some i > 0, then i is necessarily a multiple of q (see Swan [15]). It has been observed by Lundmark [8] that

$$H^{i}(\pi; \mathbf{Z}_{(p)}) = 0$$
 for  $0 < i < q$ .

Namely, suppose i is an integer with 0 < i < q and let  $\pi_p$  denote a p-Sylow subgroup of  $\pi$ . Then from the decomposition

$$H^{i}(\pi_{p}; \mathbf{Z}) \cong \operatorname{im} \iota(\pi_{p}, \pi) \oplus \ker \iota(\pi, \pi_{p})$$

(cf. [4]) and the fact that the map induced by inclusion  $\iota(\pi_p, \pi): H^i(\pi; \mathbf{Z}) \to H^i(\pi_p; \mathbf{Z})$  is monic on the *p*-primary subgroup, we infer, because  $\pi_p$  is cyclic, that  $H^i(\pi; \mathbf{Z}_{(p)}) \cong \mathbf{Z}/p^k$  or  $H^i(\pi; \mathbf{Z}_{(p)}) = 0$ . The former case is impossible since *i* is not a multiple of *q* and hence  $H^i(\pi; \mathbf{Z}_{(p)}) = 0$  for 0 < i < q.

Let  $\pi$  be an arbitrary finite group of order n and  $N = \sum x \in \mathbb{Z}\pi$ ,  $x \in \pi$ . Then

$$\begin{array}{ccc}
\mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi/N \\
\downarrow & \downarrow \\
\mathbf{Z} \longrightarrow \mathbf{Z}/n
\end{array}$$

is a pullback square of rings (with obvious maps). Hence there is a short exact sequence of  $\mathbb{Z}\pi/N$ -modules

$$0 \rightarrow IZ\pi \rightarrow Z\pi/N \rightarrow Z/n \rightarrow 0$$

where  $IZ\pi$  denotes the augmentation ideal. Notice that a  $Z\pi/N$ -module may be considered as a  $\pi$ -module via the projection  $Z\pi \to Z\pi/N$ .

DEFINITION 1.1. A  $\mathbb{Z}\pi/N$ -module M is said to be *trivial*, if it is trivial as a  $\pi$ -module; M is called *nilpotent*, if M possesses a finite filtration with associated graded module a trivial  $\mathbb{Z}\pi/N$ -module.

If M is a  $\mathbb{Z}\pi/N$ -module, then we will write IM for  $(I\mathbb{Z}\pi)M$  and  $I^kM$  for  $I(I^{k-1}M)$ ,  $k \ge 2$ . Obviously, M is then nilpotent if and only if  $I^kM = 0$  for some k, (if and only if M is nilpotent as a  $\pi$ -module, respectively). Furthermore, M is a trivial  $\mathbb{Z}\pi/N$ -module if and only if IM = 0; hence a trivial  $\mathbb{Z}\pi/N$ -module is the same as a  $\mathbb{Z}/n$ -module. It is plain that the underlying abelian group of a nilpotent  $\mathbb{Z}\pi/N$ -module is an n-torsion group.

LEMMA 1.2. Let  $\pi$  denote a finite group whose p-Sylow subgroups are cyclic of order  $p^k$ , p a fixed prime. Then, for  $\mathbb{Z}/p^k$  considered as a trivial  $\mathbb{Z}\pi/N$ -module

proj. 
$$\dim_{\mathbf{Z}_{\pi/N}} (\mathbf{Z}/p^k) \leq q$$

where q denotes the minimal p-period of  $\pi$ .

*Proof.* By [14] there exists a periodic resolution

$$\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbf{Z}_{(p)} \rightarrow 0$$

with  $P_i$  projective  $\mathbf{Z}_{(p)}\pi$ -modules,  $P_i = P_{i+q}$  and  $P_q \to P_{q-1}$  factoring through  $\mathbf{Z}_{(p)}$ . Let  $\Lambda = \mathbf{Z}\pi/N$  and  $\Lambda_p = \Lambda \otimes \mathbf{Z}_{(p)}$ . From the short exact sequence  $\mathbf{Z}_{(p)} \xrightarrow{N} \mathbf{Z}_{(p)}\pi \to \Lambda_p$  we deduce  $H_i(\pi; \Lambda_p) \cong H_{i-1}(\pi; \mathbf{Z}_{(p)})$  for  $i \ge 2$ , and an exact sequence

$$0 \to H_1(\pi; \Lambda_p) \to \mathbf{Z}_{(p)} \xrightarrow{n} \mathbf{Z}_{(p)} \to H_0(\pi; \Lambda_p) \to 0.$$

Since  $H_i(\pi; \mathbf{Z}_{(p)}) = 0$  for 0 < i < q - 1 we conclude that

$$H_i(\pi; \Lambda_p) = \begin{cases} \mathbf{Z}/p^k & \text{if } i = 0, q \\ 0 & \text{if } 0 < i < q. \end{cases}$$

These groups are the homology groups of the complex  $\cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to 0$  in dimension  $\leq q$ , where  $Q_i = \Lambda_p \otimes_{\pi} P_i$ . Notice that  $Q_i$  is torsionfree as an abelian group, since it is  $\Lambda_p$ -projective. We know that  $d_q: Q_q \to Q_{q-1}$  factors through  $\Lambda_p \otimes_{\pi} \mathbf{Z}_{(p)} \cong \mathbf{Z}/p^k$  and therefore, since im  $(d_q)$  is a torsionfree abelian

group, we infer  $d_q = 0$ . Thus

$$0 \rightarrow Q_{q-1} \rightarrow Q_{q-2} \rightarrow \cdots \rightarrow Q_0 \twoheadrightarrow \mathbf{Z}/p^k$$

is a projective resolution of the trivial  $\Lambda_p$ -module  $\mathbb{Z}/p^k$ . As a result proj.  $\dim_{\Lambda_p}(\mathbb{Z}/p^k) \leq q-1$ . Of course proj.  $\dim_{\Lambda}(\Lambda_p) = 1$ , as one can see by tensoring a free abelian presentation of  $\mathbb{Z}_{(p)}$  with  $\Lambda$ . As a consequence

proj. 
$$\dim_{\Lambda} (\mathbf{Z}/p^k) \leq \operatorname{proj.} \dim_{\Lambda_p} (\mathbf{Z}/p^k) + \operatorname{proj.} \dim_{\Lambda} (\Lambda_p) \leq q$$

which completes the proof of the lemma.

An immediate consequence is the following theorem which was mentioned in the introduction.

THEOREM 1.3. Suppose  $\pi$  is a finite group of order n with cyclic Sylow subgroups. Then  $\mathbb{Z}/n$  considered as a trivial  $\mathbb{Z}\pi/N$ -module has finite projective dimension.

*Proof.* Write  $\mathbb{Z}/n = \oplus \mathbb{Z}/p^{k(p)}$ , the sum taken over all prime divisors of n. Then

proj. 
$$\dim_{\mathbf{Z}_{\pi/N}}(\mathbf{Z}/n) = \max (\text{proj. } \dim_{\mathbf{Z}_{\pi/N}}(\mathbf{Z}/p^{k(p)}) | p/n) < \infty$$

Remark. From the short exact sequence  $I\mathbf{Z}\pi \to \mathbf{Z}\pi/N \to \mathbf{Z}/n$  we see that proj.  $\dim_{\mathbf{Z}\pi/N}(I\mathbf{Z}\pi) = \text{proj. } \dim_{\mathbf{Z}\pi/N}(\mathbf{Z}/n) - 1$ . Hence, if  $\pi$  has cyclic Sylow subgroups, we get from Theorem 1.3

proj. 
$$\dim_{\mathbf{Z}\pi/N}(I\mathbf{Z}\pi)<\infty$$

This generalizes a well known fact on the augmentation ideal of a finite cyclic group, in which case  $IZ\pi$  is free of rank 1 over  $Z\pi/N$ .

We will apply later Lemma 1.2 and Theorem 1.3 in case  $\pi$  has square free order; for such a  $\pi$  the Sylow subgroups are of course cyclic of prime order.

LEMMA 1.4. Let  $\pi$  be a finite group of square free order n and let M denote a nilpotent  $\mathbb{Z}\pi/N$ -module. Then

(i) proj.  $\dim_{\mathbf{Z}_{\pi/N}}(M) < \infty$ ;

if, in addition, M is finitely generated, then

# (ii) M is of type FP and card (M) is a unit in $\mathbb{Z}[1/n]$ .

**Proof.** We first assume that M is a trivial  $\mathbb{Z}\pi/N$ -module. Then M is a direct sum of modules of the form  $\mathbb{Z}/p$ , p dividing n. From Lemma 1.2 we see then that proj. dim  $M < \infty$ . If M is a general nilpotent  $\mathbb{Z}\pi/N$ -module, we choose a finite filtration of M such that gr(M) is a trivial  $\mathbb{Z}\pi/N$ -module. Clearly proj. dim  $gr(M) \ge \operatorname{proj}$ . dim M and i) follows. If M is finitely generated then,  $\mathbb{Z}\pi/N$  being noetherian, we can find a projective resolution of M of finite length, which is also of finite type; by definition, M is therefore of type FP. Finally, a finitely generated nilpotent  $\mathbb{Z}\pi/N$ -module has as underlying abelian group a finitely generated n-torsion group. Hence card (M) is a unit in  $\mathbb{Z}[1/n]$ .

## 2. The transfer homomorphism $T: u(\mathbb{Z}[1/n]) \to (K_0\mathbb{Z}\pi)/\text{im } S$

Let  $\pi$  denote a finite group of order n with cyclic Sylow subgroups. Then according to Theorem 1.3, proj.  $\dim_{\mathbf{Z}\pi/N}(\mathbf{Z}/n) < \infty$ , and therefore the canonical projection  $j: \mathbf{Z}\pi/N \to \mathbf{Z}/n$  gives rise to a transfer map (cf. Bass [1, Chapter IX, 1.7])

$$j^*: K_0(\mathbf{Z}/n) \to K_0(\mathbf{Z}\pi/N).$$

The map  $j^*$  is defined on a generator  $[\mathbb{Z}/p^k]$  of  $K_0(\mathbb{Z}/n)$  by choosing a  $\mathbb{Z}\pi/N$ -projective resolution of finite type

$$0 \to P_m \to P_{m-1} \to \cdots \to P_0 \to \mathbb{Z}/p^k \to 0$$

of the trivial  $\mathbb{Z}\pi/N$ -module  $\mathbb{Z}/p^k$ , and setting

$$j^*[\mathbf{Z}/p^k] = \sum (-1)^i [P_i] \in K_0(\mathbf{Z}\pi/N).$$

Let  $j_*: K_0(\mathbf{Z}\pi/N) \to K_0(\mathbf{Z}/n)$  denote the map induced by the projection  $j: \mathbf{Z}\pi/N \to \mathbf{Z}/n$ .

LEMMA 2.1. 
$$j_*j^*: K_0(\mathbf{Z}/n) \to K_0(\mathbf{Z}/n)$$
 is the 0-homomorphism.

*Proof.* Let q denote the minimal p-periode of  $\pi$  and let  $[\mathbb{Z}/p^k] \in K_0(\mathbb{Z}/n)$  denote a generator. Choose a  $\mathbb{Z}\pi/N$ -projective resolution of finite type of  $\mathbb{Z}/p^k$ 

which has length q (cf. Lemma 1.2)

$$0 \to L_q \to L_{q-1} \to \cdots \to L_0 \to \mathbf{Z}/p^k \to 0.$$

Then

$$j_*j^*[\mathbf{Z}/p^k] = j_*(\sum (-1)^i [L_i])$$

$$= \sum (-1)^i [\mathbf{Z}/n \otimes_{\pi} L_i]$$

$$= \sum_{r/n} (\sum (-1)^i [\mathbf{Z}/n(r) \otimes_{\pi} L_i])$$

where n(r) stands for the highest power of the prime r, which divides n. For  $r \neq p$  we have

$$\operatorname{Tor}_{\mathbf{Z}\pi/N}^{*}(\mathbf{Z}/n(r),\mathbf{Z}/p^{k})=0$$

and therefore the complex

$$0 \rightarrow \mathbf{Z}/n(r) \otimes_{\pi} L_a \rightarrow \cdots \rightarrow \mathbf{Z}/n(r) \otimes_{\pi} L_0 \rightarrow 0$$

is exact. Hence  $\sum (-1)^i [\mathbf{Z}/n(r) \otimes_{\pi} L_i] = 0$  for  $r \neq p$ , and therefore  $j_*j^*[\mathbf{Z}/p^k] = \sum (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} L_i]$ . To compute  $\sum (-1)^i [\mathbf{Z}/p^k \otimes_{\pi} L_i]$  and the homology of  $\{\mathbf{Z}/p^k \otimes_{\pi} L_i\}$  we can as well use the  $\mathbf{Z}\pi/N \otimes \mathbf{Z}_{(p)}$ -projective resolution  $\{Q_i\}$  of  $\mathbf{Z}/p^k$ , which was considered in the proof of Lemma 1.2. Hence

$$\sum_{i=0}^{q-1} (-1)^{i} [\mathbf{Z}/p^{k} \otimes_{\pi} L_{i}] = \sum_{i=0}^{q-1} (-1)^{i} [\mathbf{Z}/p^{k} \otimes_{\pi} Q_{i}]$$

and plainly for  $0 \le i \le q-1$  one has

$$\operatorname{Tor}_{\mathbf{Z}_{\pi/N}}^{i}(\mathbf{Z}/p^{k},\mathbf{Z}/p^{k}) = H_{i}(\pi;\mathbf{Z}/p^{k}) = \begin{cases} 0 & \text{for } 0 < i < q-1 \\ \mathbf{Z}/p^{k} & \text{for } i = 0, q-1 \end{cases}$$

Therefore  $j_*j^*[\mathbf{Z}/p^k] = [\mathbf{Z}/p^k] + (-1)^{q-1}[\mathbf{Z}/p^k] = 0$  because the *p*-period *q* of  $\pi$  is an even number [15].

If  $\pi$  denotes an arbitrary group of order n then associated with the square of rings

$$\begin{array}{ccc}
\mathbf{Z}\pi & \xrightarrow{pr} & \mathbf{Z}\pi/N \\
\downarrow & & \downarrow^{i} \\
\mathbf{Z} & \longrightarrow & \mathbf{Z}/n
\end{array}$$

there is an exact sequence (cf. Milnor [9]) which reduces to

$$u(\mathbf{Z}/n) \xrightarrow{S} K_0(\mathbf{Z}\pi) \xrightarrow{pr_*} K_0(\mathbf{Z}\pi/N) \xrightarrow{\tilde{\jmath}_*} \tilde{K}_0(\mathbf{Z}/n) \to 0$$
 (2.2)

We call S the Swan homomorphism (cf. [14]). S can be described in the following way: for k a unit mod n, S(k) = [(k, N)] where (k, N) denotes the projective ideal in  $\mathbb{Z}\pi$  generated by k and N.

Consider now the case of a  $\pi$  with cyclic Sylow subgroups. Then  $j_*j^*=0$  by Lemma 2.1 and, by the exactness of (2.2), the transfer  $j^*$  gives therefore rise to a homomorphism

$$t: K_0(\mathbf{Z}/n) \to (K_0\mathbf{Z}\pi)/\text{im } S$$

such that  $\overline{pr_*}t = j^*$ ,  $\overline{pr_*}: (K_0\mathbb{Z}\pi)/\text{im } S \to K_0(\mathbb{Z}\pi/N)$  denoting the map induced by  $pr_*$ .

If  $n = p_1^{k_1} \cdots p_m^{k_m}$  then  $K_0(\mathbf{Z}/n)$  is a free abelian group, freely generated by  $\{[\mathbf{Z}/p_i^{k_i}], 1 \le i \le m\}$ . Hence there is a unique group homomorphism

$$\varphi: u\left(\mathbf{Z}\left[\frac{1}{n}\right]\right) \to K_0(\mathbf{Z}/n)$$

such that  $\varphi(\pm p_i) = [\mathbb{Z}/p_i^{k_i}]$ . If we compose  $\varphi$  with t we get a map  $T = t\varphi$  which we will also call a *transfer*, since it is induced by  $j^*$ . For  $\pi$  a group with cyclic Sylow subgroups we get therefore a commutative diagram

$$u(\mathbf{Z}[1/n]) \xrightarrow{T} (K_0 \mathbf{Z} \pi) / \text{im } S$$

$$\downarrow^{\varphi} \qquad \qquad \downarrow^{\bar{p}\bar{r}_*}$$

$$K_0(\mathbf{Z}/n) \xrightarrow{j^*} \qquad K_0(\mathbf{Z} \pi/N)$$

$$(2.3)$$

We will sometimes consider  $K_0(\mathbb{Z}\pi/N)$  to be the range of T; this should not give rise to any confusion, since  $\overline{pr_*}$  is injective.

It is well known that if R is a ring and M an R-module of type FP, then M defines an element  $[M] \in K_0R$  (depending only upon the isomorphism class of M) by choosing any finite projective resolution of finite type

$$0 \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

and setting  $[M] = \sum (-1)^i [P_i] \in K_0 R$ ; if  $0 \to M' \to M \to M'' \to 0$  is an exact sequence of modules of type FP, then [M] = [M'] + [M''] (cf. [1] and [11]).

LEMMA 2.4. Let  $\pi$  denote a finite group of square free order n and let M denote a finitely generated nilpotent  $\mathbb{Z}\pi/N$ -module. Then

$$T(\operatorname{card} M) = [M] \in K_0(\mathbf{Z}\pi/N).$$

*Proof.* Notice that M is of type FP over  $\mathbb{Z}\pi/N$  and card  $M \in u(\mathbb{Z}[1/n])$  by Lemma 1.4. Hence  $T(\operatorname{card} M)$  and [M] are well defined elements of  $K_0(\mathbb{Z}\pi/N)$ . If M is a trivial  $\mathbb{Z}\pi/N$ -module, then  $T(\operatorname{card} M) = j^*\varphi(\operatorname{card} M) = [M]$  where the second equation follows from the definition of  $\varphi$ ,  $j^*$  and [M] respectively. For the general case we choose a finite filtration of M with gr(M) a trivial  $\mathbb{Z}\pi/N$ -module. Clearly card  $M = \operatorname{card} gr(M)$  and [M] = [gr(M)]; therefore  $T(\operatorname{card} M) = [M]$ .

For the applications in the next section we will be particularly interested in groups  $\pi$  for which im S=0. The following theorem gives some information on T for such cases.

THEOREM 2.5. Let  $\pi$  denote a finite group of order n with cyclic Sylow subgroups. Then

- (i)  $T \equiv 0$  in case n is a prime power or n = 2p, p an odd prime.
- (ii)  $T(p_1 \cdots p_m) = 0$  if  $\pi$  is cyclic of order  $p_1^{k_1} \cdots p_m^{k_m}$ .
- (iii)  $T(3) = T(5) \neq 0$  if n = 15, and T(3) has order 2.

Furthermore, in all three cases listed above one has im S = 0, and T can therefore be considered as a map  $T: u(\mathbf{Z}[1/n]) \to K_0 \mathbf{Z} \pi$ .

We will break the proof up into a couple of lemmas.

LEMMA 2.6. Let  $\pi$  be a cyclic group of order n. Then  $j^*: K_0(\mathbb{Z}/n) \to K(\mathbb{Z}\pi/N)$  factors through  $\tilde{K}_0(\mathbb{Z}/n)$ .

*Proof.* We may assume n > 1. Let x denote a generator of  $\pi$ . Then  $I\mathbf{Z}\pi$  is freely generated by (1-x) over  $\mathbf{Z}\pi/N$  and hence there is an exact sequence  $0 \to \mathbf{Z}\pi/N \to \mathbf{Z}\pi/N \to \mathbf{Z}/n \to 0$ , from which we infer that  $j^*[\mathbf{Z}/n] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N] = 0$ . Thus  $j^*$  factors through  $\tilde{K}_0(\mathbf{Z}/n) = K_0(\mathbf{Z}/n)/\langle [\mathbf{Z}/n] \rangle$ .

LEMMA 2.7. Let p denote an odd prime and  $\pi = \mathbb{Z}/2p$  or the dihedral group  $D_{2p}$ . Then

$$j^* = 0: K_0(\mathbb{Z}/2p) \rightarrow K_0(\mathbb{Z}\pi/N)$$

**Proof.** We will first consider the case  $\pi = \mathbb{Z}/2p$ . Since  $j^*$  factors through  $\tilde{K}_0(\mathbb{Z}/2p)$  which is cyclic, generated by the equivalence class of  $[\mathbb{Z}/2]$ , it suffices to prove that  $j^*[\mathbb{Z}/2] = 0$ . Let  $\pi = \langle x, y \mid x^2 = y^p = 1, xy = yx \rangle$ ,  $R = \mathbb{Z}[\omega]$  with  $\omega = \exp(2\pi i/p)$  and  $R[\mathbb{Z}/2] \cong \mathbb{Z}\pi/(1 + y + \cdots + y^{p-1})$  the obvious isomorphism (mapping  $\omega$  to y). Consider the pullback square of rings

$$\begin{array}{ccc}
\mathbf{Z}\pi & \xrightarrow{\lambda} & R[\mathbf{Z}/2] \\
\downarrow & & \downarrow \\
\mathbf{Z}[\mathbf{Z}/2] & \longrightarrow & \mathbf{F}_p[\mathbf{Z}/2]
\end{array}$$

with obvious maps. Since  $u(R[\mathbb{Z}/2]) \to u(\mathbb{F}_p[\mathbb{Z}/2])$  is surjective (cf. Reiner-Ullom [12, §7]) we get from the associated Milnor-Mayer-Vietoris sequence a monomorphism

$$\lambda_*: K_0 \mathbb{Z} \pi \to K_0 R[\mathbb{Z}/2].$$

Let  $P \subset \mathbb{Z}\pi$  be the ideal generated by (1-y) and 2. Then  $\mathbb{Z}\pi/P \cong F_2[\mathbb{Z}/2]$  is certainly cohomologically trivial and hence P is projective (cf. Rim [13]). Since  $F_2[\mathbb{Z}/2]/N = \mathbb{Z}/2$  we see that  $j^*[\mathbb{Z}/2] = [\mathbb{Z}\pi/N] - [\mathbb{Z}\pi/N \otimes_{\pi} P]$ . It suffices therefore to show that  $[P] = [\mathbb{Z}\pi] \in K_0\mathbb{Z}\pi$ . But  $\lambda_*[P] = [(1-\omega,2)] = [R[\mathbb{Z}/2]]$  since  $R/(1-\omega)R \cong \mathbb{Z}/p$  and p odd. Hence  $[P] = [\mathbb{Z}\pi]$  because  $\lambda_*$  is injective, from where we conclude that  $j^*[\mathbb{Z}/2] = 0$ . In case  $\pi = D_{2p}$  we proceed in a similar way. Notice that  $K_0(\mathbb{Z}/2p)$  is freely generated by  $[\mathbb{Z}/2]$  and  $[\mathbb{Z}/p]$ . From Corollary 3.5 we infer that  $j^*[\mathbb{Z}/p] = 0$  and we are therefore left showing that  $j^*[\mathbb{Z}/2] = 0$ . Let  $D_{2p} = \langle x, y \mid x^2 = y^p, yxy = x \rangle$ . Notice that  $P = (1-y)\mathbb{Z}\pi + 2\mathbb{Z}\pi$  is a twosided ideal with  $\mathbb{Z}\pi/P \cong F_2[\mathbb{Z}/2]$ , which is cohomologically trivial. Hence P is a projective  $\pi$ -module and clearly  $j^*[\mathbb{Z}/2] = [\mathbb{Z}\pi/N] - [\mathbb{Z}\pi/N \otimes_{\pi} P]$ . In order to see that  $[P] = [\mathbb{Z}\pi]$  we consider the square of rings

$$\begin{array}{ccc}
\mathbf{Z}\pi & \xrightarrow{\lambda} R_t[\mathbf{Z}/2] \\
\downarrow & & \downarrow \\
\mathbf{Z}[\mathbf{Z}/2] & \longrightarrow \mathbf{F}_p[\mathbf{Z}/2]
\end{array}$$

with  $R_t[\mathbf{Z}/2] = \mathbf{Z}\pi/(1+y+\cdots y^{p-1})$  a twisted group ring. By [12, §7]  $u(R_t[\mathbf{Z}/2]) \rightarrow u(\mathbf{F}_p[\mathbf{Z}/2])$  is surjective and hence

$$\lambda_*: K_0 \mathbf{Z} \pi \to K_0 R_t [\mathbf{Z}/2]$$

is injective. Since  $\mathbf{F}_2[\mathbf{Z}/2] \otimes_{\pi} R_t[\mathbf{Z}/2] \cong \mathbf{F}_2[\mathbf{Z}/2] \otimes_{\pi} \mathbf{F}_p[\mathbf{Z}/2] = 0$  we infer that  $\lambda_*[P] = [R_t[\mathbf{Z}/2]]$  and whence  $[P] = [\mathbf{Z}\pi]$  from the injectivity of  $\lambda_*$ . Therefore  $j^*[\mathbf{Z}/2] = 0$ , which completes the proof.

LEMMA 2.8. Let  $\pi = \mathbb{Z}/15$ . Then T(3) is the element of order 2 in  $K_0\mathbb{Z}[\mathbb{Z}/15]$ .

**Proof.** Let  $\pi = \langle x, y \mid x^3 = y^5 = 1, xy = yx \rangle$  and let  $P = (x+2)\mathbb{Z}\pi + (y-1)\mathbb{Z}\pi$ . Then  $\mathbb{Z}\pi/P = M$  is cyclic of order 9 with y operating trivially and x operating by multiplication with 7 mod 9. One checks easily that M is cohomologically trivial using the criterion of [13]. Hence P is projective. Since  $M/NM \cong \mathbb{Z}/3$  as trivial  $\mathbb{Z}\pi/N$ -module we infer that

$$j^*[\mathbf{Z}/3] = [\mathbf{Z}\pi/N] - [\mathbf{Z}\pi/N \otimes_{\pi} P] \in K_0(\mathbf{Z}\pi/N).$$

Notice that im S = 0 since  $\pi$  is cyclic. Hence we can think of T(3) to be the element  $[\mathbf{Z}\pi] - [P] \in K_0(\mathbf{Z}\pi)$ . Recall that  $K_0\mathbf{Z}\pi \cong \mathbf{Z} \oplus \mathbf{Z}/2$  by Kervaire-Murthy [7]. Since P is projective of rank 1, it remains therefore to prove that  $[P] \neq 0$  in  $\tilde{K}_0(\mathbf{Z}\pi) \cong \mathbf{Z}/2$ . For this we consider the pullback square (with obvious maps)

$$\begin{array}{ccc}
\mathbf{Z}\pi & \longrightarrow & R[\mathbf{Z}/3] \\
\downarrow & & \downarrow \\
\mathbf{Z}[\mathbf{Z}/3] & \longrightarrow & \mathbf{F}_{5}[\mathbf{Z}/3]
\end{array}$$

where  $R = \mathbb{Z}[\exp{(2\pi i/5)}]$ . The associated Milnor-Mayer-Vietoris sequence yields a map

$$\partial: u(\mathbf{F}_5[\mathbf{Z}/3]) \to \tilde{K}_0\mathbf{Z}[\mathbf{Z}/15]$$

By [7]  $\partial$  factors through  $u(\mathbf{F}_5(\omega)) \cong \mathbb{Z}/24$  where  $\mathbf{F}_5(\omega)$  is the field  $\mathbf{F}_5[\mathbb{Z}/3]/(1+x+x^2)$ ,  $\omega$  is the residue class of the generator  $x \in \mathbb{Z}/3$ . Furthermore,  $\partial$  is surjective (cf. [7]). Notice that  $R[\mathbb{Z}/3] \otimes_{\pi} M = 0$  and therefore  $P' = R[\mathbb{Z}/3] \otimes_{\pi} P \cong R[\mathbb{Z}/3]$ . Furthermore  $P'' = \mathbb{Z}[\mathbb{Z}/3] \otimes_{\pi} P \subset \mathbb{Z}[\mathbb{Z}/3]$  is the principal ideal generated by (x+2). Hence (cf. [9])

$$P = \{((x+2)a, b) \in \mathbb{Z}[\mathbb{Z}/3] \times R[\mathbb{Z}/3] \mid \overline{(x+2)}\bar{a} = \bar{b} \in \mathbb{F}_5[\mathbb{Z}/3] \}.$$

Notice that  $x+2 \in u(\mathbf{F}_5[\mathbf{Z}/3])$  corresponds to  $(3, \omega+2) \in u(\mathbf{F}_5) \times u(\mathbf{F}_5(\omega))$  under the obvious isomorphism  $\mathbf{F}_5[\mathbf{Z}/3] \cong \mathbf{F}_5 \times \mathbf{F}_5(\omega)$ . It follows therefore that  $\partial(x+2) = \partial(\omega+2) = [P]$  and, since  $(\omega+2)$  has order 24 in  $u(\mathbf{F}_5(\omega))$ , we conclude that [P] must have order 2. This completes the proof of the lemma.

We can now complete the proof of Theorem 2.5: First, if n is a prime power,  $\pi$  (having cyclic Sylow subgroups) is necessarily cyclic and therefore  $j^*: K_0(\mathbf{Z}/n) \to K_0(\mathbf{Z}\pi/N)$  factors through  $\tilde{K}_0(\mathbf{Z}/n)$  by Lemma 2.6. But  $\tilde{K}_0(\mathbf{Z}/n) = 0$  for n a prime power. Hence  $T \equiv 0$  in this case. If n = 2p, p an odd prime, then  $\pi = \mathbf{Z}/2p$  or  $D_{2p}$  and it follows from Lemma 2.7 that  $T \equiv 0$ . Thus (i) holds. Assume now that  $\pi$  is cyclic of order  $\prod p_i^{k_i} = n$ . Then  $T(\prod p_i) = t\varphi(\prod p_i) = t[\mathbf{Z}/n] = 0$ , since  $j^*[\mathbf{Z}/n] = 0$  by Lemma 2.6. Therefore (ii) holds. For (iii) notice that in case n = 15,  $\pi$  is necessarily cyclic. Hence, by applying Lemma 2.6,  $j^*[\mathbf{Z}/3] = -j^*[\mathbf{Z}/5]$  or T(3) = -T(5). Moreover T(3) has order 2 by Lemma 2.8 and therefore  $T(3) = T(5) \neq 0$ . Finally im S = 0 for  $\pi$  cyclic or  $\pi = D_{2p}$ , p an odd prime (cf. Ullom [16]) and hence im S = 0 for the groups considered in (i)–(iii) of Theorem 2.5.

## 3. Applications to homologically nilpotent spaces

Following [3] we call a connected space X homologically nilpotent, if  $\pi_1 X$  operates nilpotently on  $H_i \tilde{X}$  for all i. In particular, if X is a nilpotent space, then X is homologically nilpotent and conversely, a homologically nilpotent space X is nilpotent if and only if  $\pi_1 X$  is a nilpotent group (cf. [10]). Let  $H_*^{\pi}(X; I\mathbf{Z}\pi)$  denote the homology of the (left)  $\mathbf{Z}\pi/N$ -complex  $I\mathbf{Z}\pi \otimes_{\pi} C_* \tilde{X}$ , where  $\pi = \pi_1 X$ ; similarly for  $H_*^{\pi}(X; \mathbf{Z}\pi/N)$ .

LEMMA 3.1. If X is homologically nilpotent with  $\pi_1 X$  of finite order n, then  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  and  $H_i^{\pi}(X; \mathbf{Z}\pi/N)$  are nilpotent  $\mathbf{Z}\pi/N$ -modules for all i. If, in addition, n is square free and X finitely dominated, then  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  and  $H_i^{\pi}(X; \mathbf{Z}\pi/N)$  are of type FP over  $\mathbf{Z}\pi/N$ .

Proof. Consider the long exact homology sequence associated with the exact sequence of chain complexes  $0 \to I\mathbf{Z}\pi \otimes_{\pi} C_{*}\tilde{X} \to C_{*}\tilde{X} \to C_{*}X \to 0$ . Standard results on nilpotent actions (cf. [6, Chapter I.4]) imply then that  $H_{i}^{\pi}(X; I\mathbf{Z}\pi)$  is a nilpotent  $\pi$ -module. Since the  $\pi$ -action on  $H_{i}^{\pi}(X; I\mathbf{Z}\pi)$  factors through  $\mathbf{Z}\pi/N$ , we conclude that  $H_{i}^{\pi}(X; I\mathbf{Z}\pi)$  is a nilpotent  $\mathbf{Z}\pi/N$ -module. If X is dominated by a finite complex and  $\pi_{1}X$  finite, then certainly  $H_{i}^{\pi}(X; I\mathbf{Z}\pi)$  is a finitely generated  $\mathbf{Z}\pi/N$ -module. Hence, by Lemma 1.4,  $H_{i}^{\pi}(X; I\mathbf{Z}\pi)$  is of type FP. The proof for  $H_{i}^{\pi}(X; \mathbf{Z}\pi/N)$  is similar, using  $0 \to C_{*}X \to C_{*}\tilde{X} \to \mathbf{Z}\pi/N \otimes_{\pi} C_{*}\tilde{X} \to 0$ .

LEMMA 3.2. Let X be a finitely dominated homologically nilpotent space with finite fundamental group of order n and let  $H_*(X, \tilde{X})$  denote the homology of the mapping cylinder of  $\tilde{X} \to X \mod \tilde{X}$ . Then the groups  $H_i^{\pi}(X, I\mathbf{Z}_{\pi})$  and  $H_{i+1}(X, \tilde{X})$  have the same finite cardinality c(i) for  $i \ge 0$ , and c(i) is a unit in  $\mathbf{Z}[1/n]$ .

**Proof.** Since  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  is a finitely generated nilpotent  $\mathbf{Z}\pi/N$ -module, it has by Lemma 1.4 a cardinality c(i) which is a unit in  $\mathbf{Z}[1/n]$ . From long exact homology sequences it is obvious that c(i) is also given by

$$c(i) = (\text{card coker } (H_{i+1}\tilde{X} \rightarrow H_{i+1}X)) \cdot (\text{card ker } (H_i\tilde{X} \rightarrow H_iX))$$

which equals card  $H_{i+1}(X, \tilde{X})$ .

It follows that for X as in Lemma 3.2, the rational number

$$\rho(X) = \text{card } H_{\text{odd}}(X, \tilde{X})/\text{card } H_{\text{ev}}(X, \tilde{X})$$

is a well defined unit in  $\mathbb{Z}[1/n]$ . This unit  $\rho(X)$  is related to the finiteness obstruction  $wX \in K_0\mathbb{Z}\pi_1X$  of Wall (cf. [17], [10]) in the following way.

THEOREM 3.3. Let X be a homologically nilpotent space with non-trivial fundamental group  $\pi$  of square free order n. Suppose further that X is dominated by a finite complex and let  $\bar{w}X$  denote the image of wX in  $(K_0\mathbf{Z}\pi_1X)/\text{im }S$ . Then

$$T\rho(X) = \bar{w}X$$

If X is in addition nilpotent (i.e.  $\pi_1 X$  is cyclic) then

- (i)  $T\rho(X) = wX$ , and
- (ii) wX = 0 in case  $\rho(X)$  is a power (positive or negative) of n.

Before proving Theorem 3.3 we will establish a different way of computing  $\rho(X)$ .

LEMMA 3.4. Let X be as in (3.2) and let  $H_*^{\pi}(X; \mathbf{Z}\pi/N)$  denote the homology of  $\mathbf{Z}\pi/N \otimes_{\pi} C_*(\tilde{X})$ . Then

$$\rho(X) = \operatorname{card} H_{\text{ev}}^{\pi}(X; \mathbf{Z}\pi/N)/\operatorname{card} H_{\text{odd}}^{\pi}(X; \mathbf{Z}\pi/N)$$

*Proof.* We may of course assume that  $n = \operatorname{card} \pi_1 X > 1$ . Since  $H_*(\tilde{X}; \mathbf{Q})$  is semisimple and nilpotent as  $\pi$ -module, we have  $H_*(\tilde{X}; \mathbf{Q}) \cong H_*(X; \mathbf{Q})$  and therefore the Euler characteristic of X vanishes. Thus, for all primes p,  $\chi_p(X) = \sum_i (-1)^i \dim H_i(X; \mathbf{Z}/p) = 0$ . Since n is square free, we infer then that

card 
$$H_{\text{ev}}(X; \mathbf{Z}/n) = \text{card } H_{\text{odd}}(X; \mathbf{Z}/n)$$
.

The long exact homology sequence associated with the exact sequence  $I\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi/N \rightarrow \mathbf{Z}/n$  then yields

card 
$$H^{\pi}_{\text{ev}}(X; \mathbf{Z}\pi/N)/\text{card } H^{\pi}_{\text{odd}}(X; \mathbf{Z}\pi/N)$$
  
= card  $H^{\pi}_{\text{ev}}(X; I\mathbf{Z}\pi)/\text{card } H^{\pi}_{\text{odd}}(X; I\mathbf{Z}\pi)$ 

from where the result follows, using Lemma 3.2.

Proof of Theorem 3.3. Let  $pr_*: K_0(\mathbf{Z}\pi) \to K_0(\mathbf{Z}\pi/N)$  be the map induced by the projection. If  $\bar{C}_*$  denotes a chain complex of type FP, homotopy equivalent to the singular complex of  $\tilde{X}$ , then  $pr_*wX = \sum (-1)^i [\mathbf{Z}\pi/N \otimes_{\pi} \bar{C}_i]$ . Since  $H_i^{\pi}(X; \mathbf{Z}\pi/N)$  is of type FP (cf. Lemma 3.1) we infer that (cf. [11])

$$\sum_{i} (-1)^{i} [\mathbf{Z} \pi/N \otimes_{\pi} \overline{C}_{i}] = \sum_{i} (-1)^{i} [H_{i}^{\pi}(X; \mathbf{Z} \pi/N)]$$

and therefore, since  $[H_i^{\pi}(X; \mathbf{Z}\pi/N)] = T(\text{card } H_i^{\pi}(X; \mathbf{Z}\pi/N))$  by Lemma 2.4, we get

$$pr_*wX = \bar{w}X = T(\operatorname{card} H^{\pi}_{\operatorname{ev}}(X; \mathbf{Z}\pi/N)/\operatorname{card} H^{\pi}_{\operatorname{odd}}(X; \mathbf{Z}\pi/N)) = T\rho(X).$$

In case X is in addition nilpotent,  $\pi_1 X$  is necessarily cyclic and therefore im S = 0. Hence  $T\rho(X) = wX$  in this case. Furthermore, if  $\rho(X) = n^k$ , we conclude from Theorem 2.5(ii) that  $wX = T(n^k) = 0$ . This completes the proof of Theorem 3.3.

As a first application we will prove the following algebraic result, which is used to prove Lemma 2.7.

COROLLARY 3.5. Let  $\pi$  denote the dihedral group  $D_{2p}$ , p an odd prime. Then  $j^*[\mathbf{Z}/p] = 0 \in K_0(\mathbf{Z}D_{2p}/N)$ .

*Proof.* By Theorem A [14] there is a finite simplicial complex  $\tilde{X}$  of the homotopy type of  $S^3$  on which  $D_{2p}$  acts freely and simplicially. Let  $X = \tilde{X}/D_{2p}$ . Then

$$H_i(X, \tilde{X}) = \begin{cases} 0, & \text{if } i = 0, 2 \\ \mathbb{Z}/2, & \text{if } i = 1 \\ \mathbb{Z}/2p, & \text{if } i = 3 \end{cases}$$

Thus  $\rho(X) = 4p$ . Since X is a finite complex with trivial action of  $\pi_1 X$  on  $H_* \tilde{X}$ , we infer that wX = 0 and therefore T(4p) = 0. We have already observed in course of the proof of Lemma 2.7 that  $j^*[\mathbb{Z}/2] = 0$ . Hence

$$j^*[\mathbf{Z}/p] = j^*[(\mathbf{Z}/2) \oplus (\mathbf{Z}/2) \oplus (\mathbf{Z}/p)] = T(4p) = 0.$$

Recall that two nilpotent spaces X and Y of finite type are of the same genus (cf. [6]) if their p-localizations  $X_p$ ,  $Y_p$  are homotopy equivalent for all primes p. In [10] one finds an example of two finitely dominated nilpotent spaces X and Y of the same genus with fundamental groups of order 8 such that  $wX \neq 0$  but wY = 0. For fundamental groups of square free orders, such an example is impossible. Namely one has

COROLLARY 3.6. Let X and Y be two finitely dominated nilpotent spaces of the same genus, with non trivial fundamental groups of square free order n. Then  $\rho(X) = \rho(Y)$  and wX, wY have the same finite orders.

*Proof.* Notice that  $\pi_1 X$  and  $\pi_1 Y$  are abelian and whence  $\pi_1 X \cong \pi_1 Y$ , since X and Y are of the same genus (cf. [6]). Furthermore, there are for all primes p commutative diagrams

$$egin{array}{cccc} ilde{X}_p & \longrightarrow & ilde{Y}_p \ & & & \downarrow^{pr} \ ilde{X}_p & \longrightarrow & Y_p \end{array}$$

with the horizontal maps being homotopy equivalences. Thus

$$H_i(X, \tilde{X}; \mathbf{Z}_{(p)}) \cong H_i(X_p, \tilde{X}_p; \mathbf{Z}_{(p)}) \cong H_i(Y_p, \tilde{Y}_p; \mathbf{Z}_{(p)}) \cong H_i(Y, \tilde{Y}; \mathbf{Z}_{(p)})$$

and therefore  $H_i(X, \tilde{X}) \cong H_i(Y, \tilde{Y})$  since the groups  $H_i(X, \tilde{X})$  and  $H_i(Y, \tilde{Y})$  are finite. Hence  $\rho X = \rho Y$  and, since  $\pi_1 X \cong \pi_1 Y$ , it follows from Theorem 3.3(i) that wX and wY have the same orders; the orders must be finite, because X and Y must have vanishing Euler characteristic (cf. proof of Lemma 3.4).

Another application of Theorem 3.3 is the following vanishing theorem for the Wall obstruction.

COROLLARY 3.7. Let X be a finitely dominated homologically nilpotent space with fundamental group of order p or 2p, p a prime. Then wX = 0 and X is therefore of the homotopy type of a finite complex.

**Proof.** First consider the case p=2. By a result of Fröhlich (cf. [5, Theorem 6(i)]) we infer that  $\tilde{K}_0\pi_1X=0$  and, since the Euler characteristic of X is necessarily 0 (cf. proof of Lemma 3.4), it follows that wX=0. Second, let p denote an odd prime. Then we may apply Theorem 3.3 and obtain  $T\rho(X)=\bar{w}X$ . Since  $T\equiv 0$  and im S=0 for the  $\pi_1X$  in question (cf. Theorem 2.5), we infer that wX=0.

It is sometimes possible to compute  $\rho(X)$  directly from  $H_*X$ , giving rise to a particular simple formula for wX. We will treat one such case in the next section and plan to treat other cases in a forthcoming paper.

## 4. The Wall obstruction for H-spaces

We want to prove the Theorem II mentioned in the introduction.

Suppose X is a space with  $\bigoplus H_i(X; \mathbb{Z})$  finitely generated and let p denote a fixed prime. Then we write

$$\chi_p(X, t) = \sum_i (-1)^i \beta_i t^i, \qquad \beta_i = \dim_i H_i(X; \mathbb{Z}/p)$$

for the Poincaré polynomial of X with respect to  $\mathbb{Z}/p$ . Define

$$e_p(X) = \frac{d}{dt} \chi_p(X, t) \Big|_{t=-1}$$

and, in case  $\pi_1 X$  has finite order n, define

$$e(X) = \prod_{p/n} p^{e_p(X)}$$

THEOREM 4.1. Let X be a finitely dominated nilpotent space with fundamental group  $\pi$  of square free order. Suppose that  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  is a trivial  $\pi$ -module for all i. Then

$$\rho(X) = e(X)$$

*Proof.* We may assume that  $\pi \neq \{1\}$ . Let x denote a generator of the necessarily cyclic group  $\pi$ . Then there is a exact sequence of  $\mathbb{Z}\pi/N$ -modules

$$0 \rightarrow I\mathbf{Z}\pi \xrightarrow{1-x} I\mathbf{Z}\pi \rightarrow \mathbf{Z}/n \rightarrow 0$$

where n denotes the order of  $\pi$ . Since (1-x) induces 0 in  $H_*^{\pi}(X; I\mathbb{Z}\pi)$ , the associated long exact homology sequence breaks up into short exact sequences

$$0 \rightarrow H_i^{\pi}(X; I\mathbf{Z}\pi) \rightarrow H_i(X; \mathbf{Z}/n) \rightarrow H_{i-1}^{\pi}(X; I\mathbf{Z}\pi) \rightarrow 0$$

for  $i \ge 0$ . Since  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  is a trivial  $\mathbf{Z}\pi/N$ -module and n is square free,  $H_i^{\pi}(X; I\mathbf{Z}\pi) \otimes \mathbf{Z}_{(p)}$  is a  $\mathbf{Z}/p$ -vector space. Define

$$\beta_i = \dim H_i(X; \mathbf{Z}/p), \qquad \gamma_i = \dim H_i^{\pi}(X; I\mathbf{Z}\pi) \otimes \mathbf{Z}_{(p)}.$$

Then

$$\gamma_i = \beta_i - \gamma_{i-1} = \beta_i - \beta_{i-1} + \cdots + (-1)^i \beta_0$$

and

$$\sum (-1)^{j} \gamma_{j} = \sum (\beta_{0} - \beta_{1} + \dots + (-1)^{j} \beta_{j})$$

$$= (m+1)\beta_{0} - m\beta_{1} + \dots + (-1)^{m} \beta_{m}$$

where m denotes the largest integer k with  $\beta_k \neq 0$ . Hence

$$\sum_{j=0}^{\infty} (-1)^{j} \gamma_{j} = (-1)^{m} \frac{d}{dt} \left( t^{m+1} \chi_{p} \left( X, \frac{1}{t} \right) \right) \Big|_{t=-1}$$
$$= (m+1) \chi_{p}(X, -1) + \chi'_{p}(X, -1).$$

Since X is homologically nilpotent with non-trivial finite fundamental group, the Euler characteristic of X is 0 (cf. proof of Lemma 3.4) and hence  $\chi_p(X, -1) = 0$ . The above equation reduces therefore to

$$\sum_{i} (-1)^{j} \gamma_{i} = \chi'_{p}(X, -1) = e_{p}(X).$$

Hence card  $H^{\pi}_{\text{ev}}(X; I\mathbf{Z}\pi)/\text{card } H^{\pi}_{\text{odd}}(X; I\mathbf{Z}\pi) = \prod_{p/n} p^{e_p(X)} = e(X)$  and therefore  $\rho(X) = e(X)$ .

In order to prove that for an H-space X the  $\pi$ -operation on  $H_*(X; I\mathbf{Z}\pi)$  is trivial, we will need the following lemma.

LEMMA 4.2. Let X be an H-space and  $a \in \pi_1 X$ . Then the induced covering transformation  $a_* : \tilde{X} \to \tilde{X}$  is equivariantly homotopic to the identity.

**Proof.** Without loss of generality we may assume that the H-structure  $\mu: X \times X \to X$  has the base point as a strict identity. Equip  $\tilde{X}$  with the canonical H-structure  $\tilde{\mu}$ . Then  $pr^{-1}(*)$  is a central subgroup of  $(\tilde{X}, \tilde{\mu})$ , naturally isomorphic to  $\pi_1 X$ . We may thus think of a as an element of  $pr^{-1}(*)$  which acts on  $\tilde{X}$  by left multiplication. Choosing a path from  $a \in pr^{-1}(*)$  to \* we get a homotopy of the

map  $a_*$  to Id, which is equivariant with respect to the  $\pi_1 X = pr^{-1}(*)$ -action, because  $pr^{-1}(*)$  is central.

COROLLARY 4.3. If X is an H-space, then  $H_*^{\pi}(X; I\mathbf{Z}\pi)$  is a trivial  $\mathbf{Z}\pi/N$ -module.

This is clear since by 4.2, the operation of  $a \in \pi_1 X$  on  $C_* \tilde{X}$  is chain homotopic to Id as map of  $\pi_1 X$ -complexes, and therefore the induced action of a on  $I\mathbf{Z}\pi \otimes_{\pi} C_* \tilde{X}$  is chain homotopic to Id.

Proof of Theorem II of the Introduction. If X is of rank 1, then X is equivalent to one of the spaces  $S^1$ ,  $S^3$ ,  $S^7$ ,  $\mathbb{R}P^3$  or  $\mathbb{R}P^7$  (cf. Browder [2]). Hence wX = 0 in these cases. If rank  $(X) \ge 2$ , then  $\chi_p(X, t)$  contains a factor  $(1 - t^{n_1})(1 - t^{n_2})$  with  $n_1$  and  $n_2$  odd. Therefore

$$e_p(X) = \chi'_p(X, -1) = 0$$

for all primes p. In particular we obtain e(X) = 1 and, since  $H_i^{\pi}(X; I\mathbf{Z}\pi)$  is a trivial  $\pi$ -module for all i we infer from Theorem 4.1 that  $\rho(X) = e(X)$ . Hence  $wX = T\rho(X) = 0$ .

# 5. Appendix

If X denotes a homologically nilpotent space with finite fundamental group, then there is a simple criterion for deciding whether X is dominated by a finite complex.

THEOREM 5.1. Let X be a homologically nilpotent complex with finite fundamental group. Then the following are equivalent.

- (i) X is dominated by a finite complex
- (ii)  $H_i(\tilde{X}; \mathbb{Z})$  and  $H_i(X; \mathbb{Z})$  are finitely generated abelian groups for all i and zero for i sufficiently large.

*Proof.* Certainly (i) implies (ii). If (ii) is given, then from [3, Corollary 3.4] we infer that X has the homotopy type of a finite dimensional complex. Since  $\mathbb{Z}\pi_1X$  is noetherian and  $H_i(\tilde{X}; \mathbb{Z})$  finitely generated for all i, Theorems B and F of [17] imply that X is dominated by a finite complex.

A more general result of this type in case X is nilpotent was proved in [11].

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Received October 11, 1976.