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On the inverse limit of free nilpotent groups

G. BAUMSLAG and U. STAMMBACH

0. Introduction

A group P is called parafree (see [B1], [B2]) if it is residually nilpotent and if there exists a free group F and a homomorphism $\varphi: F \to P$ such that φ induces isomorphisms $\varphi_i: F/F \xrightarrow{\sim} P/P_i$, $i \ge 2$ modulo the terms F_i , P_i of the lower central series. Since F is residually nilpotent, the map φ is injective, so that F may be thought of as a subgroup of P. If F = F(X) is free on the set X, then P is called parafree on X. It is plain (see [B2]) that a parafree group on X can be embedded in $\hat{F} = \lim_{\leftarrow} F/F_i$. The group \hat{F} thus certainly merits some interest. This paper is a contribution to the study of the group \hat{F} .

In section 1 we introduce some notation and prove a basic lemma which enables us to identify the subgroups of $\hat{F} = \hat{F}(X)$ which are parafree on X. In section 2 we deal with the case of finitely generated free groups F = F(X). It turns out (Corollary 2.2) that in this case the group \hat{F} is parafree on X. This result contrasts with the case where X is infinite. In section 3 we deal in detail with the case where X is countably infinite. We prove the following results: \hat{F} is not parafree on X (Corollary 3.5; see also [BK]); \hat{F}_{ab} contains uncountably many linearly independent divisible elements (Theorem 3.9); \hat{F} contains a free subgroup of uncountable rank with a generating set which is linearly independent mod \hat{F}_2 (Theorem 3.6); the 2-generator subgroups of \hat{F} are free (Theorem 3.11). We note that the restriction to the countable case in all of the main results of this section is not decisive. The conclusions remain true if X is allowed to be uncountable.

In section 4 we define two subgroups \bar{F} , \tilde{F} of $\hat{F} = \hat{F}(X)$. The group \bar{F} is the union of the subgroups of \hat{F} which are parafree on X. It is shown that \bar{F} is parafree on X, so that \bar{F} is the universal parafree group on X in the sense that it contains all groups which are parafree on X (Theorem 4.1). The subgroup \tilde{F} of \hat{F} consists of all elements of \hat{F} which can be expressed by finitely many elements of X. It is shown that \tilde{F} too is parafree on X (Proposition 4.4). Hence clearly $\tilde{F} \subseteq \bar{F}$. But we show that $\tilde{F} \neq \bar{F}$ if X is countably infinite (Proposition 4.5).

We also show that the group \bar{F} is freely indecomposable (Corollary 4.3). We were however unable to settle the question whether \tilde{F} and \hat{F} are freely indecomposable in case X is infinite.

1. The inverse limit

Let G be a group and let $\{G_i\}$ denote its lower central series, i.e.

$$G_1 = G, \quad G_i = [G, G_{i-1}], \quad i \ge 2.$$
 (1.1)

As usual we shall denote G/G_2 by G_{ab} . We consider the inverse system of the canonical projections

$$\{G/G_i \to G/G_{i-1}\}. \tag{1.2}$$

Its inverse limit is denoted by

$$\hat{G} = \lim G/G_i \tag{1.3}$$

and its canonical maps by $\tau_i: \hat{G} \to G/G_i$. We may regard \hat{G} as the subgroup of the (categorical) product $\prod_{i=2}^{\infty} G/G_i$ consisting of the elements

$$\Lambda = (\lambda_1 G_2, \lambda_2 G_3, \ldots) \tag{1.4}$$

with $\lambda_i \in G$ and $\lambda_{i+1} \equiv \lambda_i$ modulo G_{i+1} . Then clearly $\tau_i(\Lambda) = \lambda_{i-1} G_i \in G/G_i$. By universality of the inverse limit the family $\pi_i : G \to G/G_i$ of canonical projections induces a homomorphism $h : G \to \hat{G}$ such that $\pi_i = \tau_i h$. Plainly it is given by

$$h(x) = (xG_2, xG_3, \ldots), \qquad x \in G.$$
 (1.5)

The homomorphism h is injective if and only if G is residually nilpotent (i.e. if $G_{\omega} = \bigcap_{i \ge 2} G_i = e$).

In the sequel we shall be interested in subgroups P of \hat{G} with $hG \subseteq P \subseteq \hat{G}$ and with the property that h induces isomorphisms $h_i: G/G_i \xrightarrow{\sim} P/P_i$, $i \ge 2$. The following lemma characterizes these subgroups.

LEMMA 1.1. Let P be a group with $hG \subseteq P \subseteq \hat{G}$. Then the following statements are equivalent

- (i) $h: G \to P$ induces isomorphisms $h_i: G/G_i \xrightarrow{\sim} P/P_i$, $i \ge 2$;
- (ii) $\tau_i: P \to \hat{G} \to G/G_i$ induces isomorphisms $\sigma_i: P/P_i \to G/G_i$, $i \ge 2$;
- (iii) $h: G \to P$ induces an epimorphism $h_2: G_{ab} \longrightarrow P_{ab}$;
- (iv) $\tau_2: P \to G_{ab}$ induces a monomorphism $\sigma_2: P_{ab} \rightarrowtail G_{ab}$.

Proof. We consider the map

$$\pi_i = \tau_i h: G \to P \subseteq \hat{G} \to G/G_i, \qquad i \ge 2.$$
 (1.6)

It induces the identity $G/G_i \xrightarrow{h_i} P/P_i \xrightarrow{\sigma_i} G/G_i$, $i \ge 2$. Hence h_i is always injective and σ_i surjective. Moreover h_i is surjective if and only if σ_i is injective. This proves the equivalence of (i) and (ii). Also, it is well known that h_i is surjective if and only if h_2 is, proving the equivalence of (i) and (iii). Finally h_2 is surjective if and only if σ_2 is injective, proving the equivalence of (iii) and (iv).

COROLLARY 1.2. Let F = F(X) be the free group on the set X. Then a group P with $F \subseteq P \subseteq \hat{F}$ is parafree on X if and only if $h_2: F_{ab} \to P_{ab}$ is surjective.

Proof. This is immediate from Lemma 1.1, since the groups F and \hat{F} are clearly residually nilpotent.

2. The case of finitely generated free groups

Let F = F(X) be the free group on the set $X = \{x_1, x_2, \ldots, x_n\}$. The following result is due to Bousfield-Kan [BK]. Since its proof to be found in [BK] uses topological methods we shall include, for completeness, a purely algebraic proof; it is also due to Bousfield-Kan.

THEOREM 2.1. Let F be a finitely generated free group. Then $h: F \to \hat{F}$ induces isomorphisms $h_i: F/F_i \cong \hat{F}/\hat{F}_i$, $i \ge 2$.

COROLLARY 2.2. If $F = F(x_1, ..., x_n)$, then \hat{F} is parafree on $X = \{x_1, ..., x_n\}$.

Proof. By Lemma 1.1 we only have to show that $\sigma_2: \hat{F}_{ab} \to F_{ab}$ induced by $\tau_2: \hat{F} \to F/F_2$ is injective. We thus have to show that an element

$$\Lambda^* = (\lambda_1 F_2, \lambda_2 F_3, \ldots) \in \hat{F}$$
 (2.1)

with $\lambda_1 \in F_2$ is in \hat{F}_2 . In the course of the proof we shall need the following two wellknown results which we mention without proof.

LEMMA 2.3.

$$[ab, c] = [a, c]^b [b, c],$$

 $[c, ab] = [c, b][c, a]^b.$

LEMMA 2.4. Let $F = F(x_1, ..., x_n)$. Then given $a \in F_k$, $k \ge 2$ there exist

 $u_1, \ldots, u_n \in F_{k-1}$ such that

$$a \equiv [u_1, x_1][u_2, x_2] \cdot \cdot \cdot [u_n, x_n] \mod F_{k+1}.$$

We shall construct elements

$$\Gamma^{(i)} = (\gamma_1^{(i)} F_2, \gamma_2^{(i)} F_3, \ldots) \in \hat{F}, \qquad 1 \le i \le n$$
(2.2)

such that in \hat{F}

$$\Lambda^* = [\Gamma^{(1)}, h(x_1)][\Gamma^{(2)}, h(x_2)] \cdot \cdot \cdot [\Gamma^{(n)}, h(x_n)]. \tag{2.3}$$

In order to find $\gamma_k^{(i)}$ we proceed by induction on k. Since $\lambda_1 \in F_2$ and hence $\lambda_2 \in F_2$ there are, by Lemma 2.4, elements $u_1, \ldots, u_n \in F_1$ such that

$$\lambda_2 \equiv [u_1, x_1] \cdot \cdot \cdot [u_n, x_n] \mod F_3. \tag{2.4}$$

Set $\gamma_1^{(i)} = u_1$. We then have

$$\lambda_1 \equiv [\gamma_1^{(1)}, x_1] \cdots [\gamma_1^{(n)}, x_n] \mod F_2$$
 (2.5)

and also

$$\lambda_2 \equiv [\gamma_1^{(1)}, x_1] \cdots [\gamma_1^{(n)}, x_n] \mod F_3.$$
 (2.6)

Suppose now that $\gamma_1^{(i)}, \ldots, \gamma_k^{(i)}, 1 \le i \le n$ have already been determined such that

$$\gamma_{l+1}^{(i)} \equiv \gamma_l^{(i)} \mod F_{l+1}, \qquad 1 \le l \le k-1,$$
(2.7)

$$\lambda_l \equiv [\gamma_l^{(1)}, x_1] \cdots [\gamma_l^{(n)}, x_n] \mod F_{l+1}, \qquad 1 \le l \le k,$$
 (2.8)

and in addition

$$\lambda_{k+1} \equiv [\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n] \mod F_{k+2}. \tag{2.9}$$

Since $\lambda_{k+2} \equiv \lambda_{k+1} \mod F_{k+2}$ there exists $r_{k+2} \in F_{k+2}$ such that

$$\lambda_{k+2} = ([\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n]) r_{k+2}. \tag{2.10}$$

By Lemma 2.4 we can find $v_i \in F_{k+1}$ such that

$$r_{k+2} \equiv [v_1, x_1] \cdots [v_n, x_n] \mod F_{k+3}.$$
 (2.11)

We may thus set

$$\gamma_{k+1}^{(i)} = \gamma_k^{(i)} \cdot v_i, \qquad 1 \le i \le n. \tag{2.12}$$

We then clearly have

$$\gamma_{k+1}^{(i)} \equiv \gamma_k^i \bmod F_{k+1}. \tag{2.13}$$

Moreover

$$[\gamma_{k+1}^{(1)}, x_1] \cdots [\gamma_{k+1}^{(n)}, x_n] = [\gamma_k^{(1)} v_1, x_1] \cdots [\gamma_k^{(n)} v_n, x_n]$$

$$= [\gamma_k^{(1)}, x_1]^{v_1} [v_1, x_1] \cdots [\gamma_k^n, x_n]^{v_n} [v_n, x_n], \text{ by Lemma 2.3}$$

$$\equiv ([\gamma_k^{(1)}, x_1] \cdots [\gamma_k^{(n)}, x_n]) ([v_1, x_1] \cdots [v_n, x_n]) \mod F_{k+3}$$

$$\equiv \lambda_{k+2} \mod F_{k+3}, \text{ by (2.10), (2.11).}$$
(2.14)

A fortiori we have

$$[\gamma_{k+1}^{(1)}, x_1] \cdots [\gamma_{k+1}^{(n)}, x_n] \equiv \lambda_{k+1} \mod F_{k+2}.$$
 (2.15)

This completes the proof of Theorem 2.1.

For the proof of Corollary 2.2 we only have to remark that \hat{F} , being a subgroup of $\prod_{i\geq 2} F/F_i$, is residually nilpotent. Next we recall that the group \hat{F} contains any parafree group on X (see [B2]). Since, by Corollary 2.2, the group \hat{F} is itself parafree on X if X is finite, we see that \hat{F} is the biggest parafree group on the (finite) set X.

We note that the proof of Theorem 2.1 works equally well if the free group F is replaced by an arbitrary group G generated by the set $X = \{x_1, \ldots x_n\}$. We may thus state

COROLLARY 2.5. If G is finitely generated, then $\hat{G} = \hat{G}$.

Finally we note

COROLLARY 2.6. If F is finitely generated, then \hat{F} is freely indecomposable.

Proof. Suppose $\hat{F} = A * B$ with $A \neq \{e\} \neq B$. Then there are surjective maps $A \twoheadrightarrow C_1$, $B \twoheadrightarrow C_2$ with C_i , i = 1, 2 infinite cyclic. Thus we obtain an epimorphism $f: \hat{F} \to C_1 * C_2 = F(x, y)$ onto the free group on two generators. Take elements $a, b \in \hat{F}$ with fa = x, fb = y. Since F(x, y) is free there exists $g: F(x, y) \to \hat{F}$ with gx = a, gy = b and $fg = Id_F$. Using Corollary 2.5 we obtain an epimorphism $p: \hat{F}(x, y) \to \hat{F} \cong \hat{F} \to F(x, y)$. But, by Corollary 2.2 the groups $\hat{F}(x, y)$ and F(x, y)

are parafree of the same rank, so that by Theorem 1.1 of [B2] p is an isomorphism and $\hat{F}(x, y)$ is free. This is a contradiction.

3. The case of free groups of countably infinite rank

Let $F = F(x_1, x_2 \cdots)$ be a free group on the countably infinite set $X = \{x_1, x_2, \ldots\}$. In order to obtain results on \hat{F} we shall first construct a metabelian group W.

Let A be the free abelian group on $X = \{x_1, x_2, ...\}$ and let IA be the augmentation ideal of the integral group ring of A. Clearly IA/IA^2 is free abelian on $\{x_1-1, x_2-1, ...\}$. The following lemma is a generalization of this fact (see [BG]).

LEMMA 3.1. IA^n/IA^{n+1} is the free abelian group on the set $\{\prod_{j=1}^n (x_{i(j)}-1)\}$.

We now define W as the semi-direct product $W = IA \ 1$ A where IA is regarded as right A-module in the usual way. We prove

LEMMA 3.2. $W_n = IA^n$, $n \ge 2$.

Proof. Let $u, v \in IA$, $x, y \in A$; then using Lemma 2.3 we obtain

$$[ux, vy] = [u, vy]^{x}[x, vy] = [u, y]^{x}[u, v]^{yx}[x, y][x, v]^{y} = [u, y]^{x}[x, v]^{y},$$

since both IA and A are abelian. Thus we have $W_2 = [W, W] = [IA, A] = IA^2$. Using induction it is now easy to prove $W_n = IA^n$ for n > 2. We leave the details to the reader. We now consider $\hat{W} = \lim_{\leftarrow} W/W_n = \lim_{\leftarrow} (IA/IA^n \downarrow A) = (\lim_{\leftarrow} IA/IA^n) \downarrow A$. By Lemma 3.1 the group IA^n/IA^{n+1} is free abelian on the n-fold products $\prod_{i=1}^n (x_{i(i)}-1)$. We may thus identify IA/IA^n , as abelian group, with the augmentation ideal of the quotient of the polynomial ring on $y_i = x_i - 1$, $i = 1, 2, \ldots$ modulo the ideal generated by the n-fold products. As a consequence we see that $\lim_{\leftarrow} IA/IA^n$, as an abelian group, is isomorphic to the augmentation ideal I of the power series ring on I is given by

$$(x_i - 1) \circ x_i = (x_i - 1)(x_i - 1) + (x_i - 1) = y_i y_i + y_i, \tag{3.1}$$

$$(x_i - 1) \circ x_i^{-1} = y_i (1 - y_i + y_i^2 - y_i^3 + \cdots). \tag{3.2}$$

We note for further reference that in $\hat{W} = J \uparrow A$

$$[x_i - 1, x_j] = -(x_i - 1) + (x_i - 1) \circ x_j = y_i y_j, \tag{3.3}$$

$$[x_i - 1, x_j^{-1}] = -y_i y_j + y_i y_j^2 - y_i y_j^3 + \cdots$$
 (3.4)

Here we have used the fact that conjugation of $x_i - 1$ by x_j , x_j^{-1} in \hat{W} is just operation of x_j , x_j^{-1} on y_i in J. Since

$$\hat{W}_2 = [J, A]$$
 $\hat{W}_3 = [J, A, A]$, etc. (3.5)

We obtain from (3.3), (3.4) the following key result.

LEMMA 3.3. $\hat{W}_k = J \cdot IA^{k-1}$, $k \ge 2$. In particular, an element $v \in J$ is in \hat{W}_2 if and only if it can be written as a finite linear combination

$$v = \sum_{i=1}^{n} w_i y_i, \qquad w_i \in J.$$
 (3.6)

With this result it is now possible to settle various questions about our group \hat{F} . We first reprove a result of Bousfield-Kan [BK, p. 114].

THEOREM 3.4. Let F = F(X) be a free group where X is countably infinite. Then $h_2: F_{ab} \to \hat{F}_{ab}$ is not surjective.

COROLLARY 3.5. Let F = F(X) be a free group where X is countably infinite. Then \hat{F} is not parafree on X.

Proof. We enumerate the elements of X as follows

$$X = \{x_{11}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}, \ldots\}$$
 (3.7)

and consider (see [BK]) the element $\Lambda = (\lambda_1 F_2, \lambda_2 F_3, \ldots) \in \hat{F}$ where

$$\lambda_1 = e, \lambda_k = [x_{21}, x_{22}][x_{31}, x_{32}, x_{33}] \cdot \cdot \cdot [x_{k1}, \dots, x_{kk}], k \ge 2.$$
 (3.8)

We shall show that $\Lambda \notin \hat{F}_2$ but $\tau_2(\Lambda) = e \in F_{ab}$; hence τ_2 is not injective. This implies, by Lemma 1.1, that h_2 is not surjective.

Consider the free abelian group A on X and the group $W = IA \ 1$ A. Define a

map $f: F \to W$ by

$$f(x_{kj}) = \begin{cases} x_{kj} - 1 \in IA \subseteq W & \text{for } j = 1, \\ x_{kj} \in A \subseteq W & \text{for } 2 \le j \le k. \end{cases}$$
 (3.9)

We then obtain a map $\hat{f}:\hat{F}\to\hat{W}$ with

$$\hat{f}(\Lambda) = (x_{21} - 1)(x_{22} - 1) + (x_{31} - 1)(x_{32} - 1)(x_{33} - 1) + \cdots$$
(3.10)

(see (3.3)). It is then clear from Lemma 3.3 that $\hat{f}(\Lambda) \not\in \hat{W}_2$, so that $\Lambda \not\in \hat{F}_2$. This completes the proof of Theorem 3.4.

We note that it might conceivably be the case that \hat{F} is parafree on some set other than X. That this is not the case follows from Corollary 3.9 where the existence of non-trivial divisible elements in \hat{F}_{ab} is proved. We first state

THEOREM 3.6. Let F = F(X) be a free group where X is countably infinite. Then \hat{F} contains a subgroup which is free on an uncountable set Y of elements which are linearly independent mod \hat{F}_2 .

For the proof of this result we shall need the following

LEMMA 3.7. There exists an uncountable set Σ of sequences $\sigma = (\sigma_0, \sigma_1, ...)$ of natural numbers σ_i with the following properties:

- (i) $\sigma_0 \ge 2$, $\sigma_{i+1} > \sigma_i$, $i \ge 0$;
- (ii) if $\{\sigma^{(1)}, \ldots, \sigma^{(n)}\}$ is a finite subset of Σ then there exists $i \ge 0$ such that for every $k \ge i$ the entries $\sigma_k^{(1)}, \ldots, \sigma_k^{(n)}$ are different.

Proof. Let Ω be an uncountable set of sequences $\omega = (\omega_0, \omega_1, \ldots)$ of numbers $0, 1, 2, \ldots, 9$ with $\omega_0 \ge 2$. Define, for any such ω , a sequence

$$\sigma(\omega) = (\sigma_0(\omega), \, \sigma_1(\omega), \, \ldots) \tag{3.11}$$

by setting

$$\sigma_i(\omega) = \omega_0 \cdot 10^i + \omega_1 \cdot 10^{i-1} + \cdots + \omega_{i-1} \cdot 10^1 + \omega_i \cdot 10^0.$$
 (3.12)

It is plain that $\omega \neq \omega'$ implies $\sigma(\omega) \neq \sigma(\omega')$. Also, it is clear that $\sigma_0(\omega) = \omega_0 \geq 2$ and that $\sigma_{i+1}(\omega) > \sigma_i(\omega)$, $i \geq 0$. Moreover, if $\{\sigma^{(1)}, \ldots, \sigma^{(n)}\}$ is a finite subset of Σ with $\sigma^{(l)} = \sigma(\omega^{(l)})$, then there exists $i \geq 0$ such that for $k \geq i$ the elements $\sigma_k^{(1)}, \ldots, \sigma_k^{(n)}$ are different. We may thus set $\Sigma = \{\sigma(\omega) \mid \omega \in \Omega\}$.

Proof (of Theorem 3.6). Let F = F(X) where

$$X = \{x_{01}, x_{02}, \dots, x_{11}, x_{12}, \dots, x_{21}, x_{22}, \dots\}$$
(3.13)

Define for any sequence $\sigma = (\sigma_0, \sigma_1, \ldots) \in \Sigma$ an element

$$\Lambda(\sigma) = (\lambda_1 F_2, \lambda_2 F_3, \ldots) \in \hat{F}$$
(3.14)

by setting

$$\begin{cases} \lambda_{i} = e & \text{for } i < \sigma_{0}, \\ \lambda_{i} = [x_{01}, \dots, x_{0\sigma_{0}}][x_{11}, \dots, x_{1\sigma_{1}}] \cdots [x_{l1}, \dots, x_{l\sigma_{l}}] & \text{for } \sigma_{l} \leq i < \sigma_{l+1}. \end{cases}$$
(3.15)

Note that $\Lambda(\sigma)$ is an element of \hat{F} since σ is strictly increasing. Next we shall show that the elements $\Lambda(\sigma)$, $\sigma \in \Sigma$ generate a free subgroup of \hat{F} . For this it is enough to show that any *finite* set of elements

$$\Lambda^{(l)} = \Lambda(\sigma^{(l)}), \qquad 1 \leq l \leq n$$

freely generate a free subgroup. By Lemma 3.7 we may conclude that there exists an $i \ge 0$ such that the entries $\sigma_i^{(1)}, \ldots, \sigma_i^{(n)}$ are all different. We now consider the projection $p: F(X) \to F(x_{i1}, x_{i2}, x_{i3}, \ldots)$; then p induces a map $\hat{p}: \hat{F}(X) \to \hat{F}(x_{i1}, x_{i2}, x_{i3}, \ldots)$ with

$$\hat{p}(\Lambda(\sigma^{(l)})) = [x_{i1}, \ldots, x_{i\sigma_i^{(l)}}] \in F(x_{i1}, x_{i2}, x_{i3}, \ldots) \subseteq \hat{F}(x_{i1}, x_{i2}, x_{i3}, \ldots)$$
(3.16)

It follows at once that $\hat{p}(\Lambda(\sigma^{(l)}))$ freely generate a free subgroup of $F(x_{i1}, x_{i2}, \ldots)$. Hence the elements $\Lambda(\sigma^{(l)})$, $1 \le l \le n$ freely generate a free subgroup of $\hat{F}(X)$.

It remains to show that the elements $\Lambda(\sigma)$, $\sigma \in \Sigma$ are linearly independent mod \hat{F}_2 . For this we consider the group $W = IA \ 1$ A where A is the free abelian group on X and the map $f: F \to W$ defined by

$$f(x_{ik}) = \begin{cases} (x_{i1} - 1) \in IA \subseteq W & \text{for } k = 1, & i \ge 0, \\ x_{ik} \in A \subseteq W & \text{for } k \ge 2, & i \ge 0. \end{cases}$$

$$(3.17)$$

For the induced map $\hat{f}:\hat{F}\to\hat{W}$ we then obtain

$$\hat{f}(\Lambda(\sigma)) = (x_{01} - 1) \cdot \cdot \cdot (x_{0\sigma_0} - 1) + (x_{11} - 1) \cdot \cdot \cdot (x_{1\sigma_1} - 1) + \cdot \cdot \cdot . \tag{3.18}$$

It follows from Lemma 3.3 that $\hat{f}(\Lambda(\sigma)) \notin \hat{W}_2$. Moreover no non-trivial linear

combination of elements $\hat{f}(\Lambda(\sigma))$ lies in \hat{W}_2 . For, if $\sigma^{(1)}, \ldots, \sigma^{(n)}$ are different elements of Σ then, by Lemma 3.7, there exists $i \ge 0$ such that for all $k \ge i$ the entries $\sigma_k^{(1)}, \ldots, \sigma_k^{(n)}$ are different. Hence a nontrivial linear combination of $\hat{f}(\Lambda(\sigma^{(1)})), \ldots, \hat{f}(\Lambda(\sigma^{(n)}))$ cannot lie in \hat{W}_2 .

We note that in the course of the above proof we also have proved the following result

COROLLARY 3.8. Let F = F(X) be a free group where X is countably infinite. Then \hat{F}_{ab} is uncountably infinite.

We next prove

THEOREM 3.9. Let F = F(X) be a free group where X is countably infinite. Then \hat{F}_{ab} contains uncountably many linearly independent divisible elements.

Proof. Let $X = \{x_{01}, x_{02}, \dots, x_{11}, x_{12}, \dots, x_{21}, x_{22}, \dots\}$ and consider for $\sigma \in \Sigma$ (Lemma 3.7) the element

$$\Gamma(\sigma) = (\gamma_1 F_2, \gamma_2 F_3, \ldots) \in \hat{F}$$
(3.19)

defined by

$$\begin{cases} \gamma_{i} = e & \text{for } i < \sigma_{2}, \\ \gamma_{i} = ([x_{21}, \dots, x_{2\sigma_{2}}]([x_{31}, \dots, x_{3\sigma_{3}}] \cdots [x_{l1}, \dots, x_{l\sigma_{l}}]^{l \cdots})^{3})^{2} & \text{for } \sigma_{l} \leq i < \sigma_{l+1}. \end{cases}$$
(3.20)

(The fact that we start with σ_2 is merely a notational convenience.) We shall show that each $\Gamma(\sigma)$ gives rise to a divisible element in \hat{F}_{ab} . We first recall that F_{ab} is a direct summand of \hat{F}_{ab} . Thus in order to exhibit divisible elements in \hat{F}_{ab} we may consider the quotient of \hat{F}_{ab} by F_{ab} , in other words we may consider the quotient of \hat{F} by the normal subgroup N generated by F and \hat{F}_2 . In order to show that for $k \ge 2$ the element $\Gamma(\sigma)$ is a k-th power modulo N we consider the element

$$\Delta = \Delta(\sigma) = (\delta_1 F_2, \delta_2 F_3, \ldots) \in \hat{F}$$
(3.21)

where

$$\begin{cases} \delta_{i} = e & \text{for } i < \sigma_{k} \\ \delta_{i} = ([x_{k1}, \dots, x_{k\sigma_{k}}]([x_{l1}, \dots, x_{l\sigma_{l}}]^{l \dots})^{k+1})^{k!} & \text{for } \sigma_{i} \leq i < \sigma_{l \times 1}, \ l \geq k. \end{cases}$$
 (3.22)

Clearly, modulo N the elements $\Gamma(\sigma)$ and $\Delta(\sigma)$ are equivalent. But $\Delta(\sigma)$ is a k-th power. Hence $\Gamma(\sigma)$ is a k-th power modulo N. An argument similar to the one used in the proof of Theorem 3.6 shows that the elements $\Gamma(\sigma)$, $\sigma \in \Sigma$ are linearly independent in \hat{F}_{ab} .

COROLLARY 3.10. Let F = F(X) be a free group, where X is countably infinite. Then \hat{F}_k/\hat{F}_{k+1} , $k \ge 1$ contains non-trivial divisible elements.

Proof. Let $X = \{x_{01}, x_{02}, \dots, x_{11}, x_{12}, \dots, x_{21}, x_{22}, \dots\}$ and let $\Gamma = \Gamma(\sigma) \in \hat{F}$ be one of the elements defined in (3.19), (3.20). We consider the element

$$\Gamma^* = [\Gamma, x_{01}, x_{02}, \dots, x_{0,k-1}] \in \hat{F}_k. \tag{3.23}$$

Since the k-fold commutator is a linear map from the k-fold tensor product of \hat{F}_{ab} into \hat{F}_k/\hat{F}_{k+1}

$$[\cdot\cdot\cdot]:\hat{F}_{ab}\otimes\cdot\cdot\cdot\otimes\hat{F}_{ab}\to\hat{F}_{k}/\hat{F}_{k+1} \tag{3.24}$$

the element Γ^* gives certainly rise to a divisible element in \hat{F}_k/\hat{F}_{k+1} . It remains to prove that Γ^* is non-trivial. We use the map $f: F \to W$ defined by

$$\begin{cases}
f(x_{0l}) = x_{0l} \in A \subseteq W, & l \ge 1. \\
f(x_{il}) = x_{il} \in A \subseteq W, & i \ge 1, & l \ge 2. \\
f(x_{i1}) = (x_{i1} - 1) \in IA \subseteq W, & i \ge 1.
\end{cases}$$
(3.25)

For the induced map $\hat{f}:\hat{F}\to\hat{W}$ we obtain

$$\hat{f}(\Gamma^*) = [2(x_{21}-1)\cdots(x_{2\sigma_2}-1)+3!(x_{31}-1)\cdots(x_{3\sigma_3}-1)+\cdots](x_{01}-1)\cdots(x_{0,k-1}-1)$$
(3.26)

By Lemma 3.3, $\hat{f}(\Gamma^*) \notin \hat{W}_{k+1}$ and hence $\Gamma^* \notin \hat{F}_{k+1}$.

The following result should be compared with Theorem 4.2 of [B2].

THEOREM 3.11. The 2-generator subgroups of \hat{F} are free.

Proof. Let $a, b \in \hat{F} = \hat{F}(X)$. We have to consider two cases.

(i) Let $[a, b] \neq e$. In this case we shall deduce our result from Theorem 4.2 of [B2]. Clearly there exists $i \geq 1$ such that for $\tau_i : \hat{F} \to F/F_i$ we have $\tau_i[a, b] \neq e$. Consider then a finite subset $Y \subseteq X$ and the projection $p : F(X) \to F(Y)$, such that for $p\tau_i : \hat{F}(X) \to F(Y)/F_i(Y)$ we have $[p\tau_i a, p\tau_i b] \neq e$. We may thus consider $\hat{p} : \hat{F}(X) \to \hat{F}(Y)$. Since $\hat{F}(Y)$ is parafree and $[pa, pb] \neq e$ we may conclude from

Theorem 4.2 of [B2] that $\hat{p}a$, $\hat{p}b$ generate a free subgroup. Hence a, b generate a free subgroup in $\hat{F}(X)$.

(ii) Let [a, b] = e. We shall proceed as in the proof of Theorem 4.2 of [B2]. By Theorem 3.1 of [B2] the group $\hat{F} = \hat{F}(X)$ can be embedded in the power series ring $\mathbb{Z}[[X]]$ and hence in $\mathbb{Q}[[X]]$. Under that embedding let

$$\begin{cases}
a = 1 + a_i + \cdots, & a_i \neq 0 \\
b = 1 + b_i + \cdots, & b_j \neq 0
\end{cases}$$
(3.27)

i.e. a_i , b_j are the first non-zero terms in the power series corresponding to a, b. The elements a_i , b_j are Lie elements in $\mathcal{L}[X]$, the Lie algebra over \mathbb{Q} generated by X in $\mathbb{Q}[X]$. Since $\mathcal{L}[X]$ is a free Lie algebra over \mathbb{Q} it follows that a_i , b_j generate a free sub- \mathbb{Q} -Lie algebra (see Sirsov [S], Witt [W]). Since [a, b] = e we have in $\mathbb{Q}[[X]]$

$$ab = 1 + a_i + b_j + a_i b_j + \cdots = 1 + a_i + b_j + b_j a_i + \cdots = ba.$$
 (3.28)

Hence in $\mathcal{L}[X]$ we have $[a_i, b_j] = 0$, so that the Lie algebra generated by a_i, b_j is abelian. Since it is free it must be isomorphic to \mathbb{Q} . It follows that there exist integers m, n > 0 such that

$$ma_i = nb_i. (3.29)$$

In particular $i = \deg a_i = \deg b_i = j$. We now compute $c = a^{-m} \cdot b^n$ in $\mathbb{Z}[[X]]$.

$$c = a^{-m}b^{n} = (1 - ma_{i} + \cdots)(1 + nb_{j} + \cdots)$$

$$= 1 - ma_{i} + nb_{j} + \{\text{terms of degree} \ge i + 1\}$$

$$= 1 + \{\text{terms of degree} \ge i + 1\}.$$
(3.30)

But c, being an element in the subgroup generated by a, b commutes with a and b and hence satisfies $[a, c] = e \cdot A$ repetition of the above argument with c at the place of b shows that the power series expansion of c has the form

$$c = 1 + c_k + \cdots, \qquad c_k \neq 0 \tag{3.31}$$

with k = i. This is a contradiction to (3.30) so that c = 1, i.e. $a^m = b^n$. Since \hat{F} is torsion-free it follows that the subgroup generated by a, b is infinite cyclic, hence free.

4. Two subgroups of \hat{F}

Let F = F(X) be the free group on the set X. Here we shall exhibit two subgroups \bar{F} , \tilde{F} of $\hat{F} = \hat{F}(X)$ which are parafree on X. Note that \hat{F} is not itself parafree on X if X is at least countably infinite.

THEOREM 4.1. There exists a subgroup $\bar{F} \subseteq \hat{F} = \hat{F}(X)$ which is parafree on X and has the property that it contains all subgroups of \hat{F} which are parafree on X.

Proof. We first show that the system of subgroups of \hat{F} which are parafree on X is directed. Thus let U, V be two subgroups of \hat{F} which are parafree on X. Let W be the subgroup of \hat{F} generated by U, V. We claim that W is parafree on X. Since $F \subseteq W \subseteq \hat{F}$ we may apply Lemma 1.1, so that we have to show that $F_{ab} \to W_{ab}$ is surjective. Consider the obvious epimorphism $U_{*_F}V \twoheadrightarrow W$. We then have the series of maps

$$F_{ab} \xrightarrow{\sim} U_{ab} \rightarrow (U_{*v}V)_{ab} \rightarrow W_{ab}$$
 (4.1)

so that we only have to show that $U_{ab} \to (U_{*_F}V)_{ab}$ is surjective. But this is trivial since U, V are parafree on X. We may thus define \bar{F} by $\bar{F} = \lim_{\to} U$, where U is a subgroup of \hat{F} which is parafree on X. It is then clear that $F_{ab} \to \bar{F}_{ab}$ is surjective, so that \bar{F} is parafree on X, by Lemma 1.1.

It is obvious that the above construction is independent of the fact that the group we start with is free. Thus if G is an arbitrary group we may find in \hat{G} a group \bar{G} with $hG \subseteq \bar{G} \subseteq \hat{G}$ such that h induces isomorphisms $h_i: G/G_i \cong \bar{G}/\bar{G}_i$, $i \ge 2$, and with the following universal property. If $f: G \to H$ is a homomorphism such that $f_i: G/G_i \cong H/H_i$, $i \ge 2$ and H is residually nilpotent, then there exists a unique $\bar{f}: H \to \bar{G}$ such that $\bar{f} \circ f = h: G \to \bar{G}$. By construction of \bar{G} we have

COROLLARY 4.2. $\bar{\bar{G}} = \bar{G}$.

As in Corollary 2.6 we obtain from Corollary 4.2

COROLLARY 4.3. Let F be free, then \overline{F} is freely indecomposable.

We shall now construct another subgroup \tilde{F} of $\hat{F} = \hat{F}(X)$ which is parafree on X. Thus we have $\tilde{F} \subseteq \tilde{F}$, by Theorem 4.1, but we shall later show that $\tilde{F} \neq \bar{F}$ if X is (at least) countably infinite.

Consider the directed system of finite subsets Y of X and the associated

directed system of groups $\hat{F}(Y)$. Define

$$F = \lim_{\longrightarrow} \hat{F}(Y), \qquad Y \subseteq X, Y \text{ finite}$$
 (4.2)

If we consider $\hat{F}(X)$ as a subgroup of the power series ring $\mathbb{Z}[[X]]$ (see Theorem 3.1 of [B2]), then \tilde{F} may be described as the subgroup of those power series of \hat{F} which involve only finitely many elements of X.

PROPOSITION 4.4. The group \tilde{F} is parafree on X.

Proof. Clearly $F \subseteq \tilde{F} \subseteq \hat{F}$. Since $\tilde{F}_{ab} = \lim_{\to} (\hat{F}(Y))_{ab} = \lim_{\to} (F(Y))_{ab}$ is free abelian on X, the map $F_{ab} \to \tilde{F}_{ab}$ is surjective. By Lemma 1.1 we conclude that \tilde{F} is parafree on X.

PROPOSITION 4.5. Let F = F(X) where X is countably infinite. Then \tilde{F} is a proper subgroup of \bar{F} .

Proof. We shall exhibit a subgroup U of \hat{F} which is parafree on X, but not contained in \tilde{F} . Since $U \subseteq \bar{F}$ by Theorem 4.1 it then follows that $\tilde{F} \neq \bar{F}$.

Let $X = \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$. Define elements

$$Z^{(i)} = (\zeta_1^{(i)} F_2, \zeta_2^{(i)} F_3, \ldots) \in \hat{F}, \qquad i = 0, 1, \ldots$$
(4.3)

by setting

$$\begin{cases} \zeta_1^{(i)} = x_i, \\ \zeta_2^{(i)} = [x_{i+1}, y_{i+1}]x_i, \\ \zeta_3^{(i)} = [[x_{i+2}, y_{i+2}]x_{i+1}, y_{i+1}]x_i, \text{ etc.} \end{cases}$$

$$(4.4)$$

where $x_0 = e$. Modulo any F_k and hence in \hat{F} we have

$$Z^{(i)} = [Z^{(i+1)}, y_{i+1}]x_i, \qquad i = 0, 1, \dots$$
 (4.5)

Consider now the subgroup U of \hat{F} generated by $x_1, x_2, \ldots, y_1, y_2, \ldots, Z^{(0)}, Z^{(1)}, Z^{(2)}, \ldots$. We claim that U is parafree on X. By Lemma 1.1 we only have to show that $F_{ab} \to U_{ab}$ is surjective. But it is clear that

$$\begin{cases}
Z^{(0)} \equiv e \mod U_2, \\
Z^{(i)} \equiv x_i \mod U_2, & i \ge 1.
\end{cases}$$
(4.6)

Finally it is plain that none of the elements $Z^{(i)}$ is contained in \tilde{F} .

BIBLIOGRAPHY

- [BG] F. BACHMANN, L. GRÜNENFELDER, Ueber Lie-Ringe von Gruppen und ihre universelle Enveloppen, Comment. Math. Helv. 47, (1972), 332-340.
- [B1] G. BAUMSLAG, Groups with the same lower central sequence as a relatively free group. I. The groups, Trans. Amer. Math. Soc. 129 (1967), 308-321.
- [B2] ——, Groups with the same lower central sequence as a relatively free group. II. Properties. Trans. Amer. Math. Soc. 142 (1969), 507-538.
- [BK] A. K. Bousfield, D. M. Kan, Homotopy Limits, Completions and Localizations. Lecture Notes in Mathematics, vol. 304; Springer 1972.
 - [S] A Sirsov, Unteralgebren von freien Liealgebren (russisch), Math. Sbornik N.S. 33 (75), (1953), 441-452.
- [W] E. WITT, Die Unterringe der freien Lieschen Ringe, Math. Z. 64 (1956), 195-216.

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