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Slowly growing subharmonic functions I

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Dedicated to Albert Pfluger on his 70th Birthday

1. Introduction

Let $u(z)$ be subharmonic (s.h.) and not constant in the open z-plane and let

$$B(r, u) = \sup_{|z|=r} u(z). \tag{1.1}$$

If $B(r, u)$ grows sufficiently slowly then it is known that, for "most" values of $z = re^{i\theta}$, $u(z)$ is not much smaller than $B(r)$. For instance if

$$B(r) = O(\log r)^2 \quad \text{as} \quad r \to \infty, \tag{1.2}$$

then Barry [3] showed that

$$u(z) > B(r) - O(1), \qquad |z| = r \tag{1.3}$$

except for a set of r of small upper logarithmic density. Further Hayman [5] showed that if (1.2) holds, then for any positive ε we have

$$u(z) > (1-\varepsilon)B(r), \qquad |z| = r \tag{1.4}$$

outside a sequence of disks $|z - z_n| < \rho_n$, such that

$$\Sigma \rho_n/(1+|z_n|) < \infty. \tag{1.5}$$

Both these results fail if the condition (1.2) is weakened. Thus Barry [3] showed that if

$$\psi(r)/(\log r)^2 \to +\infty, \tag{1.6}$$

then there exists $u(z)$ satisfying

$$B(r, u) = O\{\psi(r)\} \tag{1.7}$$

and

$$u(-r) - u(r) \to -\infty, \quad \text{as} \quad r \to \infty.$$

Further Piranian [6] showed under the hypothesis (1.6) that there exists $u(z)$ satisfying (1.7) but such that $u(z)$ does not tend to $+\infty$ as $z \to \infty$ along any ray. On the other hand if (1.4) holds outside a set of disks satisfying (1.5) then almost all rays through the origin meet these disks only on a bounded set, so that (1.4) holds on such rays from a certain point onwards.

2. Statement of results

In this paper we consider in more detail the nature of the set of points where (1.4) fails, when $u(z)$ satisfies (1.7) for a given function $\psi(r)$. When

$$\psi(r) = O(\log r)^2 \tag{2.1}$$

our results are fairly complete.

The corresponding result for (1.3) under the hypothesis (1.7) with $\psi(r) = \log r$ has been obtained by Arsove and Huber [2], and our method is closely related to theirs. However, the present paper also contains converse results.

To state our theorems we make the following

DEFINITION. Let $\psi(r)$ be a positive increasing function of r satisfying (2.1) and suppose that K is fixed, $K > 1$. Let c_ν be any nonnegative, nondecreasing and convex sequence, such that $c_\nu = 0$, $\nu \leq 0$,

$$c_\nu \to \infty \quad \text{as} \quad \nu \to +\infty \tag{2.2}$$

and

$$c_\nu = O\{\psi(K^\nu)\} \quad \text{as} \quad \nu \to +\infty. \tag{2.3}$$

Then

$$\delta_\nu = \frac{c_\nu - 2c_{\nu-1} + c_{\nu-2}}{c_\nu} \tag{2.4}$$

will be called a ψ-sequence. (If $c_\nu = 0$, we define $\delta_\nu = 0$.) In terms of ψ-sequences our results can be stated as follows.

THEOREM 1. *Suppose that $u(z)$ is a s.h. and non-constant function in the open plane which satisfies* (1.7). *Then given $K > 1$, there exists $r_0 = r_0(K)$ where $1 < r_0 < K$, with the following properties. For a fixed positive ε let $E_\nu(\varepsilon)$ be the set of all points z in the annulus*

$$A_\nu : r_0 K^\nu \leq |z| < r_0 K^{\nu+1}, \tag{2.5}$$

such that

$$u(z) < (1-\varepsilon) B(|z|). \tag{2.6}$$

Then the capacities $\operatorname{Cap}(E_\nu)$ *of E_ν satisfy*

$$\log \left\{ \frac{r_0 K^{\nu+2}}{\operatorname{Cap}(E_\nu)} \right\} \geq \frac{\varepsilon \log K + o(1)}{\delta_\nu}, \tag{2.7}$$

where δ_ν is a ψ-sequence.

THEOREM 2. *Suppose given numbers $K > 1$, $r_0 > 1$, $\varepsilon > 0$ and $\varepsilon' > \varepsilon$, and a ψ-sequence δ_ν. Let E_ν be any sets in the annuli A_ν whose outer capacities satisfy for all large ν*

$$\log \left\{ \frac{r_0 K^{\nu+2}}{\operatorname{Cap} E_\nu} \right\} > \frac{\varepsilon' \log K}{\delta_\nu}. \tag{2.8}$$

If $\delta_\nu = 0$, (2.8) *is to be interpreted so that E_ν is empty.*

Then there exists $u(z)$ s.h. and non-constant in the plane, and satisfying (1.7) *and also* (2.6) *in* $\bigcup (E_\nu)$.

Theorems 1 and 2 show that the sets where $u(z)$ satisfies (2.6) for some positive ε can be very precisely characterized in terms of ψ-sequences. Taking for instance $\varepsilon = 1$ we obtain a characterisation of the sets on which $u(z)$ remains bounded. Thus our problem is reduced to a problem in the theory of series, namely the characterisation of ψ-sequences. Clearly if $c_n = 0$ for $n < n_0$, $c_{n_0} > 0$ in the definition of a ψ-sequence, then

$$\delta_n = 0, \qquad n < n_0, \qquad \delta_{n_0} = 1, \qquad 0 \leq \delta_n < 1, \qquad n > n_0. \tag{2.9}$$

Conversely if δ_n is any sequence satisfying (2.9) then we may define b_n inductively by

$$b_n = 0, \qquad n < n_0, \tag{2.10}$$

$$b_{n_0} > 0, \tag{2.11}$$

$$b_n = b_{n-1} + \delta_n \sum_{\nu=n_0}^{n} b_\nu, \qquad n > n_0 \tag{2.12}$$

and then define $c_n = 0$, $n < n_0$

$$c_n = \sum_{\nu=n_0}^{n} b_\nu, \qquad n \geq n_0. \tag{2.13}$$

Then evidently b_n is positive and non-decreasing for $n \geq n_0$, since $\delta_n < 1$, so that c_n is non-decreasing and convex. Further (2.2) evidently holds and so does (2.4). Thus δ_n is a ψ-sequence if and only if the sequence c_n defined in the above way satisfies (2.3).

If $\psi(r) = \log r$ we can give a simple criterion for this to happen. The corresponding functions $u(z)$ are said to be of polynomial growth and are the smallest non-constant subharmonic functions. The following theorem is a restatement of the main result of Arsove and Huber [2]. It expresses the fact that $\bigcup E_\nu$ is thin at infinity.

THEOREM 3. *The sequence δ_n satisfying* (2.9) *is a ψ-sequence with $\psi(r) = \log r$ if and only if*

$$\sum_{1}^{\infty} n\delta_n < \infty. \tag{2.14}$$

COROLLARY. *Let u be as in Theorem* 1 *with $\psi(r) = \log r$. Then the sets E_ν satisfy*

$$\sum_{1}^{\infty} \nu \left\{ \log \frac{1}{\operatorname{Cap} E_\nu} \right\}^{-1} < \infty$$

This is the famous criterion of Wiener [8].

For general ψ-sequences, i.e. those with $\psi(r)=(\log r)^2$ we have only a less precise result.

THEOREM 4. *If δ_ν is a ψ-sequence and $\lambda>\frac{1}{2}$ then*

$$\sum_1^\infty \delta_\nu^\lambda<\infty. \tag{2.15}$$

COROLLARY. *If E_ν are the sets in Theorem 1 then*

$$\sum_1^\infty\left\{\log\frac{r_0K^{\nu+2}}{\operatorname{Cap} E_\nu}\right\}^{-\lambda}<\infty, \qquad \text{if } \lambda>\tfrac{1}{2}.$$

Setting $\delta_n=1/\{n(1+\log n)\}^2$, $n\geq 1$, and using Theorem 3, we see that Theorem 4 and its corollary fail for $\lambda=\frac{1}{2}$ even for functions of polynomial growth.

2.1. *Some consequences and examples.* To illustrate the above results we give some examples and simple consequences of Theorems 1 to 4, before embarking on the proofs of these latter results.

EXAMPLE 1. Take $\psi(r)=(\log r)^\alpha$, $1<\alpha\leq 2$ and for any integer t, set $c_n=(n+t)^\alpha$, $n>\max(0,-t)$, $c_n=0$ otherwise. Then

$$\delta_n=\frac{c_n-2c_{n-1}+c_{n-2}}{c_n}=\alpha(\alpha-1)\left\{\frac{1}{n^2}-\frac{2t+(\alpha-2)}{n^3}+O(n^{-4})\right\}.$$

It is clear from (2.10) to (2.13) that if δ_n is a ψ-sequence, then so is any sequence δ_n', such that $\delta_n'\leq\delta_n$, for large n. We deduce that if

$$\delta_n=\alpha(\alpha-1)\frac{1}{n^2}+\frac{O(1)}{n^3},$$

then δ_n is a ψ-sequence for $\psi(r)=(\log r)^\alpha$, but not for any function $\psi(r)$, such that $\psi(r)=o(\log r)^\alpha$.

In the above example (2.14) fails, so that the corresponding exceptional sets are no longer thin at ∞. In fact it follows from Theorem 3 that whenever $\psi(r)$ tends to infinity more rapidly than $\log r$, then ψ-sequences will not in general satisfy (2.14) and so the corresponding exceptional sets will not be thin at ∞. However if $\psi(r)=(\log r)^\alpha$, so that $c_n=O(n^\alpha)$, we deduce from the convexity of c_n

and (2.13) that

$$nb_n \leq c_{2n} = O(n^{\alpha}), \tag{2.16}$$

and, if $c_n/n \to \infty$, that

$$\delta_n = \frac{b_n - b_{n-1}}{c_n} = o(n^{\alpha-2}). \tag{2.17}$$

If c_n/n does not tend to ∞, then $c_n = O(n)$ and (2.17) follows from (2.14).

The order condition (2.17) cannot be further sharpened as the next example shows.

EXAMPLE 2. Suppose that $1 \leq \alpha \leq 2$ and let ε_n be any sequence of positive numbers, such that $\varepsilon_n \to 0$ as $n \to \infty$. Then there exists a ψ-sequence δ_n corresponding to $\psi(r) = (\log r)^{\alpha}$, such that

$$\delta_n \geq \varepsilon_n n^{\alpha-2}$$

for infinitely many n.

If $\alpha = 1$, we choose an increasing sequence of positive integers n_k, $k = 0, 1, 2, \ldots$, $\varepsilon_{n_k} < 2^{-k}$, $k = 1, 2$. We then set $\delta_{n_0} = 1$, $\delta_n = \varepsilon_n/n$, for $n = n_k$, where $k \geq 1$, and $\delta_n = 0$ otherwise. Then

$$\sum_1^{\infty} n\delta_n = \sum_{k=0}^{\infty} n_k\delta_k = n_0 + \sum_{k=1}^{\infty} \varepsilon_{n_k} < n_0 + 1.$$

and so δ_n is a ψ-sequence in view of Theorem 3 and (2.9).

If $\alpha > 1$, we define an increasing sequence n_k, $k = 0, 1, 2$, such that $n_{k+1}/n_k \to \infty$ and set $b_n = 0$, $n < n_0$,

$$b_n = n_k^{\alpha-1}, \qquad n_k \leq n < n_{k+1}.$$

We then define c_n, δ_n by (2.12) and (2.13), and note that

$$c_n \leq nb_n \leq n^{\alpha}, \qquad n \geq 1,$$

so that δ_n is a ψ-sequence with $\psi(r)=(\log r)^\alpha$. Also for $n=n_k$

$$\begin{aligned}\delta_n &= (b_n-b_{n-1})/c_n=(b_n-b_{n-1})/(c_{n-1}+b_n)\\ &\geq ((n_k)^{\alpha-1}-(n_{k-1})^{\alpha-1})/(n_k(n_{k-1})^{\alpha-1}+(n_k)^{\alpha-1})\\ &= (1+o(1))(n_k)^{\alpha-2}/(n_{k-1})^{\alpha-1}.\end{aligned}$$

We now choose the sequence n_k inductively, by setting $n_0=1$, and if n_{k-1} has already been defined we choose n_k so large that

$$\varepsilon_{n_k}<\frac{1}{2(n_{k-1})^{\alpha-1}}\quad\text{and}\quad n_k>kn_{k-1}.$$

Thus we have for all large k

$$\delta_{n_k}>\varepsilon_{n_k}n_k^{\alpha-2}$$

as required.

Similarly we have

EXAMPLE 3. If $\psi(r)/(\log r)\to\infty$, there exists a ψ-sequence δ_n such that

$$\overline{\lim_{n\to\infty}}\, n\delta_n=\infty.$$

We set

$$B_n=\inf_{\nu\geq n}\frac{\psi(K^\nu)}{\nu},$$

so that B_n increases with n and $B_n\to\infty$ as $n\to\infty$. We choose a sequence n_k which tends to ∞ with n and is such that

$$\frac{B_{n_{k+1}}}{B_{n_k}}\to\infty$$

as $k\to\infty$.

Then we define $b_n=0$, $n<n_0$ and

$$b_n=B_{n_k},\qquad n_k\leq n<n_{k+1}.$$

We deduce that $c_n \le nb_n \le nB_n$ for $n \ge 1$, so that

$$c_n \le \psi(K^n),$$

and c_n is a ψ-sequence. On the other hand for $n = n_k$, we have

$$\delta_n = \frac{b_n - b_{n-1}}{c_n} = \frac{b_n - b_{n-1}}{c_{n-1} + b_n} \ge \frac{(1+o(1))B_{n_k}}{n_k B_{n_{k-1}} + B_{n_k}},$$

so that

$$\frac{1}{n_k \delta_{n_k}} \le (1+o(1))\left\{\frac{B_{n_{k-1}}}{B_{n_k}} + \frac{1}{n_k}\right\} \to 0 \quad \text{as} \quad k \to \infty.$$

If we combine the result of example 3 with Theorem 2, we deduce that if $\psi(r)/\log r \to \infty$, we can find $u(z)$ s.h. in the plane and satisfying (1.7) and (2.6) with any fixed ε on a sequence of sets E_ν, in $K^\nu < |z| < K^{\nu+1}$, such that

$$\text{Cap } E_\nu = \exp\{\nu(\log K + o(1))\}$$

for infinitely many ν. In particular Cap E_ν can be exponentially large for a sequence of ν. On the other hand if $\psi(r) = O(\log r)$ it follows from Theorem 3 that $n\delta_n \to 0$, so that in view of (2.7) we have

$$\frac{1}{\nu} \log\left\{\frac{r_0 K^{\nu+2}}{\text{Cap } E_\nu}\right\} = \frac{1}{\nu} \log \frac{1}{\text{Cap } E_\nu} + O(1) \to \infty.$$

Thus in this case

$$\log\left\{\frac{r_0 K^{\nu+2}}{\text{Cap } E_\nu}\right\} \sim \log\left\{\frac{1}{\text{Cap } E_\nu}\right\},$$

and so

$$\sum_1^\infty \nu\left(\log \frac{1}{\text{Cap } E_\nu}\right)^{-1} < \infty,$$

in view of Theorem 3. Thus Cap E_ν is exponentially small for all large ν.

In view of Theorem 4 the sum (2.15) converges for $\lambda > \frac{1}{2}$ but not in general for $\lambda = \frac{1}{2}$, even when $\psi(r) = \log r$. However we can obtain a fairly precise estimate for the partial sums of the series in this case.

THEOREM 5. *Suppose that δ_n is a ψ-sequence with $\psi(r)=(\log r)^\alpha$, where $1<\alpha$. Then*

$$\sum_{\nu=1}^{n} \delta_\nu^{1/2} \leq \{\alpha(\alpha-1)\}^{1/2} \log n + O(1), \quad \text{as} \quad n\to\infty.$$

We note that the result remains valid for $\alpha>2$, when (2.1) is not satisfied. We write $a_\nu = b_\nu - b_{\nu-1}$, and deduce that for $n \geq n_0$

$$\sum_{n_0}^{n} \delta_\nu^{1/2} = \sum_{n_0}^{n} (a_\nu/c_\nu)^{1/2} \leq \left(\sum_{n_0}^{n} a_\nu/b_\nu\right)^{1/2}\left(\sum_{n_0}^{n} b_\nu/c_\nu\right)^{1/2}. \tag{2.18}$$

We note that

$$\log\frac{b_n}{b_{n-1}} = \log\frac{b_n}{b_n-a_n} = -\log\left(1-\frac{a_n}{b_n}\right) \geq \frac{a_n}{b_n}.$$

Thus

$$\sum_{n_0+1}^{n} \frac{a_\nu}{b_\nu} \leq \log\frac{b_n}{b_{n_0}}, \qquad \sum_{n_0+1}^{n} \frac{b_\nu}{c_\nu} \leq \log\frac{c_n}{c_{n_0}}. \tag{2.19}$$

Using (2.16), (2.18) and (2.19) we obtain

$$\sum_{n_0}^{n} \delta_\nu^{1/2} \leq (\alpha \log n + K_1)^{1/2}((\alpha-1)\log n + K_2)^{1/2} \leq \{\alpha(\alpha-1)\}^{1/2} \log n + K_3,$$

where K_1, K_2, K_3 are positive constants and this is Theorem 5. Example 1 shows that for any α equality is possible in Theorem 5. In particular the constant $\alpha(\alpha-1)$ cannot be replaced by any smaller quantity.

We have seen that the exceptional set E_ν need not always be small. However it is small compared with the annulus A_ν. The following result is an immediate consequence of the case $\lambda=1$ of Theorem 4, Corollary.

THEOREM 6. *For all θ in $[0,2\pi]$ apart from a set of capacity zero the ray $\arg z=\theta$ meets $\bigcup E_\nu$ in a bounded set.*

In fact let F_ν be the radial projection of E_ν onto the circle $|z|=\frac{1}{2}$. Then

$$\operatorname{Cap} F_\nu \leq \frac{A \operatorname{Cap} E_\nu}{r_0 K^{\nu+2}},$$

where A is a constant. Thus Theorem 4 shows that

$$\sum_{\nu=1}^{\infty}\left\{\log\frac{2A}{\operatorname{Cap} F_\nu}\right\}^{-1}<\infty.$$

If $G_n=\bigcup_{\nu=n}^{\infty} F_\nu$, we deduce from the subadditivity of capacity that

$$\left\{\log\frac{1}{\operatorname{Cap} G_n}\right\}^{-1}\le\sum_{\nu=n}^{\infty}\left\{\log\frac{1}{\operatorname{Cap} F_\nu}\right\}^{-1}\to 0, \quad \text{as} \quad n\to\infty.$$

Thus $\operatorname{Cap} G_n\to 0$, and if $G=\bigcap_{n=1}^{\infty} G_n$, then $\operatorname{Cap} G=0$. Clearly $\arg z=\theta$, meets $\bigcup E_\nu$ in a bounded set, unless $\frac{1}{2}e^{i\theta}\in G$.

Theorem 6 sharpens a previous result of Hayman [4], where the exceptional set had measure zero. An example of Ahlfors and Heins [1, p. 344] shows that the present result is best possible. We also obtain an improvement of (1.5).

THEOREM 7. *For every $p>1$ we can include $\bigcup(E_n)$ in a sequence of disks $|z-z_k|<\rho_k$, such that*

$$\sum_{1}^{\infty}\{\log(|z_k|/\rho_k)\}^{-p}<\infty.$$

Our proof of Theorem 7 is similar to that of Theorem 5 in Essén and Jackson [4]. We set $r_1=(4K^2)^{-1}$,

$$h(r)=\min\{(\log^+(r_1/r))^{-p},1\}, \quad r>0.$$

Then $h(r)$ increases with r and

$$A=\int_0^{\infty}\log\frac{1}{r}\,dh(r)=\log(4K^2e)+\frac{1}{p-1}.$$

Let G_ν be the set $E_\nu/(2r_0K^{\nu+2})$, i.e. the set of all points $\xi=z/(2r_0K^{\nu+2})$, where z lies in E_ν. Then, in view of (2.5), G_ν lies in the annulus

$$2r_1<|\xi|<\tfrac{1}{2}. \tag{2.20}$$

Also

$$\log\left(\frac{1}{\operatorname{Cap} G_\nu}\right)=\log\left(\frac{2r_0K^{\nu+2}}{\operatorname{Cap} E_\nu}\right)>\log\left(\frac{r_0K^{\nu+2}}{\operatorname{Cap} E_\nu}\right).$$

It now follows from a result of Essén and Jackson [4, Lemma 1, p. 339] that there exists a constant A_1 depending only on A, i.e. on p and K, such that G can be included in the union of a set of disks

$$|z-z_{\nu,k}|<r_{\nu,k}, \qquad k=1,2,\ldots, \tag{2.21}$$

with

$$\sum_k h(r_{\nu,k}) \le A_1\{\log(1/\text{Cap } G_\nu)\}^{-1}+2^{-\nu} \le A_1\left\{\log\left(\frac{r_0K^{\nu+2}}{\text{Cap } E_\nu}\right)\right\}^{-1}+2^{-\nu}. \tag{2.22}$$

From the Corollary of Theorem 4 with $\lambda=1$, we see that

$$\sum_\nu\left\{\log\left(\frac{r_0K^{\nu+2}}{\text{Cap } E_\nu}\right)\right\}^{-1}<\infty. \tag{2.23}$$

In particular

$$\frac{\text{Cap } E_\nu}{r_0K^{\nu+2}}\to 0, \quad \text{as} \quad \nu\to\infty,$$

so that for $\nu\ge\nu_0$ and all k we have $h(r_{\nu,k})<1$, i.e. $r_{\nu,k}<r_1/e$. We may then assume that $|z_{\nu,k}|\ge r_1$ in (2.21), since otherwise the corresponding disk does not meet the annulus (2.20) in which G_ν lies and can be omitted from our covering. Thus (2.22) yields for $\nu\ge\nu_0$.

$$\sum_k\left(\log\frac{|z_{\nu,k}|}{r_{\nu,k}}\right)^{-p}\le\sum_k\left(\log\frac{r_1}{r_{\nu,k}}\right)^{-p}=\sum_k h(r_{\nu,k})\le A_1\left\{\log\frac{r_0K^{\nu+2}}{\text{Cap } E_\nu}\right\}^{-1}+2^{-\nu}.$$

Since the disks with centres $2r_0K^{\nu+2}z_{\nu,k}$ and radii $2r_0K^{\nu+2}r_{\nu,k}$ cover E_ν, we deduce Theorem 7 from (2.23).

In a later paper we hope to consider the case (1.6). Here the situation is more complicated and our results are less complete.

3. Results on ψ-sequences; Proof of Theorem 3

We proceed to investigate further the nature of the sequences δ_n. We write

$$b_n=c_n-c_{n-1} \tag{3.1}$$

$$a_n=b_n-b_{n-1}=c_n-2c_{n-1}+c_{n-2}. \tag{3.2}$$

Since c_n is convex and increasing, $b_n \geq 0$, $a_n \geq 0$. Since $c_n = 0$ for $n \leq 0$, we deduce that $a_n = b_n = 0$, $n \leq 0$. Also it follows from (2.2) that b_n cannot be zero for all n, so that b_n is finally positive. Thus

$$b_n \to b \quad \text{as} \quad n \to \infty, \quad \text{where} \quad 0 < b \leq \infty. \tag{3.3}$$

We have

$$c_n = \sum_{\nu=1}^{n} b_\nu \tag{3.4}$$

so that c_n/n increases with n and

$$\frac{c_n}{n} \to b \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Suppose first that b is finite. Then

$$\sum_1^\infty a_n = b. \tag{3.6}$$

Also if $n = n_0$ is the first integer for which c_n and so a_n and b_n are positive, we have, since c_n/n increases,

$$\sum_1^\infty n\delta_n = \sum_{n_0}^\infty \frac{na_n}{c_n} \leq \frac{n_0}{c_{n_0}} \sum_{n_0}^\infty a_n = \frac{n_0 b}{c_{n_0}}.$$

This proves (2.14) in this case. We note that if δ_n is a ψ-sequence with $\psi = \log r$, then $c_n = O(n)$ so that b must be finite and the above conclusions hold.

To complete the proof of Theorem 3, suppose that δ_n satisfies (2.9) and (2.14) and define b_n, c_n by (2.10) to (2.13). Then (2.12) gives

$$(1-\delta_n)b_n = (1+\delta_n)b_{n-1} + \delta_n \sum_1^{n-2} b_\nu \leq \{1+(n-1)\delta_n\}b_{n-1},$$

since b_n is increasing. Thus

$$\frac{b_n}{b_{n_0}} \leq \prod_{n=n_0+1}^\infty \left\{\frac{1+(n-1)\delta_n}{1-\delta_n}\right\} = C < \infty, \quad \text{in view of (2.14).}$$

Thus $b_n = O(1)$, $c_n = O(n)$ in this case, so that δ_n is a ψ-sequence with $\psi(r) = \log r$. This proves Theorem 3.

3.1. We have shown that if b is finite in (3.5) then (2.14) holds. This also implies (2.15). For if $\frac{1}{2} < \lambda < 1$ we define

$$p = 1/\lambda, \qquad q = p/(p-1) = 1/(1-\lambda), \quad \text{so that} \quad 1 < q < 2 < p < \infty.$$

Then it follows from Hölder's inequality that

$$\sum_1^\infty \delta_n^\lambda = \sum_1^\infty (n\delta_n)^{1/p} n^{-1/p} \le \left(\sum_1^\infty n\delta_n\right)^{1/p} \left(\sum_1^\infty n^{-q/p}\right)^{1/q} < \infty.$$

Thus (2.15) holds in this case.

We proceed to prove (2.15) in the more difficult case when b is infinite. We note that in view of (3.4) and since b_n increases

$$nb_n \le \sum_{n+1}^{2n} b_\nu \le c_{2n} = O\{\psi(K^{2n})\} = O(n^2),$$

in view of (2.1) and (2.3). Thus

$$b_n = O(n), \tag{3.7}$$

and so

$$\frac{b_n}{c_n} \to 0, \quad \text{as} \quad n \to \infty \tag{3.8}$$

if b is infinite. If b is finite (3.8) follows immediately from (3.5) so that (3.8) holds in any case. The inequality (2.15) is contained in the following somewhat more precise

THEOREM 8. *Suppose that* $a_1 > 0$, $0 \le a_n \le n$, $n \ge 1$ *and set*

$$b_n = \sum_1^n a_\nu, \qquad c_n = \sum_{\nu=1}^n b_\nu, \qquad n \ge 1.$$

Then for $\frac{1}{2} < \lambda \le 1$ *we have*

$$s_\lambda = \sum_{\nu=1}^\infty \left(\frac{a_\nu}{c_\nu}\right)^\lambda < A_1 + \frac{A_2}{a_1(\frac{1}{2}\lambda - \frac{1}{4})},$$

where the constants A_1, A_2 *depend only on* λ.

Before proving Theorem 8 we deduce Theorem 4 from Theorem 8. Let δ_n be the ψ-sequence of Theorem 4 and define a_n, b_n by (3.1) and (3.2). Then a_n cannot be zero for all n. We suppose that $m+1$ is the first positive integer such that $a_{m+1}>0$ and define

$$a_n' = a_{m+n}, \qquad n \geq 1.$$

Then it follows from (3.7) that $a_n = O(n)$ and so $a_n' = O(n)$. Thus if δ is a suitable constant and we set $a_n'' = \delta a_n'$, we have

$$a_n'' \leq n, \qquad n = 1, 2, \dots, a_1'' > 0.$$

Also if

$$b_n'' = \sum_1^n a_\nu'', \qquad c_n'' = \sum_1^n b_\nu'',$$

we clearly have $b_n'' = \delta b_{n+m}$, $c_n'' = \delta c_{n+m}$. Thus

$$\sum_1^\infty \delta_n^\lambda = \sum_1^\infty \left(\frac{a_n}{c_n}\right)^\lambda = \sum_{m+1}^\infty \left(\frac{a_n}{c_n}\right)^\lambda = \sum_1^\infty \left(\frac{a_n''}{c_n''}\right)^\lambda$$
$$\leq A_1 + \frac{A_2}{(a_1'')^{\frac{1}{2}}\lambda - \frac{1}{4}} = A_1 + \frac{A_2}{(\delta a_{m+1})^{\frac{1}{2}}\lambda - \frac{1}{4}}.$$

Thus (2.15) follows from Theorem 8 and we obtain a bound for the series, depending on the first integer m, for which $a_{m+1} = c_{m+1} - 2c_m + c_{m-1} > 0$, and on the constants implicit in (2.1) and (2.3).

3.2. To prove Theorem 8 we need a subsidiary result. We denote by A_3, $A_4 \cdots$ constants depending on λ only.

LEMMA 1. *If $\lambda = \frac{1}{2} + 2\varepsilon$, where $0 < \varepsilon \leq \frac{1}{4}$ then we have, with the hypotheses of Theorem* 8,

$$\sum_N^{2N} \left(\frac{a_n}{c_n}\right)^\lambda < A_3 \left(\frac{N}{c_N}\right)^\varepsilon, \qquad N \geq 1.$$

We write $t = \lambda - \varepsilon = \frac{1}{2} + \varepsilon$, $p = 1/t$, $q = p/(p-1)$. Then

$$c_n = \sum_{r=1}^n b_r = \sum_{r=1}^n (n - r + 1) a_r.$$

Thus we have, for $1 \leq m \leq n$,

$$\sum_{r=m}^{n} a_r^t \leq \sum_{r=1}^{n} a_r^t = \sum_{r=1}^{n} \{(n-r+1)a_r\}^{1/p}(n-r+1)^{-1/p}$$

$$\leq \left\{\sum_{1}^{n}(n-r+1)a_r\right\}^{1/p}\left\{\sum_{1}^{n}(n-r+1)^{-q/p}\right\}^{1/q}$$

$$\leq c_n^t\left\{\sum_{k=1}^{\infty} k^{-q/p}\right\}^{1/q} = A_4 c_n^t.$$

Suppose now that $c_n \leq 2c_m$. We deduce that

$$\sum_{r=m}^{n}\left(\frac{a_r}{c_r}\right)^t \leq \sum_{r=m}^{n}\left(\frac{a_r}{c_m}\right)^t \leq 2^t \sum_{r=m}^{n}\left(\frac{a_r}{c_n}\right)^t \leq 2A_4.$$

Further

$$\left(\frac{a_r}{c_r}\right)^{\lambda} = \left(\frac{a_r}{c_r}\right)^{t+\varepsilon} = \left(\frac{a_r}{c_r}\right)^{\varepsilon}\left(\frac{a_r}{c_r}\right)^{t} \leq \left(\frac{n}{c_m}\right)^{\varepsilon}\left(\frac{a_r}{c_r}\right)^{t},$$

since $a_n \leq n$. Thus

$$\sum_{r=m}^{n}\left(\frac{a_r}{c_r}\right)^{\lambda} \leq 2A_4\left(\frac{n}{c_m}\right)^{\varepsilon}. \tag{3.9}$$

We now set $m_0 = N$ and if m_k has already been defined, we define m_{k+1} to be the smallest integer m such that $c_m \geq 2c_{m_k}$. If $m_{k+1} \leq 2N$, we define $n_k = m_{k+1} - 1$. If s is the smallest integer k, for which $m_{k+1} > 2N$, we define $n_s = 2N$. Thus

$$c_{m_k} \geq 2^k c_N, \qquad k = 0, 1, 2, \ldots, s$$

Also in view of (3.9) we have

$$\sum_{N}^{2N}\left(\frac{a_n}{c_n}\right)^{\lambda} = \sum_{k=0}^{s}\sum_{r=m_k}^{n}\left(\frac{a_r}{c_r}\right)^{\lambda} \leq \sum_{k=0}^{s} 2A_4\left(\frac{n_k}{c_{m_k}}\right)^{\varepsilon}$$

$$\leq 4A_4 N^{\varepsilon}\sum_{k=0}^{s}(2^k c_N)^{-\varepsilon} \leq 4A_4\left(\frac{N}{c_N}\right)^{\varepsilon}\sum_{k=0}^{\infty} 2^{-k\varepsilon}.$$

This proves Lemma 1.

3.3. We can now complete the proof of Theorem 8. We set $m_k = 2^k$, $k = 0, 1, 2, \ldots$ and consider the sums

$$\sigma_k = \sum_{m_k}^{m_{k+1}-1} \left(\frac{a_n}{c_n}\right)^{\lambda}.$$

We write

$$B_k = b_{m_k}, \qquad C_k = c_{m_k},$$

and consider in turn the cases

$$\text{(i)} \quad C_k > \tfrac{1}{4}B_{k+1}m_k, \qquad \text{(ii)} \quad C_k \le \tfrac{1}{4}B_{k+1}m_k.$$

We denote sums over k in the ranges (i) and (ii) by Σ' and Σ'' respectively.

In case (i) we have, using Hölder's inequality

$$\begin{aligned}\sigma_k &\le \sum_{m_k}^{m_{k+1}-1} \left(\frac{a_n}{c_n}\right)^t \le C_k^{-t} \sum_{m_k}^{m_{k+1}-1} a_n^t = C_k^{-t} \sum a_n^{1/p} \\ &\le C_k^{-t}\left(\sum a_n\right)^{1/p} (m_k)^{1/q} \le C_k^{-t} B_{k+1}^t m_k^{1-t} \le 4 m_k^{1-2t}.\end{aligned}$$

Thus

$$\sum{}' \sigma_k \le 4 \sum_{k=0}^{\infty} m_k^{1-2t} = 4 \sum_{k=0}^{\infty} 4^{-k\varepsilon} = A_1. \tag{3.10}$$

In case (ii) we note that

$$C_{k+2} \ge m_{k+1}B_{k+1} \ge 4\frac{m_{k+1}}{m_k}C_k = 8C_k.$$

Thus

$$\frac{C_{k+2}}{m_{k+2}} \ge 2\frac{C_k}{m_k}.$$

As we remarked after (3.4), c_n/n increases with n. Thus if $k = k_\nu$ are the positive integers for which case (ii) holds, we deduce that $k_{\nu+2} \ge k_\nu + 2$, so that

$$\frac{C_{k_{\nu+2}}}{m_{k_{\nu+2}}} \ge 2\frac{C_{k_\nu}}{m_{k_\nu}}.$$

Thus we prove by induction, separately over even and odd ν, that

$$\frac{C_{k_\nu}}{m_{k_\nu}} \geq 2^{(\nu/2)-1} \frac{C_{k_1}}{m_{k_1}} \geq a_1 2^{(\nu/2)-1}, \qquad \nu \geq 1.$$

Now we deduce from Lemma 1 that for $k = k_\nu$ in the case (ii) we have

$$\sigma_{k_\nu} < A_3(a_1 2^{(\nu/2)-1})^{-\varepsilon} = A_3\left(\frac{2}{a_1}\right)^{\varepsilon} 2^{-\nu\varepsilon/2}.$$

On summing over ν we deduce that

$$\sum{}'' \sigma_k \leq A_2 a_1^{-\varepsilon}. \tag{3.11}$$

On adding (3.10) and (3.11) we deduce Theorem 8. Since we deduced Theorem 4 from Theorem 8, the proof of Theorem 4 is also complete. The corollary of Theorem 4 is an immediate consequence of (2.15) and (2.7). Thus it remains to prove Theorems 1 and 2.

4. Proof of Theorem 1

Suppose that $u(z)$ is s.h. in the plane. We replace $u(z)$ in $|z| < 1$ by the Poisson integral of the values of $u(z)$ on $|z| = 1$ and leave $u(z)$ unaltered for $|z| \geq 1$. The resulting function has the same asymptotic behaviour as $u(z)$ and is harmonic and in particular finite at $z = 0$. By subtracting a constant if necessary we may arrange that $u(0) = 0$. We further assume that $u(z)$ has order zero. Thus $u(z)$ has a representation of the form (see e.g. [5])

$$u(z) = \int \log \left|1 - \frac{z}{\xi}\right| d\mu(\xi), \tag{4.1}$$

where μ is the Riesz mass of u and the integral is taken over the open plane. Let $n(t)$ be the mass in $|z| < t$. Since $u(z)$ is harmonic in $|z| < 1$, $n(1) = 0$. We define

$$N(r) = \int_0^r \frac{n(t)\, dt}{t}. \tag{4.2}$$

The Jensen formula gives

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta \leq B(r, u). \tag{4.3}$$

Also it follows from (4.1) that

$$\int \log\left|\frac{z-\xi}{\xi}\right| d\mu(\xi) = u(z) \le \int \log\left(\frac{|z|+|\xi|}{|\xi|}\right) d\mu(\xi). \tag{4.4}$$

We start by proving a result which is a simple consequence of (1.5).

LEMMA 2. *With the hypotheses of Theorem* 1, *we can choose*, $r_0 = r_0(K)$, *such that on the circles* $|z| = r_\nu = r_0 K^\nu$, *we have*

$$u(z) > (1-\varepsilon_\nu)B(r_\nu), \tag{4.5}$$

where $\varepsilon_\nu \to 0$ *as* $\nu \to \infty$.

It follows from (1.5) that if $F(\varepsilon)$ is the set of all $r \ge 1$, such that

$$u(z) \le (1-\varepsilon)B(r)$$

at some point on $|z| = r$, then

$$\int_{F(\varepsilon)} \frac{dr}{r} < \infty.$$

Choose $\varepsilon_\nu = 1/\nu$ and let ρ_ν be so large that if F_ν is the part of $F(\varepsilon_\nu)$ in $r \ge \rho_\nu$ then we have

$$\int_{F_\nu} \frac{dr}{r} < \frac{\log K}{2^\nu}.$$

Let $F = \bigcup_{\nu=1}^{\infty} F_\nu$. Then

$$\int_F \frac{dr}{r} < \log K.$$

Let F_0 be the set of all r_0, such that $1 < r_0 < K$ and $r_0 K^\nu$ belongs to F for some ν. Clearly

$$\int_{F_0} \frac{dr}{r} \le \int_F \frac{dr}{r} < \log K.$$

Thus there exists r_0 in the interval $(1, K)$ and not belonging to F_0 and so for this r_0, r_0K^ν does not belong to F_ν for any ν, i.e. (4.5) holds with $r_\mu = r_0K^\mu$, provided that $r_\mu \geq \rho_\nu$.

This proves Lemma 2.

4.1. We now return to the representation (4.1) and suppose that z lies in the annulus A_ν given by (2.5). We write

$$b_\nu = n(r_\nu), \qquad c_\nu = \sum_{\mu=1}^{\nu} b_\mu, \qquad a_\nu = b_\nu - b_{\nu-1}. \tag{4.6}$$

We note that

$$\delta_\nu = \frac{a_{\nu+1}}{c_{\nu+1}} = \frac{c_{\nu+1} - 2c_\nu + c_{\nu-1}}{c_{\nu+1}} \tag{4.7}$$

is a ψ-sequence in this case. In fact

$$c_\nu \leq \sum_{\mu=1}^{\nu} \frac{1}{\log K} \int_{r_\mu}^{r_{\mu+1}} \frac{n(t)\,dt}{t} \leq \frac{1}{\log K} N(r_{\nu+1}) = O\{\psi(K^{\nu+2})\} \tag{4.8}$$

in view of (4.3) and (1.7). Thus, in view of (2.1), $c_\nu = O(\nu^2)$ and so c_ν satisfies (2.3) with $(\log r)^2$ instead of $\psi(r)$. This in turn implies (3.7) and (3.8), so that

$$c_{\nu+1} \sim c_\nu \sim c_{\nu-2} = O\{\psi(K^\nu)\}. \tag{4.9}$$

in view of (4.8). Thus (2.2) to (2.4) are satisfied. Next we show that

$$B(r) = B(r, u) = (\log K + o(1))c_{\nu+1}, \qquad r_\nu \leq r \leq r_{\nu+1}. \tag{4.10}$$

uniformly as $r \to \infty$. In fact it follows from (4.8), (4.9) and (4.3) that

$$c_{\nu+1} \log K \sim c_{\nu-1} \log K \leq N(r_\nu) \leq N(r) \leq B(r)$$

in this case. In the opposite direction we note that

$$c_\nu = \sum_{\mu=1}^{\nu} b_\mu = \sum_{\mu=1}^{\nu} n(r_\mu) \geq \frac{1}{\log K} \sum_{\mu=1}^{\nu} \int_{r_{\mu-1}}^{r_\mu} \frac{n(t)\,dt}{t} = \frac{1}{\log K} N(r_\nu) + O(1).$$

On the other hand we deduce from (4.3) and Lemma 2 that

$$N(r_\nu)=\frac{1}{2\pi}\int_0^{2\pi} u(r_\nu e^{i\theta})\,d\theta \geq (1+o(1))B(r_\nu).$$

Thus for $r_\nu \leq r \leq r_{\nu+1}$ we have

$$B(r)\leq B(r_{\nu+1})\leq(1+o(1))N(r_{\nu+1})\leq(\log K+o(1))c_{\nu+1}.$$

This proves (4.10). We deduce

LEMMA 3. *We have, uniformly as $\nu\to\infty$, for z in A_ν*

$$u(z)=\int_{A_\nu}\log|z-\xi|\,d\mu(\xi)+c_{\nu+1}(\log K+o(1))-\nu a_{\nu+1}\log K. \tag{4.11}$$

We write

$$I_\nu(z)=\int_{A_\nu}\log\left|1-\frac{z}{\xi}\right|d\mu(\xi), \tag{4.12}$$

$$u_\nu(z)=u(z)-I_\nu(z). \tag{4.13}$$

Then for $r_\nu\leq|z|\leq r_{\nu+1}$ we have, using (3.8) and (4.6),

$$I_\nu(z)\leq\log\left(1+\frac{r_{\nu+1}}{r_\nu}\right)\int_{A_\nu}d\mu(\xi)\leq b_{\nu+1}\log(1+K)=o(c_{\nu+1}).$$

Using (4.10) and Lemma 2, we deduce that for $|z|=r_\nu$, $r_{\nu+1}$ we have

$$u_\nu(z)\geq(\log K+o(1))c_{\nu+1}.$$

Since $u_\nu(z)$ is harmonic for $r_\nu<|z|<r_{\nu+1}$ it follows that this inequality remains valid in the whole annulus A_ν. Next we note that on $|z|=r_{\nu+2}$ we have, in view of (4.9), (4.10)

$$u(z)<(\log K+o(1))c_{\nu+1}$$

and further

$$I_\nu(z)>\log\left(\frac{r_{\nu+1}}{r_\nu}-1\right)\int_{A_\nu}d\mu(\xi)=O(b_{\nu+1})=o(c_{\nu+1}).$$

Thus for $|z| = r_{\nu+2}$ we have

$$u_\nu(z) < (\log K + o(1))c_{\nu+1}, \tag{4.14}$$

and since $u_\nu(z)$ is subharmonic this inequality remains valid in $|z| < r_{\nu+1}$ and so in A_ν. Thus

$$u_\nu(z) = (\log K + o(1))c_{\nu+1}, \qquad r_\nu \le |z| \le r_{\nu+1}. \tag{4.15}$$

Also

$$\begin{aligned} I_\nu(z) &= \int_{A_\nu} \log|z-\xi|\, d\mu(\xi) - \int_{A_\nu} \log|\xi|\, d\mu(\xi) \\ &= \int_{A_\nu} \log|z-\xi|\, d\mu(\xi) - \int_{A_\nu} (\nu \log K + O(1))\, d\mu(\xi) \\ &= \int_{A_\nu} \log|z-\xi|\, d\mu(\xi) - \nu a_{\nu+1} \log K + O(a_{\nu+1}) \\ &= \int_{A_\nu} \log|z-\xi|\, d\mu(\xi) - \nu a_{\nu+1} \log K + o(c_{\nu+1}). \end{aligned} \tag{4.16}$$

On combining (4.15) and (4.16) we deduce (4.11).

It follows from Lemma 3 that if $u(z)$ is any function subharmonic in the plane which satisfies (1.2) then the size of $u(z)$ is given in the annulus A_ν with great precision by (4.11). In particular $u(z)$ is much smaller than $c_{\nu+1} \log K$ if and only if the integral on the right hand side of (4.11) is large and negative.

4.2. In order to complete our proof we need another subsidiary result, whose statement is almost the definition of capacity.

LEMMA 4. *Let $\mu(\xi)$ be a positive measure of total mass μ_0 distributed over a compact set F. If G is the plane set where*

$$V(z) = \int \log|z-\xi|\, d\mu(\xi) < C. \tag{4.17}$$

then the outer capacity Cap G *of G satisfies*

$$\text{Cap } G \le \exp(C/\mu_0). \tag{4.18}$$

Conversely given any relatively open subset G of the annulus A_ν of (2.5), there exists a unit measure μ, distributed on $\bar{G}$ and such that

$$V(z)=\int \log|z-\xi|\,d\mu(\xi)=\log \operatorname{Cap} G \tag{4.19}$$

at every point of G.

We recall some facts concerning capacity.[1] Let E be a compact set and μ a unit measure distributed on E. Consider the energy integral

$$I(\mu)=\iint_E \log|a-b|\,d\mu(a)\,d\mu(b).$$

If V_0 is the maximum value of $I(\mu)$ for all such measures, then

$$\operatorname{Cap} E = e^{V_0}.$$

In particular if $I(\mu)=-\infty$ for all such measures μ, then $\operatorname{Cap} E=0$.

Suppose now that $\operatorname{Cap} E>0$. This is always the case if E has interior points. Then there exists a unit measure μ such that the conductor potential

$$V(z)=\int \log|z-a|\,d\mu(a)$$

satisfies

$$V(z)\geq V_0$$

in the whole z plane with equality at all points of E with the exception of the irregular boundary points of the unbounded component of the complement of E.

The outer capacity of more general sets is defined as follows. If G is open then $\operatorname{Cap} G$ is defined to be the upper bound of capacities of compact sets contained in G. Finally, if E is any bounded set, $\operatorname{Cap} E$ is defined to be the lower bound of capacities of open sets containing E.

Suppose now that G is the set of Lemma 4. Let F_0 be a compact subset of G, which is the union of a finite number of closed disks, so that the complement of F_0 is regular for the problem of Dirichlet. Let $V_0(z)$ be the conductor potential of F_0.

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[1] See e.g. Tsuji [7 p.p. 54 et seq.].

Then, for z in F_0,

$$V_0(z) = \int \log|z-\xi|\, d\nu(\xi) = V_0 = \log \operatorname{Cap} F_0,$$

where ν is a unit measure distributed on F_0.

Consider now, with the notation of Lemma 4,

$$u(z) = \frac{V(z)}{\mu_0} - V_0(z)$$

in the unbounded component G_0 of the complement of F_0. Then $V_0(z)$ is harmonic outside F_0 except possibly at ∞ and so $u(z)$ is s.h. at the finite points of G_0. Also near ∞

$$u(z) = \log|z| - \log|z| + o(1),$$

so that $u(z)$ is harmonic at ∞ and $u(\infty) = 0$. Thus $u(z)$ is s.h. in G_0 including ∞. Also as z approaches any finite boundary point ξ of G_0 we have

$$\overline{\lim}\, V(z) < C \quad \text{and} \quad \lim V_0(z) = \log \operatorname{Cap} F_0.$$

This shows that

$$\overline{\lim}\, u(z) \leq \frac{C}{\mu_0} - \log \operatorname{Cap} F_0.$$

Since $u(\infty) = 0$, we deduce from the maximum principle that

$$\log \operatorname{Cap} F_0 \leq C/\mu_0.$$

This inequality holds for every compact subset of G, which consists of the union of a finite number of closed disks. Any compact subset F_1 of G is contained in such a set F_0 and so

$$\operatorname{Cap} F_1 \leq \exp(C/\mu_0).$$

Now (4.18) follows from the definition of outer capacity.

Conversely let G be a relatively open subset of A_ν, let F_n be a sequence of compact sets, each of which is regular for the problem of Dirichlet and such that

$$F_n \subset F_{n+1} \subset G,$$
$$\text{Cap } F_n \to \text{Cap } G.$$

If G does not meet $|z| = r_\nu$, so that G is open, we may take for F_n the union of a finite number of closed disks. If G meets $|z| = r_\nu$, then there exists an open plane set G_0, such that

$$G_0 \cap A_\nu = G.$$

We construct the F_n as above corresponding to the set G_0. Then

$$F'_n = F_n \cap A_\nu$$

is bounded by a finite number of arcs of circles so that F'_n is still regular for the problem of Dirichlet. Also

$$F'_n \subset F'_{n+1} \quad \text{and} \quad \bigcup F'_n = G,$$

so that Cap $F'_n \to$ Cap G as required.

Let $V_n(z)$ be the conductor potential of F_n and let

$$v_n = \log \text{Cap } F_n$$

be the value of $V_n(z)$ on F_n. Then $V_{n+1}(z) - V_n(z)$ is s.h. in the unbounded complementary domain G_n of F_n, equal to $v_{n+1} - v_n$ on the boundary of G_n, and zero at ∞. Thus

$$v_{n+1} - v_n \geq 0.$$

On the other hand $V_{n+1}(z) - V_n(z)$ is harmonic in G_{n+1} and does not exceed $v_{n+1} - v_n$ on the boundary of G_{n+1}. Thus

$$V_{n+1}(z) - V_n(z) \leq v_{n+1} - v_n$$

in G_{n+1}, and also in F_{n+1}, since $V_{n+1}(z) = v_{n+1}$, $V_n(z) \geq v_n$ in F_{n+1}. Thus

$$V_n(z) - v_n$$

is a decreasing sequence of subharmonic functions and so tends to a subharmonic limit $V(z) - V_0$, where

$$V_0 = \lim v_n = \log \operatorname{Cap} G.$$

If z_0 is any point of G then $z_0 \in F_n$ for large n, and so $V_n(z_0) = v_n$ for large n. Thus $V(z_0) = V_0$.

Next the function $V(z)$ is harmonic in the exterior of G, and so by the Riesz representation theorem there exists a measure μ of total mass μ_0, say, distributed on $\bar{G}$ and such that

$$V(z) = \int_{\bar{G}} \log |z - \xi| \, d\mu(\xi) + h(z),$$

where $h(z)$ is harmonic in the open plane. Also near ∞

$$V_n(z) = \log |z| + \frac{O(1)}{z},$$

uniformly in n, and so

$$V(z) = \log |z| + \frac{O(1)}{z} \quad \text{as} \quad z \to \infty.$$

Thus

$$h(z) = O(\log |z|) \quad \text{as} \quad z \to \infty,$$

i.e. $h(z) = c = \text{const}$. Thus $\mu_0 = 1$, $c = 0$,

$$V(z) = \int_{\bar{G}} \log |z - \xi| \, d\mu(\xi)$$

and $V(z) = V_0 = \log \operatorname{Cap} G$ on G. This proves (4.19) and completes the proof of Lemma 4.

4.3. *Proof of Theorem* 1. Suppose now that $E_\nu(\varepsilon)$ is the set defined in Theorem 1. We recall the definitions (4.6) and (4.7), so that δ_ν is a ψ-sequence. It follows from (2.6), (4.9) and (4.10) that we have for z in $\bigcup E_\nu$

$$u(z) < (1 - \varepsilon)(\log K + o(1)) c_{\nu+1}.$$

In view of Lemma 3 we deduce that for z in E_ν

$$\int_{A_\nu} \log|z-\xi|\, d\mu(\xi) < -(\varepsilon \log K + o(1))c_{\nu+1} + \nu a_{\nu+1} \log K.$$

Now Lemma 4 shows that

$$\frac{\operatorname{Cap}(E_\nu)}{r_0 K^{\nu+2}} \leq \exp\left\{(-\varepsilon \log K + o(1))\frac{c_{\nu+1}}{a_{\nu+1}} + \nu \log K - (\nu+2)\log K - \log r_0\right\}$$
$$\leq \exp\left\{\frac{(-\varepsilon \log K + o(1))}{\delta_\nu} + O(1)\right\} = \exp\left\{\frac{-\varepsilon \log K + o(1)}{\delta_\nu}\right\}.$$

This yields (2.7), and completes the proof of Theorem 1.

4.4. *Proof of Theorem* 2. We proceed to prove Theorem 2. We suppose given the ψ-sequence δ_ν defined from the quantities c_ν satisfying (2.2) and (2.3) in accordance with (2.4). We define

$$a_\nu = c_\nu - 2c_{\nu-1} + c_{\nu-2}.$$

Let $E_\nu(\varepsilon)$ be the sets defined in Theorem 2. In view of the definition of outer capacity we can include E_ν in relatively open subsets G_ν of A_ν whose capacities differ by arbitrarily little from those of the E_ν, Thus we may assume without loss of generality that the E_ν are relatively open subsets of A_ν and that (2.8) is still satisfied, provided that $\delta_\nu > 0$.

Suppose now that $\delta_\nu > 0$ so that $a_\nu > 0$. Then, in view of Lemma 4 we can find a mass distribution μ_ν in A_ν of total mass a_ν, such that

$$V_\nu(z) = \int \log|z-\xi|\, d\mu_\nu(\xi) = a_\nu \log \operatorname{Cap} E_\nu \tag{4.20}$$

on E_ν. We write $\mu = \sum_1^\infty \mu_\nu$, and

$$u_0(z) = \sum_{\nu=1}^{\infty} \int \log\left|1 - \frac{z}{\xi}\right| d\mu_\nu(\xi) = \int \log\left|1 - \frac{z}{\xi}\right| d\mu(\xi).$$

We note that $u_0(z)$ is subharmonic in the plane and harmonic in $|z| < r_0$. In fact if $n(r)$ is the total mass $\sum \mu_\nu$ in $|z| < r$, then $n(r) = 0$ for $r \leq r_0$, and

$$b_\nu = n(r_{\nu+1}) = \sum_{\mu=1}^{\nu} a_\nu = c_\nu - c_{\nu-1}.$$

Again

$$N(r_\nu)=\sum_{\mu=1}^{\nu}\int_{r_{\mu-1}}^{r_\mu}\frac{n(t)\,dt}{t}\le\log K\sum_{\mu=1}^{\nu}n(r_\mu)\le c_{\nu-1}\log K=O(\nu^2), \tag{4.21}$$

in view of (2.3) and (2.1). Thus, after changing r_0 is necessary, we may apply the analysis of section 4.1 to the function $u_0(z)$ and deduce from (4.10) that the integral for $u_0(z)$ converges and

$$B(r,u_0)\sim N(r)\quad\text{as}\quad r\to\infty. \tag{4.22}$$

Since (4.9) is satisfied by our sequence c_ν, we deduce from (4.21) and (4.8) that

$$N(r)=(\log K+o(1))c_\nu,\qquad r_\nu\le r\le r_{\nu+1}.$$

On combining this with (4.22) we see that (4.10) still holds with our original choice of r_0. From this, (2.3) and (4.9) we see that

$$B(r,u_0)=O(c_{\nu+1})=O(c_\nu)=O\{\psi(K^\nu)\}=O\{\psi(r)\},\qquad r_\nu\le r\le r_{\nu+1}.$$

Thus the function $u_0(z)$ satisfies (1.7).

Next we define $I_\nu(z)$, $u_\nu(z)$ by (4.12) and (4.13) with $u_0(z)$ instead of $u(z)$ and deduce that (4.14) still holds. This shows that the upper bound implied by (4.11) still holds, so that for z in A_ν

$$u_0(z)\le\int_{A_\nu}\log\left|1-\frac{z}{\xi}\right|d\mu(\xi)+c_\nu(\log K+o(1)).$$

On combining this with (4.20) we deduce that in E_ν

$$\begin{aligned}u_0(z)&\le a_\nu\{\log\operatorname{Cap}E_\nu-\log K^{\nu+2}+O(1)\}+c_\nu\{\log K+o(1)\}\\&=c_\nu\left\{-\delta_\nu\left[\log\left(\frac{r_0K^{\nu+2}}{\operatorname{Cap}E_\nu}\right)+O(1)\right]+\log K+o(1)\right\}.\\&<c_\nu\{1-\varepsilon'+o(1)\}\log K,\end{aligned}$$

in view of (2.8). From this, (4.9) and (4.10) we deduce that for large r

$$u_0(z)<(1-\varepsilon'+o(1))B(r,u_0).$$

Since $\varepsilon'>\varepsilon$, we deduce that for some $r_0>0$, $z\in\bigcup E_\nu$, $r\ge r_0$,

$$u_0(z)<(1-\varepsilon)B(r,u_0).$$

If we now set $u(z) = u_0(z) - \max(B(r_0, u_0), 0)$, (2.6) is satisfied for all z in $\bigcup E_\nu$. This proves Theorem 2.

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