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Semi-continuity of the face-function for a convex set

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Introduction

Throughout this paper, let K be a compact convex subset of a locally convex topological vector space E. Given $x \in K$, set $F(x) = \mathrm{cl}\,\{y \in K : [y, x + \varepsilon(x-y)] \subseteq K$ for some $\varepsilon > 0\}$. We call F the face-function on K since $F(x)$ is the smallest closed face of K containing x if $F(x)$ is finite dimensional. If $f: K \to \mathbf{R}$ is continuous, we define the lower envelope f_e of f by $f_e = \sup\{g : g$ is a continuous affine function on K satisfying $g \leq f\}$. Following Klee and Martin [3], set $K_e = \{x \in K : f_e$ is continuous at x for each continuous function $f: K \to \mathbf{R}\}$ and set $K_l = \{x \in K : F$ is lower semi-continuous at $x\}$. Klee and Martin proved that $K_e \subseteq K_l$ in general and that $K = K_e = K_l$ if K is 2-dimensional. They left open whether $K_e = K_l$. We show that $K_e = K_l$ is K is finite dimensional and produce an infinite dimensional example where $K = K_l \neq K_e$.

Lower semi-continuity of F

Let $x \to F(x)$ be the face-function on K defined above. We say that F is lower semi-continuous at x if for each $y \in F(x)$ and for each neighborhood U of y, $\{z \in K : F(z)$ meets $U\}$ is a neighborhood of x. Note that $F(x)$ is a compact convex subset K for each $x \in K$. If F is lower semi-continuous on K, then the set of extreme points $ex(K)$ of K is closed. If $ex(K)$ is closed and if K is 2 or 3-dimensional, then Clausing and Magerl [1] have shown that $K_e = K$ and so $K_l = K$.

Let $P(K)$ denote the space of Radon probability measures on K and equip $P(K)$ with the weak* topology. Given $\mu \in P(K)$, there exists a unique point $r(\mu)$ in K such that $\int g\, d\mu = g(r(\mu))$ for each continuous affine function g on K. The map $r: P(K) \to K$ is the resultant or barycentric map. If $x \in K$, we let δ_x denote the point mass measure at x. The map r is an open map of $P(K)$ onto K if and only if $K = K_e$. See [2 or 5]. We say that r is open at $\mu \in P(K)$ if for each meighborhood U of μ in $P(K)$, $r(U) = \{r(\nu) : \nu \in U\}$ is a neighborhood of $r(\mu)$. We say that r is λ-open at $\mu \in P(K)$ where $0 < \lambda < 1$ if for each neighborhood U of μ

in $P(K)$, $\lambda r(U)+(1-\lambda)K$ is a neighborhood of $r(\mu)$. We first establish criteria for determining when r is open at μ. These results are of interest aside from their application to the study of the lower semi-continuity of the face-function.

LEMMA 1. *Let $\mu \in P(K)$. Then r is open at μ if r is λ-open at μ for some $0<\lambda<1$.*

Proof. Assume that r is λ-open at μ. Set $x=r(\mu)$. If $x_\alpha \to x$, then there exist $\mu_\alpha \to \mu$ and $y_\alpha \in K$ such that $\lambda r(\mu_\alpha)+(1-\lambda)y_\alpha = x_\alpha$. But $y_\alpha \to x$. Hence, there exist $\nu_\alpha \to \mu$ and $z_\alpha \in K$ such that $\lambda r(\nu_\alpha)+(1-\lambda)z_\alpha = y_\alpha$. Thus,

$$x_\alpha = \lambda(2-\lambda)\left\{r\left(\frac{\mu_\alpha}{2-\lambda}+\frac{(1-\lambda)\nu_\alpha}{2-\lambda}\right)\right\}+(1-\lambda)^2 z_\alpha.$$

One obtains that r is $\lambda(2-\lambda)$-open at μ. Hence, r is ρ-open at μ for each $0<\rho<1$. This implies that r is open at μ.

LEMMA 2. *Let $x \in K$. The following are equivalent.*

(1) *r is open at μ if $r(\mu)=x$ and*
(2) *r is open at μ if $r(\mu)=x$ and if μ is supported by 2 points.*

Proof. We only need to show $(2 \Rightarrow 1)$. Let U be a neighborhood of μ where $r(\mu)=x$ and μ is supported by n points. We show that $r(U)$ is a neighborhood of x by induction on n. The result holds for $n=2$. So assume the result holds for $n \le m$. Fix $\mu \in P(K)$ such that $r(\mu)=x$ and μ is supported by $\{x_1, \ldots, x_{m+1}\}$. Let $x=\sum_{i=1}^{m+1} \lambda_i x_i$. We assume each $\lambda_i>0$. Set $y_k=(\lambda_k x_k+\lambda_{k+1}x_{k+1})/(\lambda_k+\lambda_{k+1})$ and set $\mu_k=\sum_{i=1}^{k-1}\lambda_i\delta_{x_i}+(\lambda_k+\lambda_{k+1})\delta_{y_k}+\sum_{i>k}^{m+1}\lambda_i\delta_{x_i}$. Suppose $x_\alpha \to x$. Then there exist $\mu_k^\alpha \to \mu_k$ such that $r(\mu_k^\alpha)=x_\alpha$. Set $\nu_\alpha=\sum_{k=1}^{m+1}(1/m+1)\mu_k^\alpha$. Then $r(\nu_\alpha)=x_\alpha$. Since $\nu_\alpha \to \sum_{k=1}^{m+1}(1/m+1)\mu_k$, we have $\lim_\alpha \sup \nu_\alpha(V) \ge (m/m+1)\lambda_k$ if V is an open set containing x_k. Thus, there exist $\mu_\alpha \to \mu$ and $z_\alpha \in K$ such that $(m/m+1)r(\mu_\alpha)+(1/m+1)z_\alpha = x_\alpha$. Hence, r is open at μ by Lemma 1. By approximating measures by measures with finite support, we see that r is open at μ if $r(\mu)=x$.

THEOREM. *Let $x \in K$. The following are equivalent.*

(1) *f_e is continuous at x for each $f \in C_{\mathbf{R}}(K)$*
(2) *r is open at μ if $r(\mu)=x$*
(3) *r is open at μ if $r(\mu)=x$ and if μ is supported by 2 points.*

Proof. The implication $(1) \Rightarrow (2)$ follows from Proposition 3.1 in Phelps [4, p.

21]. The implication (2)$\Rightarrow$(1) follows from the separation form of the Hahn–Banach theorem and taking limits in the hyperspace of $P(K)$. See [2] for details. The implication (2$\Leftrightarrow$3) is simply Lemma 2.

COROLLARY. *Assume K is finite dimensional. Then $K_e = K_l$.*

Proof. The inclusion $K_e \subseteq K_l$ was established in [3]. Let $x \in K_l$. Suppose $\mu \in P(K)$ such that μ is supported by $\{y, z\}$ and $r(\mu) = x$. We only need to show that r is $\frac{1}{2}$-open at μ by Lemma 1 and the above theorem. Set $x = \lambda y + (1-\lambda)z$ where $0 \leq \lambda \leq 1$. We may assume $y \neq z$ and $0 < \lambda < 1$. Let U be a neighborhood of μ in $P(K)$. Set $\Omega = \frac{1}{2}r(U) + \frac{1}{2}K$. Assume Ω is not a neighborhood of x. Then there exist $x_n \to x$ such that $x_n \notin \Omega$ and $\dim F(x_n) = q$ where q is least possible. By taking subsequences, we may assume that there exist $y_n, z_n \in F(x_n)$ such that $y_n \to y$ and $z_n \to z$ since $\limsup F(x_n) \supseteq F(x) \supseteq \{y, z\}$. Set $w_n = \lambda y_n + (1-\lambda)z_n$. We may assume $w_n \neq x_n$. Set $\varepsilon_n = \max\{\varepsilon : x_n + \varepsilon(w_n - x_n) \in K\}$. Then $\dim F(x_n + \varepsilon_n(w_n - x_n)) < \dim F(x_n) = q$. But, $w_n \in r(U)$ for n large. If $\varepsilon_n \geq 1$ and if $w_n \in r(U)$, then

$$\frac{\varepsilon_n}{1+\varepsilon_n} w_n + \frac{1}{1+\varepsilon_n}\{x_n + \varepsilon_n(x_n - w_n)\} = x_n \in \Omega.$$

Hence, $\varepsilon_n < 1$ for n large. Thus, $x_n + \varepsilon_n(w_n - x_n) \to x$ and $x_n + \varepsilon_n(w_n - x_n) \in \Omega$ which is impossible by the minimality of q.

Example 1. Let K be the convex hull of $\{(e^{i\theta}, \pm 1) : 0 \leq \theta \leq \pi\} \cup \{(1, \pm i)\}$ in $\mathbf{C}^2$. Then K is 4-dimensional and $ex(K)$ is closed. The face-function is not lower semi-continuous at $(1, 0)$ since $F(1, 0)$ is a square and $F(e^{i\theta}, 0)$ is an interval if $0 < \theta \leq \pi$.

Example 2. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and equip X with the usual metric from $\mathbf{R}$. Let K denote the closed unit ball in the space of real Radon measures on X, i.e., the dual of $C_{\mathbf{R}}(X)$. Equip K with the weak* topology. Then K is a compact convex set. Given $\mu \in K$, set $\|\mu\| = \mu^+(X) + \mu^-(X)$. If $\|\mu\| < 1$, then $F(\mu) = K$. If $\|\mu\| = 1$, then $F(\mu)$ is the closed convex hull of $\{sgn\,(\mu(x)) \cdot \delta_x : x \in X\}$. One easily checks that the face-function is lower semi-continuous on K and so $K = K_l$. The zero measure 0 is not in K_e by criterion (2) in the theorem since $\frac{1}{2}[\delta_1 + (-1)\delta_1] = 0$ and $\frac{1}{2}[\delta_{1/n} + (-1)\delta_{1/n+1}] \to 0$.

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