

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 52 (1977)  
  
**Artikel:** Semi-continuity of the face-function for a convex set.  
**Autor:** Eifler, Larry Q.  
**DOI:** <https://doi.org/10.5169/seals-40002>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 13.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Semi-continuity of the face-function for a convex set

LARRY Q. EIFLER

### Introduction

Throughout this paper, let  $K$  be a compact convex subset of a locally convex topological vector space  $E$ . Given  $x \in K$ , set  $F(x) = \text{cl} \{y \in K : [y, x + \varepsilon(x - y)] \subseteq K \text{ for some } \varepsilon > 0\}$ . We call  $F$  the face-function on  $K$  since  $F(x)$  is the smallest closed face of  $K$  containing  $x$  if  $F(x)$  is finite dimensional. If  $f: K \rightarrow \mathbf{R}$  is continuous, we define the lower envelope  $f_e$  of  $f$  by  $f_e = \sup \{g: g \text{ is a continuous affine function on } K \text{ satisfying } g \leq f\}$ . Following Klee and Martin [3], set  $K_e = \{x \in K: f_e \text{ is continuous at } x \text{ for each continuous function } f: K \rightarrow \mathbf{R}\}$  and set  $K_l = \{x \in K: F \text{ is lower semi-continuous at } x\}$ . Klee and Martin proved that  $K_e \subseteq K_l$  in general and that  $K = K_e = K_l$  if  $K$  is 2-dimensional. They left open whether  $K_e = K_l$ . We show that  $K_e = K_l$  if  $K$  is finite dimensional and produce an infinite dimensional example where  $K = K_l \neq K_e$ .

### Lower semi-continuity of $F$

Let  $x \mapsto F(x)$  be the face-function on  $K$  defined above. We say that  $F$  is lower semi-continuous at  $x$  if for each  $y \in F(x)$  and for each neighborhood  $U$  of  $y$ ,  $\{z \in K: F(z) \text{ meets } U\}$  is a neighborhood of  $x$ . Note that  $F(x)$  is a compact convex subset  $K$  for each  $x \in K$ . If  $F$  is lower semi-continuous on  $K$ , then the set of extreme points  $\text{ex}(K)$  of  $K$  is closed. If  $\text{ex}(K)$  is closed and if  $K$  is 2 or 3-dimensional, then Clausen and Magerl [1] have shown that  $K_e = K$  and so  $K_l = K$ .

Let  $P(K)$  denote the space of Radon probability measures on  $K$  and equip  $P(K)$  with the weak\* topology. Given  $\mu \in P(K)$ , there exists a unique point  $r(\mu)$  in  $K$  such that  $\int g d\mu = g(r(\mu))$  for each continuous affine function  $g$  on  $K$ . The map  $r: P(K) \rightarrow K$  is the resultant or barycentric map. If  $x \in K$ , we let  $\delta_x$  denote the point mass measure at  $x$ . The map  $r$  is an open map of  $P(K)$  onto  $K$  if and only if  $K = K_e$ . See [2 or 5]. We say that  $r$  is open at  $\mu \in P(K)$  if for each neighborhood  $U$  of  $\mu$  in  $P(K)$ ,  $r(U) = \{r(\nu): \nu \in U\}$  is a neighborhood of  $r(\mu)$ . We say that  $r$  is  $\lambda$ -open at  $\mu \in P(K)$  where  $0 < \lambda < 1$  if for each neighborhood  $U$  of  $\mu$

in  $P(K)$ ,  $\lambda r(U) + (1 - \lambda)K$  is a neighborhood of  $r(\mu)$ . We first establish criteria for determining when  $r$  is open at  $\mu$ . These results are of interest aside from their application to the study of the lower semi-continuity of the face-function.

LEMMA 1. *Let  $\mu \in P(K)$ . Then  $r$  is open at  $\mu$  if  $r$  is  $\lambda$ -open at  $\mu$  for some  $0 < \lambda < 1$ .*

*Proof.* Assume that  $r$  is  $\lambda$ -open at  $\mu$ . Set  $x = r(\mu)$ . If  $x_\alpha \rightarrow x$ , then there exist  $\mu_\alpha \rightarrow \mu$  and  $y_\alpha \in K$  such that  $\lambda r(\mu_\alpha) + (1 - \lambda)y_\alpha = x_\alpha$ . But  $y_\alpha \rightarrow x$ . Hence, there exist  $\nu_\alpha \rightarrow \mu$  and  $z_\alpha \in K$  such that  $\lambda r(\nu_\alpha) + (1 - \lambda)z_\alpha = y_\alpha$ . Thus,

$$x_\alpha = \lambda(2 - \lambda) \left\{ r \left( \frac{\mu_\alpha}{2 - \lambda} + \frac{(1 - \lambda)\nu_\alpha}{2 - \lambda} \right) \right\} + (1 - \lambda)^2 z_\alpha.$$

One obtains that  $r$  is  $\lambda(2 - \lambda)$ -open at  $\mu$ . Hence,  $r$  is  $\rho$ -open at  $\mu$  for each  $0 < \rho < 1$ . This implies that  $r$  is open at  $\mu$ .

LEMMA 2. *Let  $x \in K$ . The following are equivalent.*

- (1)  $r$  is open at  $\mu$  if  $r(\mu) = x$  and
- (2)  $r$  is open at  $\mu$  if  $r(\mu) = x$  and if  $\mu$  is supported by 2 points.

*Proof.* We only need to show  $(2 \Rightarrow 1)$ . Let  $U$  be a neighborhood of  $\mu$  where  $r(\mu) = x$  and  $\mu$  is supported by  $n$  points. We show that  $r(U)$  is a neighborhood of  $x$  by induction on  $n$ . The result holds for  $n = 2$ . So assume the result holds for  $n \leq m$ . Fix  $\mu \in P(K)$  such that  $r(\mu) = x$  and  $\mu$  is supported by  $\{x_1, \dots, x_{m+1}\}$ . Let  $x = \sum_{i=1}^{m+1} \lambda_i x_i$ . We assume each  $\lambda_i > 0$ . Set  $y_k = (\lambda_k x_k + \lambda_{k+1} x_{k+1}) / (\lambda_k + \lambda_{k+1})$  and set  $\mu_k = \sum_{i=1}^{k-1} \lambda_i \delta_{x_i} + (\lambda_k + \lambda_{k+1}) \delta_{y_k} + \sum_{i>k}^{m+1} \lambda_i \delta_{x_i}$ . Suppose  $x_\alpha \rightarrow x$ . Then there exist  $\mu_k^\alpha \rightarrow \mu_k$  such that  $r(\mu_k^\alpha) = x_\alpha$ . Set  $\nu_\alpha = \sum_{k=1}^{m+1} (1/m + 1) \mu_k^\alpha$ . Then  $r(\nu_\alpha) = x_\alpha$ . Since  $\nu_\alpha \rightarrow \sum_{k=1}^{m+1} (1/m + 1) \mu_k$ , we have  $\lim_\alpha \sup \nu_\alpha(V) \geq (m/m + 1) \lambda_k$  if  $V$  is an open set containing  $x_k$ . Thus, there exist  $\mu_\alpha \rightarrow \mu$  and  $z_\alpha \in K$  such that  $(m/m + 1) r(\mu_\alpha) + (1/m + 1) z_\alpha = x_\alpha$ . Hence,  $r$  is open at  $\mu$  by Lemma 1. By approximating measures by measures with finite support, we see that  $r$  is open at  $\mu$  if  $r(\mu) = x$ .

THEOREM. *Let  $x \in K$ . The following are equivalent.*

- (1)  $f_e$  is continuous at  $x$  for each  $f \in C_{\mathbf{R}}(K)$
- (2)  $r$  is open at  $\mu$  if  $r(\mu) = x$
- (3)  $r$  is open at  $\mu$  if  $r(\mu) = x$  and if  $\mu$  is supported by 2 points.

*Proof.* The implication  $(1) \Rightarrow (2)$  follows from Proposition 3.1 in Phelps [4, p.

21]. The implication (2)  $\Rightarrow$  (1) follows from the separation form of the Hahn-Banach theorem and taking limits in the hyperspace of  $P(K)$ . See [2] for details. The implication (2)  $\Leftrightarrow$  (3) is simply Lemma 2.

**COROLLARY.** *Assume  $K$  is finite dimensional. Then  $K_e = K_l$ .*

*Proof.* The inclusion  $K_e \subseteq K_l$  was established in [3]. Let  $x \in K_l$ . Suppose  $\mu \in P(K)$  such that  $\mu$  is supported by  $\{y, z\}$  and  $r(\mu) = x$ . We only need to show that  $r$  is  $\frac{1}{2}$ -open at  $\mu$  by Lemma 1 and the above theorem. Set  $x = \lambda y + (1 - \lambda)z$  where  $0 \leq \lambda \leq 1$ . We may assume  $y \neq z$  and  $0 < \lambda < 1$ . Let  $U$  be a neighborhood of  $\mu$  in  $P(K)$ . Set  $\Omega = \frac{1}{2}r(U) + \frac{1}{2}K$ . Assume  $\Omega$  is not a neighborhood of  $x$ . Then there exist  $x_n \rightarrow x$  such that  $x_n \notin \Omega$  and  $\dim F(x_n) = q$  where  $q$  is least possible. By taking subsequences, we may assume that there exist  $y_n, z_n \in F(x_n)$  such that  $y_n \rightarrow y$  and  $z_n \rightarrow z$  since  $\limsup F(x_n) \supseteq F(x) \supseteq \{y, z\}$ . Set  $w_n = \lambda y_n + (1 - \lambda)z_n$ . We may assume  $w_n \neq x_n$ . Set  $\varepsilon_n = \max \{\varepsilon : x_n + \varepsilon(w_n - x_n) \in K\}$ . Then  $\dim F(x_n + \varepsilon_n(w_n - x_n)) < \dim F(x_n) = q$ . But,  $w_n \in r(U)$  for  $n$  large. If  $\varepsilon_n \geq 1$  and if  $w_n \in r(U)$ , then

$$\frac{\varepsilon_n}{1 + \varepsilon_n} w_n + \frac{1}{1 + \varepsilon_n} \{x_n + \varepsilon_n(x_n - w_n)\} = x_n \in \Omega.$$

Hence,  $\varepsilon_n < 1$  for  $n$  large. Thus,  $x_n + \varepsilon_n(w_n - x_n) \rightarrow x$  and  $x_n + \varepsilon_n(w_n - x_n) \in \Omega$  which is impossible by the minimality of  $q$ .

*Example 1.* Let  $K$  be the convex hull of  $\{(e^{i\theta}, \pm 1) : 0 \leq \theta \leq \pi\} \cup \{(1, \pm i)\}$  in  $\mathbf{C}^2$ . Then  $K$  is 4-dimensional and  $ex(K)$  is closed. The face-function is not lower semi-continuous at  $(1, 0)$  since  $F(1, 0)$  is a square and  $F(e^{i\theta}, 0)$  is an interval if  $0 < \theta \leq \pi$ .

*Example 2.* Let  $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$  and equip  $X$  with the usual metric from  $\mathbf{R}$ . Let  $K$  denote the closed unit ball in the space of real Radon measures on  $X$ , i.e., the dual of  $C_{\mathbf{R}}(X)$ . Equip  $K$  with the weak\* topology. Then  $K$  is a compact convex set. Given  $\mu \in K$ , set  $\|\mu\| = \mu^+(X) + \mu^-(X)$ . If  $\|\mu\| < 1$ , then  $F(\mu) = K$ . If  $\|\mu\| = 1$ , then  $F(\mu)$  is the closed convex hull of  $\{\text{sgn}(\mu(x)) \cdot \delta_x : x \in X\}$ . One easily checks that the face-function is lower semi-continuous on  $K$  and so  $K = K_l$ . The zero measure 0 is not in  $K_e$  by criterion (2) in the theorem since  $\frac{1}{2}[\delta_1 + (-1)\delta_1] = 0$  and  $\frac{1}{2}[\delta_{1/n} + (-1)\delta_{1/n+1}] \rightarrow 0$ .

## REFERENCES

- [1] A. CLAUSING and G. MAGERL, *Generalized Dirichlet problems and continuous selections of representing measures*, Math. Ann. 216 (1975), 71-78.

- [2] L. Q. EIFLER, *Openness of convex averaging*, Glasnik Matematicki (to appear).
- [3] V. KLEE and M. MARTIN, *Semicontinuity of the face-function of a convex set*, Comment. Math. Helvetici 46 (1971), 1–12.
- [4] R. R. PHELPS, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, 1966.
- [5] J. VESTERSTROM, *On open maps, compact convex sets, and operator algebras*, J. London Math. Soc. (2), 6 (1973), 289–297.

Received October 12, 1976