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# Slowly growing subharmonic functions I

M. Essén, W. K. Hayman and A. Huber

Dedicated to Albert Pfluger on his 70th Birthday

## 1. Introduction

Let u(z) be subharmonic (s.h.) and not constant in the open z-plane and let

$$B(r, u) = \sup_{|z|=r} u(z).$$
 (1.1)

If B(r, u) grows sufficiently slowly then it is known that, for "most" values of  $z = re^{i\theta}$ , u(z) is not much smaller than B(r). For instance if

$$B(r) = O(\log r)^2 \quad \text{as} \quad r \to \infty, \tag{1.2}$$

then Barry [3] showed that

$$u(z) > B(r) - O(1), |z| = r (1.3)$$

except for a set of r of small upper logarithmic density. Further Hayman [5] showed that if (1.2) holds, then for any positive  $\varepsilon$  we have

$$u(z) > (1 - \varepsilon)B(r), \qquad |z| = r$$
 (1.4)

outside a sequence of disks  $|z-z_n| < \rho_n$ , such that

$$\sum \rho_n/(1+|z_n|) < \infty. \tag{1.5}$$

Both these results fail if the condition (1.2) is weakened. Thus Barry [3] showed that if

$$\psi(r)/(\log r)^2 \to +\infty,\tag{1.6}$$

then there exists u(z) satisfying

$$B(r, u) = O\{\psi(r)\}\tag{1.7}$$

and

$$u(-r)-u(r) \rightarrow -\infty$$
, as  $r \rightarrow \infty$ .

Further Piranian [6] showed under the hypothesis (1.6) that there exists u(z) satisfying (1.7) but such that u(z) does not tend to  $+\infty$  as  $z \to \infty$  along any ray. On the other hand if (1.4) holds outside a set of disks satisfying (1.5) then almost all rays through the origin meet these disks only on a bounded set, so that (1.4) holds on such rays from a certain point onwards.

## 2. Statement of results

In this paper we consider in more detail the nature of the set of points where (1.4) fails, when u(z) satisfies (1.7) for a given function  $\psi(r)$ . When

$$\psi(r) = O(\log r)^2 \tag{2.1}$$

our results are fairly complete.

The corresponding result for (1.3) under the hypothesis (1.7) with  $\psi(r) = \log r$  has been obtained by Arsove and Huber [2], and our method is closely related to theirs. However, the present paper also contains converse results.

To state our theorems we make the following

DEFINITION. Let  $\psi(r)$  be a positive increasing function of r satisfying (2.1) and suppose that K is fixed, K > 1. Let  $c_{\nu}$  be any nonnegative, nondecreasing and convex sequence, such that  $c_{\nu} = 0$ ,  $\nu \le 0$ ,

$$c_{\nu} \to \infty \quad \text{as} \quad \nu \to +\infty$$
 (2.2)

and

$$c_{\nu} = O\{\psi(K^{\nu})\} \quad \text{as} \quad \nu \to +\infty. \tag{2.3}$$

Then

$$\delta_{\nu} = \frac{c_{\nu} - 2c_{\nu-1} + c_{\nu-2}}{c_{\nu}} \tag{2.4}$$

will be called a  $\psi$ -sequence. (If  $c_{\nu} = 0$ , we define  $\delta_{\nu} = 0$ .) In terms of  $\psi$ -sequences our results can be stated as follows.

THEOREM 1. Suppose that u(z) is a s.h. and non-constant function in the open plane which satisfies (1.7). Then given K > 1, there exists  $r_0 = r_0(K)$  where  $1 < r_0 < K$ , with the following properties. For a fixed positive  $\varepsilon$  let  $E_{\nu}(\varepsilon)$  be the set of all points z in the annulus

$$A_{\nu}: r_0 K^{\nu} \le |z| < r_0 K^{\nu+1}, \tag{2.5}$$

such that

$$u(z) < (1 - \varepsilon)B(|z|). \tag{2.6}$$

Then the capacities Cap  $(E_{\nu})$  of  $E_{\nu}$  satisfy

$$\log \left\{ \frac{r_0 K^{\nu+2}}{\operatorname{Cap}(E_{\nu})} \right\} \ge \frac{\varepsilon \log K + o(1)}{\delta_{\nu}}, \tag{2.7}$$

where  $\delta_{\nu}$  is a  $\psi$ -sequence.

THEOREM 2. Suppose given numbers K>1,  $r_0>1$ ,  $\varepsilon>0$  and  $\varepsilon'>\varepsilon$ , and a  $\psi$ -sequence  $\delta_{\nu}$ . Let  $E_{\nu}$  be any sets in the annuli  $A_{\nu}$  whose outer capacities satisfy for all large  $\nu$ 

$$\log \left\{ \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_{\nu}} \right\} > \frac{\varepsilon' \log K}{\delta_{\nu}}. \tag{2.8}$$

If  $\delta_{\nu} = 0$ , (2.8) is to be interpreted so that  $E_{\nu}$  is empty.

Then there exists u(z) s.h. and non-constant in the plane, and satisfying (1.7) and also (2.6) in  $\bigcup (E_{\nu})$ .

Theorems 1 and 2 show that the sets where u(z) satisfies (2.6) for some positive  $\varepsilon$  can be very precisely characterized in terms of  $\psi$ -sequences. Taking for instance  $\varepsilon = 1$  we obtain a characterisation of the sets on which u(z) remains bounded. Thus our problem is reduced to a problem in the theory of series, namely the characterisation of  $\psi$ -sequences. Clearly if  $c_n = 0$  for  $n < n_0$ ,  $c_{n_0} > 0$  in the definition of a  $\psi$ -sequence, then

$$\delta_n = 0, \quad n < n_0, \quad \delta_{n_0} = 1, \quad 0 \le \delta_n < 1, \quad n > n_0.$$
 (2.9)

Conversely if  $\delta_n$  is any sequence satisfying (2.9) then we may define  $b_n$  inductively by

$$b_n = 0, \qquad n < n_0, \tag{2.10}$$

$$b_{n_0} > 0,$$
 (2.11)

$$b_n = b_{n-1} + \delta_n \sum_{\nu = n_0}^n b_{\nu}, \qquad n > n_0$$
 (2.12)

and then define  $c_n = 0$ ,  $n < n_0$ 

$$c_n = \sum_{\nu=n_0}^n b_{\nu}, \qquad n \ge n_0.$$
 (2.13)

Then evidently  $b_n$  is positive and non-decreasing for  $n \ge n_0$ , since  $\delta_n < 1$ , so that  $c_n$  is non-decreasing and convex. Further (2.2) evidently holds and so does (2.4). Thus  $\delta_n$  is a  $\psi$ -sequence if and only if the sequence  $c_n$  defined in the above way satisfies (2.3).

If  $\psi(r) = \log r$  we can give a simple criterion for this to happen. The corresponding functions u(z) are said to be of polynomial growth and are the smallest non-constant subharmonic functions. The following theorem is a restatement of the main result of Arsove and Huber [2]. It expresses the fact that  $\bigcup E_{\nu}$  is thin at infinity.

THEOREM 3. The sequence  $\delta_n$  satisfying (2.9) is a  $\psi$ -sequence with  $\psi(r) = \log r$  if and only if

$$\sum_{1}^{\infty} n\delta_{n} < \infty. \tag{2.14}$$

COROLLARY. Let u be as in Theorem 1 with  $\psi(r) = \log r$ . Then the sets  $E_{\nu}$  satisfy

$$\sum_{1}^{\infty} \nu \left\{ \log \frac{1}{\operatorname{Cap} E_{\nu}} \right\}^{-1} < \infty$$

This is the famous criterion of Wiener [8].

For general  $\psi$ -sequences, i.e. those with  $\psi(r) = (\log r)^2$  we have only a less precise result.

THEOREM 4. If  $\delta_{\nu}$  is a  $\psi$ -sequence and  $\lambda > \frac{1}{2}$  then

$$\sum_{1}^{\infty} \delta_{\nu}^{\lambda} < \infty. \tag{2.15}$$

COROLLARY. If  $E_{\nu}$  are the sets in Theorem 1 then

$$\sum_{1}^{\infty} \left\{ \log \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_{\nu}} \right\}^{-\lambda} < \infty, \quad \text{if } \lambda > \frac{1}{2}.$$

Setting  $\delta_n = 1/\{n(1 + \log n)\}^2$ ,  $n \ge 1$ , and using Theorem 3, we see that Theorem 4 and its corollary fail for  $\lambda = \frac{1}{2}$  even for functions of polynomial growth.

2.1. Some consequences and examples. To illustrate the above results we give some examples and simple consequences of Theorems 1 to 4, before embarking on the proofs of these latter results.

EXAMPLE 1. Take  $\psi(r) = (\log r)^{\alpha}$ ,  $1 < \alpha \le 2$  and for any integer t, set  $c_n = (n+t)^{\alpha}$ ,  $n > \max(0, -t)$ ,  $c_n = 0$  otherwise. Then

$$\delta_n = \frac{c_n - 2c_{n-1} + c_{n-2}}{c_n} = \alpha(\alpha - 1) \left\{ \frac{1}{n^2} - \frac{2t + (\alpha - 2)}{n^3} + O(n^{-4}) \right\}.$$

It is clear from (2.10) to (2.13) that if  $\delta_n$  is a  $\psi$ -sequence, then so is any sequence  $\delta'_n$ , such that  $\delta'_n \leq \delta_n$ , for large n. We deduce that if

$$\delta_n = \alpha(\alpha - 1) \frac{1}{n^2} + \frac{O(1)}{n^3},$$

then  $\delta_n$  is a  $\psi$ -sequence for  $\psi(r) = (\log r)^{\alpha}$ , but not for any function  $\psi(r)$ , such that  $\psi(r) = o(\log r)^{\alpha}$ .

In the above example (2.14) fails, so that the corresponding exceptional sets are no longer thin at  $\infty$ . In fact it follows from Theorem 3 that whenever  $\psi(r)$  tends to infinity more rapidly than  $\log r$ , then  $\psi$ -sequences will not in general satisfy (2.14) and so the corresponding exceptional sets will not be thin at  $\infty$ . However if  $\psi(r) = (\log r)^{\alpha}$ , so that  $c_n = O(n^{\alpha})$ , we deduce from the convexity of  $c_n$ 

and (2.13) that

$$nb_n \le c_{2n} = O(n^{\alpha}), \tag{2.16}$$

and, if  $c_n/n \rightarrow \infty$ , that

$$\delta_n = \frac{b_n - b_{n-1}}{c_n} = o(n^{\alpha - 2}). \tag{2.17}$$

If  $c_n/n$  does not tend to  $\infty$ , then  $c_n = O(n)$  and (2.17) follows from (2.14).

The order condition (2.17) cannot be further sharpened as the next example shows.

EXAMPLE 2. Suppose that  $1 \le \alpha \le 2$  and let  $\varepsilon_n$  be any sequence of positive numbers, such that  $\varepsilon_n \to 0$  as  $n \to \infty$ . Then there exists a  $\psi$ -sequence  $\delta_n$  corresponding to  $\psi(r) = (\log r)^{\alpha}$ , such that

$$\delta_n \geq \varepsilon_n n^{\alpha-2}$$

for infinitely many n.

If  $\alpha = 1$ , we choose an increasing sequence of positive integers  $n_k$ ,  $k = 0, 1, 2, \ldots, \varepsilon_{n_k} < 2^{-k}$ , k = 1, 2. We then set  $\delta_{n_0} = 1$ ,  $\delta_n = \varepsilon_n/n$ , for  $n = n_k$ , where  $k \ge 1$ , and  $\delta_n = 0$  otherwise. Then

$$\sum_{1}^{\infty} n\delta_n = \sum_{k=0}^{\infty} n_k \delta_k = n_0 + \sum_{k=1}^{\infty} \varepsilon_{n_k} < n_0 + 1.$$

and so  $\delta_n$  is a  $\psi$ -sequence in view of Theorem 3 and (2.9).

If  $\alpha > 1$ , we define an increasing sequence  $n_k$ , k = 0, 1, 2, such that  $n_{k+1}/n_k \to \infty$  and set  $b_n = 0$ ,  $n < n_0$ ,

$$b_n = n_k^{\alpha - 1}, \qquad n_k \le n < n_{k+1}.$$

We then define  $c_n$ ,  $\delta_n$  by (2.12) and (2.13), and note that

$$c_n \leq nb_n \leq n^{\alpha}, \quad n \geq 1,$$

so that  $\delta_n$  is a  $\psi$ -sequence with  $\psi(r) = (\log r)^{\alpha}$ . Also for  $n = n_k$ 

$$\delta_{n} = (b_{n} - b_{n-1})/c_{n} = (b_{n} - b_{n-1})/(c_{n-1} + b_{n})$$

$$\geq ((n_{k})^{\alpha - 1} - (n_{k-1})^{\alpha - 1})/(n_{k}(n_{k-1})^{\alpha - 1} + (n_{k})^{\alpha - 1})$$

$$= (1 + o(1))(n_{k})^{\alpha - 2}/(n_{k-1})^{\alpha - 1}.$$

We now choose the sequence  $n_k$  inductively, by setting  $n_0 = 1$ , and if  $n_{k-1}$  has already been defined we choose  $n_k$  so large that

$$\varepsilon_{n_k} < \frac{1}{2(n_{k-1})^{\alpha-1}}$$
 and  $n_k > kn_{k-1}$ .

Thus we have for all large k

$$\delta_{n_k} > \varepsilon_{n_k} n_k^{\alpha-2}$$

as required.

Similarly we have

EXAMPLE 3. If  $\psi(r)/(\log r) \rightarrow \infty$ , there exists a  $\psi$ -sequence  $\delta_n$  such that

$$\overline{\lim}_{n\to\infty} n\delta_n = \infty.$$

We set

$$B_n=\inf_{\nu\geq n}\frac{\psi(K^{\nu})}{\nu}\,,$$

so that  $B_n$  increases with n and  $B_n \to \infty$  as  $n \to \infty$ . We choose a sequence  $n_k$  which tends to  $\infty$  with n and is such that

$$\frac{B_{n_{k+1}}}{B_{n_k}} \to \infty$$

as  $k \to \infty$ .

Then we define  $b_n = 0$ ,  $n < n_0$  and

$$b_n = B_{n_k}, \qquad n_k \le n < n_{k+1}.$$

We deduce that  $c_n \le nb_n \le nB_n$  for  $n \ge 1$ , so that

$$c_n \leq \psi(K^n),$$

and  $c_n$  is a  $\psi$ -sequence. On the other hand for  $n = n_k$ , we have

$$\delta_n = \frac{b_n - b_{n-1}}{c_n} = \frac{b_n - b_{n-1}}{c_{n-1} + b_n} \ge \frac{(1 + o(1))B_{n_k}}{n_k B_{n_{k-1}} + B_{n_k}},$$

so that

$$\frac{1}{n_k \delta_{n_k}} \leq (1 + o(1)) \left\{ \frac{B_{n_{k-1}}}{B_{n_k}} + \frac{1}{n_k} \right\} \to 0 \quad \text{as} \quad k \to \infty.$$

If we combine the result of example 3 with Theorem 2, we deduce that if  $\psi(r)/\log r \to \infty$ , we can find u(z) s.h. in the plane and satisfying (1.7) and (2.6) with any fixed  $\varepsilon$  on a sequence of sets  $E_{\nu}$ , in  $K^{\nu} < |z| < K^{\nu+1}$ , such that

$$\operatorname{Cap} E_{\nu} = \exp \left\{ \nu (\log K + o(1)) \right\}$$

for infinitely many  $\nu$ . In particular Cap  $E_{\nu}$  can be exponentially large for a sequence of  $\nu$ . On the other hand if  $\psi(r) = O(\log r)$  it follows from Theorem 3 that  $n\delta_n \to 0$ , so that in view of (2.7) we have

$$\frac{1}{\nu}\log\left\{\frac{r_0K^{\nu+2}}{\operatorname{Cap} E_{\nu}}\right\} = \frac{1}{\nu}\log\frac{1}{\operatorname{Cap} E_{\nu}} + O(1) \to \infty.$$

Thus in this case

$$\log\left\{\frac{r_0K^{\nu+2}}{\operatorname{Cap}E_{\nu}}\right\} \sim \log\left\{\frac{1}{\operatorname{Cap}E_{\nu}}\right\},\,$$

and so

$$\sum_{1}^{\infty} \nu \left( \log \frac{1}{\operatorname{Cap} E_{\nu}} \right)^{-1} < \infty,$$

in view of Theorem 3. Thus Cap  $E_{\nu}$  is exponentially small for all large  $\nu$ .

In view of Theorem 4 the sum (2.15) converges for  $\lambda > \frac{1}{2}$  but not in general for  $\lambda = \frac{1}{2}$ , even when  $\psi(r) = \log r$ . However we can obtain a fairly precise estimate for the partial sums of the series in this case.

THEOREM 5. Suppose that  $\delta_n$  is a  $\psi$ -sequence with  $\psi(r) = (\log r)^{\alpha}$ , where  $1 < \alpha$ . Then

$$\sum_{\nu=1}^{n} \delta_{\nu}^{1/2} \leq \{\alpha(\alpha-1)\}^{1/2} \log n + O(1), \text{ as } n \to \infty.$$

We note that the result remains valid for  $\alpha > 2$ , when (2.1) is not satisfied. We write  $a_{\nu} = b_{\nu} - b_{\nu-1}$ , and deduce that for  $n \ge n_0$ 

$$\sum_{n_0}^{n} \delta_{\nu}^{1/2} = \sum_{n_0}^{n} (a_{\nu}/c_{\nu})^{1/2} \le \left(\sum_{n_0}^{n} a_{\nu}/b_{\nu}\right)^{1/2} \left(\sum_{n_0}^{n} b_{\nu}/c_{\nu}\right)^{1/2}.$$
(2.18)

We note that

$$\log \frac{b_n}{b_{n-1}} = \log \frac{b_n}{b_n - a_n} = -\log \left(1 - \frac{a_n}{b_n}\right) \ge \frac{a_n}{b_n}.$$

Thus

$$\sum_{n_0+1}^{n} \frac{a_{\nu}}{b_{\nu}} \le \log \frac{b_n}{b_{n_0}}, \qquad \sum_{n_0+1}^{n} \frac{b_{\nu}}{c_{\nu}} \le \log \frac{c_n}{c_{n_0}}. \tag{2.19}$$

Using (2.16), (2.18) and (2.19) we obtain

$$\sum_{n=0}^{n} \delta_{\nu}^{1/2} \leq (\alpha \log n + K_1)^{1/2} ((\alpha - 1) \log n + K_2)^{1/2} \leq {(\alpha (\alpha - 1))^{1/2} \log n + K_3},$$

where  $K_1$ ,  $K_2$ ,  $K_3$  are positive constants and this is Theorem 5. Example 1 shows that for any  $\alpha$  equality is possible in Theorem 5. In particular the constant  $\alpha(\alpha-1)$  cannot be replaced by any smaller quantity.

We have seen that the exceptional set  $E_{\nu}$  need not always be small. However it is small compared with the annulus  $A_{\nu}$ . The following result is an immediate consequence of the case  $\lambda = 1$  of Theorem 4, Corollary.

THEOREM 6. For all  $\theta$  in  $[0, 2\pi]$  apart from a set of capacity zero the ray arg  $z = \theta$  meets  $\bigcup E_{\nu}$  in a bounded set.

In fact let  $F_{\nu}$  be the radial projection of  $E_{\nu}$  onto the circle  $|z| = \frac{1}{2}$ . Then

$$\operatorname{Cap} F_{\nu} \leq \frac{A \operatorname{Cap} E_{\nu}}{r_0 K^{\nu+2}},$$

where A is a constant. Thus Theorem 4 shows that

$$\sum_{\nu=1}^{\infty} \left\{ \log \frac{2A}{\operatorname{Cap} F_{\nu}} \right\}^{-1} < \infty.$$

If  $G_n = \bigcup_{\nu=n}^{\infty} F_{\nu}$ , we deduce from the subadditivity of capacity that

$$\left\{\log \frac{1}{\operatorname{Cap} G_n}\right\}^{-1} \leq \sum_{n=n}^{\infty} \left\{\log \frac{1}{\operatorname{Cap} F_n}\right\}^{-1} \to 0, \text{ as } n \to \infty.$$

Thus Cap  $G_n \to 0$ , and if  $G = \bigcap_{n=1}^{\infty} G_n$ , then Cap G = 0. Clearly arg  $z = \theta$ , meets  $\bigcup E_{\nu}$  in a bounded set, unless  $\frac{1}{2}e^{i\theta} \in G$ .

Theorem 6 sharpens a previous result of Hayman [4], where the exceptional set had measure zero. An example of Ahlfors and Heins [1, p. 344] shows that the present result is best possible. We also obtain an improvement of (1.5).

THEOREM 7. For every p > 1 we can include  $\bigcup (E_n)$  in a sequence of disks  $|z - z_k| < \rho_k$ , such that

$$\sum_{1}^{\infty} \left\{ \log \left( \left| z_{k} \right| / \rho_{k} \right) \right\}^{-p} < \infty.$$

Our proof of Theorem 7 is similar to that of Theorem 5 in Essén and Jackson [4]. We set  $r_1 = (4K^2)^{-1}$ ,

$$h(r) = \min \{ (\log^+(r_1/r))^{-p}, 1 \}, r > 0.$$

Then h(r) increases with r and

$$A = \int_0^\infty \log \frac{1}{r} \, dh(r) = \log (4K^2 e) + \frac{1}{p-1} \, .$$

Let  $G_{\nu}$  be the set  $E_{\nu}/(2r_0K^{\nu+2})$ , i.e. the set of all points  $\xi = z/(2r_0K^{\nu+2})$ , where z lies in  $E_{\nu}$ . Then, in view of (2.5),  $G_{\nu}$  lies in the annulus

$$2r_1 < |\xi| < \frac{1}{2}.\tag{2.20}$$

Also

$$\log\left(\frac{1}{\operatorname{Cap} G_{\nu}}\right) = \log\left(\frac{2r_0K^{\nu+2}}{\operatorname{Cap} E_{\nu}}\right) > \log\left(\frac{r_0K^{\nu+2}}{\operatorname{Cap} E_{\nu}}\right).$$

It now follows from a result of Essén and Jackson [4, Lemma 1, p. 339] that there exists a constant  $A_1$  depending only on A, i.e. on p and K, such that G can be included in the union of a set of disks

$$|z - z_{\nu,k}| < r_{\nu,k}, \qquad k = 1, 2, \dots,$$
 (2.21)

with

$$\sum_{k} h(r_{\nu,k}) \le A_1 \{ \log (1/\text{Cap } G_{\nu}) \}^{-1} + 2^{-\nu} \le A_1 \left\{ \log \left( \frac{r_0 K^{\nu+2}}{\text{Cap } E_{\nu}} \right) \right\}^{-1} + 2^{-\nu}.$$
 (2.22)

From the Corollary of Theorem 4 with  $\lambda = 1$ , we see that

$$\sum_{\nu} \left\{ \log \left( \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_{\nu}} \right) \right\}^{-1} < \infty.$$
 (2.23)

In particular

$$\frac{\operatorname{Cap} E_{\nu}}{r_0 K^{\nu+2}} \to 0, \quad \text{as} \quad \nu \to \infty,$$

so that for  $\nu \ge \nu_0$  and all k we have  $h(r_{\nu,k}) < 1$ , i.e.  $r_{\nu,k} < r_1/e$ . We may then assume that  $|z_{\nu,k}| \ge r_1$  in (2.21), since otherwise the corresponding disk does not meet the annulus (2.20) in which  $G_{\nu}$  lies and can be omitted from our covering. Thus (2.22) yields for  $\nu \ge \nu_0$ .

$$\sum_{k} \left( \log \frac{|z_{\nu,k}|}{r_{\nu,k}} \right)^{-p} \leq \sum_{k} \left( \log \frac{r_1}{r_{\nu,k}} \right)^{-p} = \sum_{k} h(r_{\nu,k}) \leq A_1 \left\{ \log \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_{\nu}} \right\}^{-1} + 2^{-\nu}.$$

Since the disks with centres  $2r_0K^{\nu+2}z_{\nu,k}$  and radii  $2r_0K^{\nu+2}r_{\nu,k}$  cover  $E_{\nu}$ , we deduce Theorem 7 from (2.23).

In a later paper we hope to consider the case (1.6). Here the situation is more complicated and our results are less complete.

## 3. Results on $\psi$ -sequences; Proof of Theorem 3

We proceed to investigate further the nature of the sequences  $\delta_n$ . We write

$$b_n = c_n - c_{n-1} (3.1)$$

$$a_n = b_n - b_{n-1} = c_n - 2c_{n-1} + c_{n-2}. (3.2)$$

Since  $c_n$  is convex and increasing,  $b_n \ge 0$ ,  $a_n \ge 0$ . Since  $c_n = 0$  for  $n \le 0$ , we deduce that  $a_n = b_n = 0$ ,  $n \le 0$ . Also it follows from (2.2) that  $b_n$  cannot be zero for all n, so that  $b_n$  is finally positive. Thus

$$b_n \to b$$
 as  $n \to \infty$ , where  $0 < b \le \infty$ . (3.3)

We have

$$c_n = \sum_{\nu=1}^n b_{\nu} \tag{3.4}$$

so that  $c_n/n$  increases with n and

$$\frac{c_n}{n} \to b \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Suppose first that b is finite. Then

$$\sum_{1}^{\infty} a_n = b. \tag{3.6}$$

Also if  $n = n_0$  is the first integer for which  $c_n$  and so  $a_n$  and  $b_n$  are positive, we have, since  $c_n/n$  increases,

$$\sum_{1}^{\infty} n\delta_n = \sum_{n_0}^{\infty} \frac{na_n}{c_n} \leq \frac{n_0}{c_{n_0}} \sum_{n_0}^{\infty} a_n = \frac{n_0b}{c_{n_0}}.$$

This proves (2.14) in this case. We note that if  $\delta_n$  is a  $\psi$ -sequence with  $\psi = \log r$ , then  $c_n = O(n)$  so that b must be finite and the above conclusions hold.

To complete the proof of Theorem 3, suppose that  $\delta_n$  satisfies (2.9) and (2.14) and define  $b_n$ ,  $c_n$  by (2.10) to (2.13). Then (2.12) gives

$$(1-\delta_n)b_n = (1+\delta_n)b_{n-1} + \delta_n \sum_{n=1}^{n-2} b_n \le \{1+(n-1)\delta_n\}b_{n-1},$$

since  $b_n$  is increasing. Thus

$$\frac{b_n}{b_{n_0}} \le \prod_{n=n_0+1}^{\infty} \left\{ \frac{1 + (n-1)\delta_n}{1 - \delta_n} \right\} = C < \infty, \text{ in view of } (2.14).$$

Thus  $b_n = O(1)$ ,  $c_n = O(n)$  in this case, so that  $\delta_n$  is a  $\psi$ -sequence with  $\psi(r) = \log r$ . This proves Theorem 3.

3.1. We have shown that if b is finite in (3.5) then (2.14) holds. This also implies (2.15). For if  $\frac{1}{2} < \lambda < 1$  we define

$$p = 1/\lambda$$
,  $q = p/(p-1) = 1/(1-\lambda)$ , so that  $1 < q < 2 < p < \infty$ .

Then it follows from Hölder's inequality that

$$\sum_{1}^{\infty} \delta_{n}^{\lambda} = \sum_{1}^{\infty} (n\delta_{n})^{1/p} n^{-1/p} \leq \left(\sum_{1}^{\infty} n\delta_{n}\right)^{1/p} \left(\sum_{1}^{\infty} n^{-q/p}\right)^{1/q} < \infty.$$

Thus (2.15) holds in this case.

We proceed to prove (2.15) in the more difficult case when b is infinite. We note that in view of (3.4) and since  $b_n$  increases

$$nb_n \leq \sum_{n+1}^{2n} b_{\nu} \leq c_{2n} = O\{\psi(K^{2n})\} = O(n^2),$$

in view of (2.1) and (2.3). Thus

$$b_n = O(n), (3.7)$$

and so

$$\frac{b_n}{c_n} \to 0$$
, as  $n \to \infty$  (3.8)

if b is infinite. If b is finite (3.8) follows immediately from (3.5) so that (3.8) holds in any case. The inequality (2.15) is contained in the following somewhat more precise

THEOREM 8. Suppose that  $a_1 > 0$ ,  $0 \le a_n \le n$ ,  $n \ge 1$  and set

$$b_n = \sum_{1}^{n} a_{\nu}, \qquad c_n = \sum_{\nu=1}^{n} b_{\nu}, \qquad n \ge 1.$$

Then for  $\frac{1}{2} < \lambda \le 1$  we have

$$s_{\lambda} = \sum_{\nu=1}^{\infty} \left(\frac{a_{\nu}}{c_{\nu}}\right)^{\lambda} < A_1 + \frac{A_2}{a_1(\frac{1}{2}\lambda - \frac{1}{4})},$$

where the constants  $A_1, A_2$  depend only on  $\lambda$ .

Before proving Theorem 8 we deduce Theorem 4 from Theorem 8. Let  $\delta_n$  be

the  $\psi$ -sequence of Theorem 4 and define  $a_n, b_n$  by (3.1) and (3.2). Then  $a_n$  cannot be zero for all n. We suppose that m+1 is the first positive integer such that  $a_{m+1} > 0$  and define

$$a_n' = a_{m+n}, \qquad n \ge 1.$$

Then it follows from (3.7) that  $a_n = O(n)$  and so  $a'_n = O(n)$ . Thus if  $\delta$  is a suitable constant and we set  $a''_n = \delta a'_n$ , we have

$$a_n'' \le n$$
,  $n = 1, 2, ..., a_1'' > 0$ .

Also if

$$b_n'' = \sum_{1}^{n} a_{\nu}'', \qquad c_n'' = \sum_{1}^{n} b_{\nu}'',$$

we clearly have  $b_n'' = \delta b_{n+m}$ ,  $c_n'' = \delta c_{n+m}$ . Thus

$$\sum_{1}^{\infty} \delta_{n}^{\lambda} = \sum_{1}^{\infty} \left( \frac{a_{n}}{c_{n}} \right)^{\lambda} = \sum_{m+1}^{\infty} \left( \frac{a_{n}}{c_{n}} \right)^{\lambda} = \sum_{1}^{\infty} \left( \frac{a_{n}''}{c_{n}''} \right)^{\lambda}$$

$$\leq A_{1} + \frac{A_{2}}{(a_{1}'')^{\frac{1}{2}}\lambda - \frac{1}{4}} = A_{1} + \frac{A_{2}}{(\delta a_{m+1})^{\frac{1}{2}}\lambda - \frac{1}{4}}.$$

Thus (2.15) follows from Theorem 8 and we obtain a bound for the series, depending on the first integer m, for which  $a_{m+1} = c_{m+1} - 2c_m + c_{m-1} > 0$ , and on the constants implicit in (2.1) and (2.3).

3.2. To prove Theorem 8 we need a subsidiary result. We denote by  $A_3$ ,  $A_4 \cdots$  constants depending on  $\lambda$  only.

LEMMA 1. If  $\lambda = \frac{1}{2} + 2\varepsilon$ , where  $0 < \varepsilon \le \frac{1}{4}$  then we have, with the hypotheses of Theorem 8,

$$\sum_{N}^{2N} \left( \frac{a_n}{c_n} \right)^{\lambda} < A_3 \left( \frac{N}{c_N} \right)^{\varepsilon}, \qquad N \ge 1.$$

We write  $t = \lambda - \varepsilon = \frac{1}{2} + \varepsilon$ , p = 1/t, q = p/(p-1). Then

$$c_n = \sum_{r=1}^n b_r = \sum_{r=1}^n (n-r+1)a_r.$$

Thus we have, for  $1 \le m \le n$ ,

$$\sum_{r=m}^{n} a_{r}^{t} \leq \sum_{r=1}^{n} a_{r}^{t} = \sum_{r=1}^{n} \left\{ (n-r+1)a_{r} \right\}^{1/p} (n-r+1)^{-1/p}$$

$$\leq \left\{ \sum_{1}^{n} (n-r+1)a_{r} \right\}^{1/p} \left\{ \sum_{1}^{n} (n-r+1)^{-q/p} \right\}^{1/q}$$

$$\leq c_{n}^{t} \left\{ \sum_{k=1}^{\infty} k^{-q/p} \right\}^{1/q} = A_{4}c_{n}^{t}.$$

Suppose now that  $c_n \leq 2c_m$ . We deduce that

$$\sum_{r=m}^{n} \left(\frac{a_r}{c_r}\right)^t \leq \sum_{r=m}^{n} \left(\frac{a_r}{c_m}\right)^t \leq 2^t \sum_{r=m}^{n} \left(\frac{a_r}{c_n}\right)^t \leq 2A_4.$$

Further

$$\left(\frac{a_r}{c_r}\right)^{\lambda} = \left(\frac{a_r}{c_r}\right)^{t+\varepsilon} = \left(\frac{a_r}{c_r}\right)^{\varepsilon} \left(\frac{a_r}{c_r}\right)^{t} \le \left(\frac{n}{c_m}\right)^{\varepsilon} \left(\frac{a_r}{c_r}\right)^{t},$$

since  $a_n \le n$ . Thus

$$\sum_{r=m}^{n} \left(\frac{a_r}{c_r}\right)^{\lambda} \le 2A_4 \left(\frac{n}{c_m}\right)^{\varepsilon}. \tag{3.9}$$

We now set  $m_0 = N$  and if  $m_k$  has already been defined, we define  $m_{k+1}$  to be the smallest integer m such that  $c_m \ge 2c_{m_k}$ . If  $m_{k+1} \le 2N$ , we define  $n_k = m_{k+1} - 1$ . If s is the smallest integer k, for which  $m_{k+1} > 2N$ , we define  $n_s = 2N$ . Thus

$$c_{m_k} \ge 2^k c_N, \qquad k = 0, 1, 2, \dots, s$$

Also in view of (3.9) we have

$$\begin{split} \sum_{N}^{2N} \left( \frac{a_n}{c_n} \right)^{\lambda} &= \sum_{k=0}^{s} \sum_{r=m_k}^{n} \left( \frac{a_r}{c_r} \right)^{\lambda} \le \sum_{k=0}^{s} 2A_4 \left( \frac{n_k}{c_{m_k}} \right)^{\varepsilon} \\ &\le 4A_4 N^{\varepsilon} \sum_{k=0}^{s} (2^k c_N)^{-\varepsilon} \le 4A_4 \left( \frac{N}{c_N} \right)^{\varepsilon} \sum_{k=0}^{\infty} 2^{-k\varepsilon}. \end{split}$$

This proves Lemma 1.

3.3. We can now complete the proof of Theorem 8. We set  $m_k = 2^k$ ,  $k = 0, 1, 2, \ldots$  and consider the sums

$$\sigma_k = \sum_{m_k}^{m_{k+1}-1} \left(\frac{a_n}{c_n}\right)^{\lambda}.$$

We write

$$B_k = b_{m_k}, \qquad C_k = c_{m_k},$$

and consider in turn the cases

(i) 
$$C_k > \frac{1}{4}B_{k+1}m_k$$
, (ii)  $C_k \le \frac{1}{4}B_{k+1}m_k$ .

We denote sums over k in the ranges (i) and (ii) by  $\Sigma'$  and  $\Sigma''$  respectively. In case (i) we have, using Hölder's inequality

$$\sigma_{k} \leq \sum_{m_{k}}^{m_{k+1}-1} \left(\frac{a_{n}}{c_{n}}\right)^{t} \leq C_{k}^{-t} \sum_{m_{k}}^{m_{k+1}-1} a_{n}^{t} = C_{k}^{-t} \sum a_{n}^{1/p}$$

$$\leq C_{k}^{-t} \left(\sum a_{n}\right)^{1/p} (m_{k})^{1/q} \leq C_{k}^{-t} B_{k+1}^{t} m_{k}^{1-t} \leq 4 m_{k}^{1-2t}.$$

Thus

$$\sum' \sigma_k \le 4 \sum_{k=0}^{\infty} m_k^{1-2t} = 4 \sum_{k=0}^{\infty} 4^{-k\varepsilon} = A_1.$$
 (3.10)

In case (ii) we note that

$$C_{k+2} \ge m_{k+1} B_{k+1} \ge 4 \frac{m_{k+1}}{m_{k}} C_k = 8 C_k.$$

Thus

$$\frac{C_{k+2}}{m_{k+2}} \ge 2 \frac{C_k}{m_k}.$$

As we remarked after (3.4),  $c_n/n$  increases with n. Thus if  $k = k_{\nu}$  are the positive integers for which case (ii) holds, we deduce that  $k_{\nu+2} \ge k_{\nu} + 2$ , so that

$$\frac{C_{k_{\nu+2}}}{m_{k_{\nu+2}}} \ge 2 \frac{C_{k_{\nu}}}{m_{k_{\nu}}}.$$

Thus we prove by induction, separately over even and odd  $\nu$ , that

$$\frac{C_{k_{\nu}}}{m_{k_{\nu}}} \ge 2^{(\nu/2)-1} \frac{C_{k_{1}}}{m_{k_{1}}} \ge a_{1} 2^{(\nu/2)-1}, \qquad \nu \ge 1.$$

Now we deduce from Lemma 1 that for  $k = k_{\nu}$  in the case (ii) we have

$$\sigma_{k_{\nu}} < A_3 (a_1 2^{(\nu/2)-1})^{-\varepsilon} = A_3 \left(\frac{2}{a_1}\right)^{\varepsilon} 2^{-\nu \varepsilon/2}.$$

On summing over  $\nu$  we deduce that

$$\sum^{\prime\prime} \sigma_k \le A_2 a_1^{-\varepsilon}. \tag{3.11}$$

On adding (3.10) and (3.11) we deduce Theorem 8. Since we deduced Theorem 4 from Theorem 8, the proof of Theorem 4 is also complete. The corollary of Theorem 4 is an immediate consequence of (2.15) and (2.7). Thus it remains to prove Theorems 1 and 2.

## 4. Proof of Theorem 1

Suppose that u(z) is s.h. in the plane. We replace u(z) in |z| < 1 by the Poisson integral of the values of u(z) on |z| = 1 and leave u(z) unaltered for  $|z| \ge 1$ . The resulting function has the same asymptotic behaviour as u(z) and is harmonic and in particular finite at z = 0. By subtracting a constant if necessary we may arrange that u(0) = 0. We further assume that u(z) has order zero. Thus u(z) has a representation of the form (see e.g. [5])

$$u(z) = \int \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi), \tag{4.1}$$

where  $\mu$  is the Riesz mass of u and the integral is taken over the open plane. Let n(t) be the mass in |z| < t. Since u(z) is harmonic in |z| < 1, n(1) = 0. We define

$$N(r) = \int_0^r \frac{n(t) dt}{t}. \tag{4.2}$$

The Jensen formula gives

$$N(r) = \frac{1}{2\pi} \int_{0}^{2\pi} u(re^{i\theta}) d\theta \le B(r, u).$$
 (4.3)

Also it follows from (4.1) that

$$\int \log \left| \frac{z - \xi}{\xi} \right| d\mu(\xi) = u(z) \le \int \log \left( \frac{|z| + |\xi|}{|\xi|} \right) d\mu(\xi). \tag{4.4}$$

We start by proving a result which is a simple consequence of (1.5).

LEMMA 2. With the hypotheses of Theorem 1, we can choose,  $r_0 = r_0(K)$ , such that on the circles  $|z| = r_v = r_0 K^v$ , we have

$$u(z) > (1 - \varepsilon_{\nu})B(r_{\nu}), \tag{4.5}$$

where  $\varepsilon_{\nu} \rightarrow 0$  as  $\nu \rightarrow \infty$ .

It follows from (1.5) that if  $F(\varepsilon)$  is the set of all  $r \ge 1$ , such that

$$u(z) \le (1 - \varepsilon)B(r)$$

at some point on |z| = r, then

$$\int_{F(r)} \frac{dr}{r} < \infty.$$

Choose  $\varepsilon_{\nu} = 1/\nu$  and let  $\rho_{\nu}$  be so large that if  $F_{\nu}$  is the part of  $F(\varepsilon_{\nu})$  in  $r \ge \rho_{\nu}$  then we have

$$\int_{I} \frac{dr}{r} < \frac{\log K}{2^{\nu}}.$$

Let  $F = \bigcup_{\nu=1}^{\infty} F_{\nu}$ . Then

$$\int_{F} \frac{dr}{r} < \log K.$$

Let  $F_0$  be the set of all  $r_0$ , such that  $1 < r_0 < K$  and  $r_0 K^{\nu}$  belongs to F for some  $\nu$ . Clearly

$$\int_{F_0} \frac{dr}{r} \le \int_{F} \frac{dr}{r} < \log K.$$

Thus there exists  $r_0$  in the interval (1, K) and not belonging to  $F_0$  and so for this  $r_0$ ,  $r_0K^{\nu}$  does not belong to  $F_{\nu}$  for any  $\nu$ , i.e. (4.5) holds with  $r_{\mu} = r_0K^{\mu}$ , provided that  $r_{\mu} \ge \rho_{\nu}$ .

This proves Lemma 2.

4.1. We now return to the representation (4.1) and suppose that z lies in the annulus  $A_y$  given by (2.5). We write

$$b_{\nu} = n(r_{\nu}), \qquad c_{\nu} = \sum_{\mu=1}^{\nu} b_{\mu}, \qquad a_{\nu} = b_{\nu} - b_{\nu-1}.$$
 (4.6)

We note that

$$\delta_{\nu} = \frac{a_{\nu+1}}{c_{\nu+1}} = \frac{c_{\nu+1} - 2c_{\nu} + c_{\nu-1}}{c_{\nu+1}} \tag{4.7}$$

is a  $\psi$ -sequence in this case. In fact

$$c_{\nu} \leq \sum_{\mu=1}^{\nu} \frac{1}{\log K} \int_{r_{\mu}}^{r_{\mu+1}} \frac{n(t) dt}{t} \leq \frac{1}{\log K} N(r_{\nu+1}) = O\{\psi(K^{\nu+2})\}$$
 (4.8)

in view of (4.3) and (1.7). Thus, in view of (2.1),  $c_{\nu} = O(\nu^2)$  and so  $c_{\nu}$  satisfies (2.3) with  $(\log r)^2$  instead of  $\psi(r)$ . This in turn implies (3.7) and (3.8), so that

$$c_{\nu+1} \sim c_{\nu} \sim c_{\nu-2} = O\{\psi(K^{\nu})\}.$$
 (4.9)

in view of (4.8). Thus (2.2) to (2.4) are satisfied. Next we show that

$$B(r) = B(r, u) = (\log K + o(1))c_{\nu+1}, \qquad r_{\nu} \le r \le r_{\nu+1}. \tag{4.10}$$

uniformly as  $r \rightarrow \infty$ . In fact it follows from (4.8), (4.9) and (4.3) that

$$c_{\nu+1}\log K \sim c_{\nu-1}\log K \leq N(r_{\nu}) \leq N(r) \leq B(r)$$

in this case. In the opposite direction we note that

$$c_{\nu} = \sum_{\mu=1}^{\nu} b_{\mu} = \sum_{\mu=1}^{\nu} n(r_{\mu}) \ge \frac{1}{\log K} \sum_{\mu=1}^{\nu} \int_{r_{\mu-1}}^{r_{\mu}} \frac{n(t) dt}{t} = \frac{1}{\log K} N(r_{\nu}) + O(1).$$

On the other hand we deduce from (4.3) and Lemma 2 that

$$N(r_{\nu}) = \frac{1}{2\pi} \int_{0}^{2\pi} u(r_{\nu}e^{i\theta}) d\theta \ge (1 + o(1))B(r_{\nu}).$$

Thus for  $r_{\nu} \le r \le r_{\nu+1}$  we have

$$B(r) \le B(r_{\nu+1}) \le (1+o(1))N(r_{\nu+1}) \le (\log K + o(1))c_{\nu+1}.$$

This proves (4.10). We deduce

LEMMA 3. We have, uniformly as  $\nu \to \infty$ , for z in  $A_{\nu}$ 

$$u(z) = \int_{A_{\nu}} \log|z - \xi| \ d\mu(\xi) + c_{\nu+1}(\log K + o(1)) - \nu a_{\nu+1} \log K. \tag{4.11}$$

We write

$$I_{\nu}(z) = \int_{A_{\nu}} \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi), \tag{4.12}$$

$$u_{\nu}(z) = u(z) - I_{\nu}(z).$$
 (4.13)

Then for  $r_{\nu} \le |z| \le r_{\nu+1}$  we have, using (3.8) and (4.6),

$$I_{\nu}(z) \leq \log\left(1 + \frac{r_{\nu+1}}{r_{\nu}}\right) \int_{A_{\nu}} d\mu(\xi) \leq b_{\nu+1} \log(1+K) = o(c_{\nu+1}).$$

Using (4.10) and Lemma 2, we deduce that for  $|z| = r_{\nu}$ ,  $r_{\nu+1}$  we have

$$u_{\nu}(z) \geq (\log K + o(1))c_{\nu+1}.$$

Since  $u_{\nu}(z)$  is harmonic for  $r_{\nu} < |z| < r_{\nu+1}$  it follows that this inequality remains valid in the whole annulus  $A_{\nu}$ . Next we note that on  $|z| = r_{\nu+2}$  we have, in view of (4.9), (4.10)

$$u(z) < (\log K + o(1))c_{\nu+1}$$

and further

$$I_{\nu}(z) > \log\left(\frac{r_{\nu+1}}{r_{\nu}} - 1\right) \int_{A_{\nu}} d\mu(\xi) = O(b_{\nu+1}) = o(c_{\nu+1}).$$

Thus for  $|z| = r_{\nu+2}$  we have

$$u_{\nu}(z) < (\log K + o(1))c_{\nu+1},$$
 (4.14)

and since  $u_{\nu}(z)$  is subharmonic this inequality remains valid in  $|z| < r_{\nu+1}$  and so in  $A_{\nu}$ . Thus

$$u_{\nu}(z) = (\log K + o(1))c_{\nu+1}, \qquad r_{\nu} \le |z| \le r_{\nu+1}.$$
 (4.15)

Also

$$I_{\nu}(z) = \int_{A_{\nu}} \log|z - \xi| \, d\mu(\xi) - \int_{A_{\nu}} \log|\xi| \, d\mu(\xi)$$

$$= \int_{A_{\nu}} \log|z - \xi| \, d\mu(\xi) - \int_{A_{\nu}} (\nu \log K + O(1)) \, d\mu(\xi)$$

$$= \int_{A_{\nu}} \log|z - \xi| \, d\mu(\xi) - \nu a_{\nu+1} \log K + O(a_{\nu+1})$$

$$= \int_{A_{\nu}} \log|z - \xi| \, d\mu(\xi) - \nu a_{\nu+1} \log K + o(c_{\nu+1}). \tag{4.16}$$

On combining (4.15) and (4.16) we deduce (4.11).

It follows from Lemma 3 that if u(z) is any function subharmonic in the plane which satisfies (1.2) then the size of u(z) is given in the annulus  $A_{\nu}$  with great precision by (4.11). In particular u(z) is much smaller than  $c_{\nu+1} \log K$  if and only if the integral on the right hand side of (4.11) is large and negative.

4.2. In order to complete our proof we need another subsidiary result, whose statement is almost the definition of capacity.

LEMMA 4. Let  $\mu(\xi)$  be a positive measure of total mass  $\mu_0$  distributed over a compact set F. If G is the plane set where

$$V(z) = \int \log|z - \xi| \ d\mu(\xi) < C. \tag{4.17}$$

then the outer capacity Cap G of G satisfies

$$\operatorname{Cap} G \le \exp\left(C/\mu_0\right). \tag{4.18}$$

Conversely given any relatively open subset G of the annulus  $A_{\nu}$  of (2.5), there exists a unit measure  $\mu$ , distributed on  $\bar{G}$  and such that

$$V(z) = \int \log|z - \xi| \ d\mu(\xi) = \log \operatorname{Cap} G \tag{4.19}$$

at every point of G.

We recall some facts concerning capacity. (1) Let E be a compact set and  $\mu$  a unit measure distributed on E. Consider the energy integral

$$I(\mu) = \iint_{\mathcal{F}} \log|a-b| \ d\mu(a) \ d\mu(b).$$

If  $V_0$  is the maximum value of  $I(\mu)$  for all such measures, then

Cap 
$$E = e^{V_0}$$
.

In particular if  $I(\mu) = -\infty$  for all such measures  $\mu$ , then Cap E = 0.

Suppose now that Cap E > 0. This is always the case if E has interior points. Then there exists a unit measure  $\mu$  such that the conductor potential

$$V(z) = \int \log|z - a| \ d\mu(a)$$

satisfies

$$V(z) \ge V_0$$

in the whole z plane with equality at all points of E with the exception of the irregular boundary points of the unbounded component of the complement of E.

The outer capacity of more general sets is defined as follows. If G is open then Cap G is defined to be the upper bound of capacities of compact sets contained in G. Finally, if E is any bounded set, Cap E is defined to be the lower bound of capacities of open sets containing E.

Suppose now that G is the set of Lemma 4. Let  $F_0$  be a compact subset of G, which is the union of a finite number of closed disks, so that the complement of  $F_0$  is regular for the problem of Dirichlet. Let  $V_0(z)$  be the conductor potential of  $F_0$ .

<sup>&</sup>lt;sup>1</sup> See e.g. Tsuji [7 p.p. 54 et seq.].

Then, for z in  $F_0$ ,

$$V_0(z) = \int \log |z - \xi| \ d\nu(\xi) = V_0 = \log \operatorname{Cap} F_0,$$

where  $\nu$  is a unit measure distributed on  $F_0$ .

Consider now, with the notation of Lemma 4,

$$u(z) = \frac{V(z)}{\mu_0} - V_0(z)$$

in the unbounded component  $G_0$  of the complement of  $F_0$ . Then  $V_0(z)$  is harmonic outside  $F_0$  except possibly at  $\infty$  and so u(z) is s.h. at the finite points of  $G_0$ . Also near  $\infty$ 

$$u(z) = \log |z| - \log |z| + o(1),$$

so that u(z) is harmonic at  $\infty$  and  $u(\infty) = 0$ . Thus u(z) is s.h. in  $G_0$  including  $\infty$ . Also as z approaches any finite boundary point  $\xi$  of  $G_0$  we have

$$\overline{\lim} V(z) < C$$
 and  $\lim V_0(z) = \log \operatorname{Cap} F_0$ .

This shows that

$$\overline{\lim} u(z) \leq \frac{C}{\mu_0} - \log \operatorname{Cap} F_0.$$

Since  $u(\infty) = 0$ , we deduce from the maximum principle that

$$\log \operatorname{Cap} F_0 \leq C/\mu_0$$
.

This inequality holds for every compact subset of G, which consists of the union of a finite number of closed disks. Any compact subset  $F_1$  of G is contained in such a set  $F_0$  and so

Cap 
$$F_1 \leq \exp(C/\mu_0)$$
.

Now (4.18) follows from the definition of outer capacity.

Conversely let G be a relatively open subset of  $A_{\nu}$ , let  $F_n$  be a sequence of

compact sets, each of which is regular for the problem of Dirichlet and such that

$$F_n \subset F_{n+1} \subset G$$

Cap 
$$F_n \rightarrow \text{Cap } G$$
.

If G does not meet  $|z| = r_{\nu}$ , so that G is open, we may take for  $F_n$  the union of a finite number of closed disks. If G meets  $|z| = r_{\nu}$ , then there exists an open plane set  $G_0$ , such that

$$G_0 \cap A_{\nu} = G$$
.

We construct the  $F_n$  as above corresponding to the set  $G_0$ . Then

$$F'_n = F_n \cap A_{\nu}$$

is bounded by a finite number of arcs of circles so that  $F'_n$  is still regular for the problem of Dirichlet. Also

$$F'_n \subset F'_{n+1}$$
 and  $\bigcup F'_n = G$ ,

so that Cap  $F'_n \rightarrow$  Cap G as required.

Let  $V_n(z)$  be the conductor potential of  $F_n$  and let

$$v_n = \log \operatorname{Cap} F_n$$

be the value of  $V_n(z)$  on  $F_n$ . Then  $V_{n+1}(z) - V_n(z)$  is s.h. in the unbounded complementary domain  $G_n$  of  $F_n$ , equal to  $v_{n+1} - v_n$  on the boundary of  $G_n$ , and zero at  $\infty$ . Thus

$$v_{n+1}-v_n\geq 0.$$

On the other hand  $V_{n+1}(z) - V_n(z)$  is harmonic in  $G_{n+1}$  and does not exceed  $v_{n+1} - v_n$  on the boundary of  $G_{n+1}$ . Thus

$$V_{n+1}(z) - V_n(z) \le v_{n+1} - v_n$$

in  $G_{n+1}$ , and also in  $F_{n+1}$ , since  $V_{n+1}(z) = v_{n+1}$ ,  $V_n(z) \ge v_n$  in  $F_{n+1}$ . Thus

$$V_n(z)-v_n$$

is a decreasing sequence of subharmonic functions and so tends to a subharmonic limit  $V(z) - V_0$ , where

$$V_0 = \lim v_n = \log \operatorname{Cap} G$$
.

If  $z_0$  is any point of G then  $z_0 \in F_n$  for large n, and so  $V_n(z_0) = v_n$  for large n. Thus  $V(z_0) = V_0$ .

Next the function V(z) is harmonic in the exterior of G, and so by the Riesz representation theorem there exists a measure  $\mu$  of total mass  $\mu_0$ , say, distributed on  $\bar{G}$  and such that

$$V(z) = \int_{\bar{G}} \log|z - \xi| d\mu(\xi) + h(z),$$

where h(z) is harmonic in the open plane. Also near  $\infty$ 

$$V_n(z) = \log|z| + \frac{O(1)}{z},$$

uniformly in n, and so

$$V(z) = \log|z| + \frac{O(1)}{z}$$
 as  $z \to \infty$ .

Thus

$$h(z) = O(\log |z|)$$
 as  $z \to \infty$ ,

i.e. h(z) = c = const. Thus  $\mu_0 = 1$ , c = 0,

$$V(z) = \int_{\bar{G}} \log|z - \xi| \ d\mu(\xi)$$

and  $V(z) = V_0 = \log \operatorname{Cap} G$  on G. This proves (4.19) and completes the proof of Lemma 4.

4.3. Proof of Theorem 1. Suppose now that  $E_{\nu}(\varepsilon)$  is the set defined in Theorem 1. We recall the definitions (4.6) and (4.7), so that  $\delta_{\nu}$  is a  $\psi$ -sequence. It follows from (2.6), (4.9) and (4.10) that we have for z in  $\bigcup E_{\nu}$ 

$$u(z) < (1-\epsilon)(\log K + o(1))c_{\nu+1}$$
.

In view of Lemma 3 we deduce that for z in  $E_{\nu}$ 

$$\int_{A_{\nu}} \log |z-\xi| \ d\mu(\xi) < -(\varepsilon \log K + o(1)) c_{\nu+1} + \nu a_{\nu+1} \log K.$$

Now Lemma 4 shows that

$$\frac{\operatorname{Cap}(E_{\nu})}{r_{0}K^{\nu+2}} \leq \exp\left\{\left(-\varepsilon \log K + o(1)\right) \frac{c_{\nu+1}}{a_{\nu+1}} + \nu \log K - (\nu+2) \log K - \log r_{0}\right\} \\
\leq \exp\left\{\frac{\left(-\varepsilon \log K + o(1)\right)}{\delta_{\nu}} + O(1)\right\} = \exp\left\{\frac{-\varepsilon \log K + o(1)}{\delta_{\nu}}\right\}.$$

This yields (2.7), and completes the proof of Theorem 1.

4.4. Proof of Theorem 2. We proceed to prove Theorem 2. We suppose given the  $\psi$ -sequence  $\delta_{\nu}$  defined from the quantities  $c_{\nu}$  satisfying (2.2) and (2.3) in accordance with (2.4). We define

$$a_{\nu} = c_{\nu} - 2c_{\nu-1} + c_{\nu-2}$$

Let  $E_{\nu}(\varepsilon)$  be the sets defined in Theorem 2. In view of the definition of outer capacity we can include  $E_{\nu}$  in relatively open subsets  $G_{\nu}$  of  $A_{\nu}$  whose capacities differ by arbitrarily little from those of the  $E_{\nu}$ , Thus we may assume without loss of generality that the  $E_{\nu}$  are relatively open subsets of  $A_{\nu}$  and that (2.8) is still satisfied, provided that  $\delta_{\nu} > 0$ .

Suppose now that  $\delta_{\nu} > 0$  so that  $a_{\nu} > 0$ . Then, in view of Lemma 4 we can find a mass distribution  $\mu_{\nu}$  in  $A_{\nu}$  of total mass  $a_{\nu}$ , such that

$$V_{\nu}(z) = \int \log|z - \xi| \, d\mu_{\nu}(\xi) = a_{\nu} \log \operatorname{Cap} E_{\nu} \tag{4.20}$$

on  $E_{\nu}$ . We write  $\mu = \sum_{1}^{\infty} \mu_{\nu}$ , and

$$u_0(z) = \sum_{\nu=1}^{\infty} \int \log \left| 1 - \frac{z}{\xi} \right| d\mu_{\nu}(\xi) = \int \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi).$$

We note that  $u_0(z)$  is subharmonic in the plane and harmonic in  $|z| < r_0$ . In fact if n(r) is the total mass  $\sum \mu_{\nu}$  in |z| < r, then n(r) = 0 for  $r \le r_0$ , and

$$b_{\nu} = n(r_{\nu+1}) = \sum_{\mu=1}^{\nu} a_{\nu} = c_{\nu} - c_{\nu-1}.$$

Again

$$N(r_{\nu}) = \sum_{\mu=1}^{\nu} \int_{r_{\mu-1}}^{r_{\mu}} \frac{n(t) dt}{t} \le \log K \sum_{\mu=1}^{\nu} n(r_{\mu}) \le c_{\nu-1} \log K = O(\nu^{2}), \tag{4.21}$$

in view of (2.3) and (2.1). Thus, after changing  $r_0$  is necessary, we may apply the analysis of section 4.1 to the function  $u_0(z)$  and deduce from (4.10) that the integral for  $u_0(z)$  converges and

$$B(r, u_0) \sim N(r)$$
 as  $r \to \infty$ . (4.22)

Since (4.9) is satisfied by our sequence  $c_{\nu}$ , we deduce from (4.21) and (4.8) that

$$N(r) = (\log K + o(1))c_{\nu}, \qquad r_{\nu} \le r \le r_{\nu+1}.$$

On combining this with (4.22) we see that (4.10) still holds with our original choice of  $r_0$ . From this, (2.3) and (4.9) we see that

$$B(r, u_0) = O(c_{\nu+1}) = O(c_{\nu}) = O\{\psi(K^{\nu})\} = O\{\psi(r)\}, \quad r_{\nu} \le r \le r_{\nu+1}.$$

Thus the function  $u_0(z)$  satisfies (1.7).

Next we define  $I_{\nu}(z)$ ,  $u_{\nu}(z)$  by (4.12) and (4.13) with  $u_0(z)$  instead of u(z) and deduce that (4.14) still holds. This shows that the upper bound implied by (4.11) still holds, so that for z in  $A_{\nu}$ 

$$u_0(z) \le \int_{A_{\nu}} \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi) + c_{\nu}(\log K + o(1)).$$

On combining this with (4.20) we deduce that in  $E_{\nu}$ 

$$u_0(z) \le a_{\nu} \{ \log \operatorname{Cap} E_{\nu} - \log K^{\nu+2} + O(1) \} + c_{\nu} \{ \log K + o(1) \}$$

$$= c_{\nu} \left\{ -\delta_{\nu} \left[ \log \left( \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_{\nu}} \right) + O(1) \right] + \log K + o(1) \right\}.$$

$$< c_{\nu} \{ 1 - \varepsilon' + o(1) \} \log K,$$

in view of (2.8). From this, (4.9) and (4.10) we deduce that for large r

$$u_0(z) < (1 - \varepsilon' + o(1))B(r, u_0).$$

Since  $\varepsilon' > \varepsilon$ , we deduce that for some  $r_0 > 0$ ,  $z \in \bigcup E_{\nu}$ ,  $r \ge r_0$ ,

$$u_0(z) < (1-\varepsilon)B(r, u_0).$$

If we now set  $u(z) = u_0(z) - \max(B(r_0, u_0), 0)$ , (2.6) is satisfied for all z in  $\bigcup E_{\nu}$ . This proves Theorem 2.

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