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A long homology localization tower

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1. Introduction

Let R be fixed as a subring of the rational numbers or a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, p prime. The purpose of this paper is to give a new description of the *R-homology localization* X_R of a space X [1]. The main ingredient is an *inverse limit* construction for X_R (complementary to Bousfield's *direct limit* construction [1, 11.5]) which is obtained by transfinitely iterating the *R-nilpotent completion* process of [3]. Thus one immediate benefit is a clearer understanding of the relationship between X_R and the R-nilpotent completion $R_\infty X$ of X.

A space X is said to be *R-Bousfield* if X_R is homotopy equivalent to X. The possibility of two constructions for X_R is suggested by the fact that the natural map $X \to X_R$ has two universal properties:

(i) $X \to X_R$ is terminal, up to homotopy, in the category of all maps $X \to Y$ which induce isomorphisms $H_*(X; R) \approx H_*(Y; R)$.

(ii) $X \to X_R$ is initial, up to homotopy, in the category of all maps $X \to Y$ which have an R-Bousfield target space Y.

In order to exploit property (ii) effectively, it is necessary to study the

1.1. *Structure of R-Bousfield Spaces.* For each ordinal $\alpha \geq 0$, let I_α be the class of R-Bousfield spaces defined inductively as follows.

(i) I_0 contains all fibrant spaces with the property that each connected component has the homotopy type of a simplicial R-module.

(ii) I_α ($\alpha > 0$) contains all fibrant spaces which are of the homotopy type of $\operatorname{holim}_{\leftarrow} D$ [3, p. 295], where D is a small diagram of spaces, each of which belongs to I_β for some $\beta < \alpha$.

The spaces in I_0 are R-Bousfield by [1, §4] and, inductively, the spaces in I_α ($\alpha > 0$) are R-Bousfield by [1, §12]. Using [2] it is not hard to prove (see §5)

1.2. PROPOSITION. *If X is an R-Bousfield space, then $X \in I_\alpha$ for some ordinal α.*

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1.3. *The Long Tower.* Let Ω be the opposite category of the category of all ordinals, that is, Ω is the category with one object for each ordinal α and one morphism $\beta \to \alpha$ for each $\beta \geq \alpha$. A *long tower* in a category C is a functor $F: \Omega \to \mathrm{C}$, usually written $\{F(\alpha)\}_\alpha$. The long tower is said to be *augmented* by the object X of C if there are compatible maps $X \to F(\alpha)$, $\alpha \in \Omega$.

For any space X we will construct a natural *long R-homology localization tower* $\{T_\alpha X\}_\alpha$ of spaces, naturally augmented by X, such that

1.4. (i) *if $f: X \to Y$ induces an isomorphism $H_*(X; R) \to H_*(Y; R)$ then f induces homotopy equivalences $T_\alpha X \sim T_\alpha Y$,*
(ii) *for any X or α, $T_\alpha X \in I_\alpha$, and*
(iii) *if $X \in I_\alpha$, then the natural map $X \to T_{\alpha+1}X$ is a homotopy equivalence.*

In view of 1.2, these properties imply that

(iv) *for each space X there is some ordinal α such that for all $\beta > \alpha$, the map $X \to T_\beta X$ is up to homotopy the R-homology localization map $X \to X_R$.*

In fact, α can be chosen to be any ordinal such that $X_R \in I_\alpha$. Thus the point at which the localization tower finally stabilizes for a given X depends explicitly on the minimal number of homotopy inverse limits needed to construct X_R from the spaces in I_0 (that is, from disjoint unions of products of R-module Eilenberg-MacLane spaces [6, 24.5]). This ordinal is an intrinsic measure of the homotopical complexity of X_R or of the homological complexity of X itself.

1.5. *Relationship to the R-completion.* The first few spaces in the tower $\{T_\alpha X\}_\alpha$ appear at least implicitly in [3]. The space $T_0 X$ is exactly RX [3, p. 14], $T_1 X$ is homotopy equivalent to $R_\infty X$, and $T_2 X$ is homotopy equivalent to the homotopy inverse limit of the cosimplicial resolution of X [3, p. 20] constructed using the triple structure of R_∞ [3, p. 26]. The whole tower $\{T_\alpha X\}_\alpha$ is obtained by imitating the process of passing from RX to $R_\infty X$ at successor ordinals and taking inverse limits at limit ordinals (see §6). The main technical innovation is the substitution of *augmented functors* (§3) for *triples* [3, p. 13].

This paper was inspired by Bousfield's algebraic work in [2], but, although we use his results heavily, our constructions do not seem to be related in a simple way to his.

1.6. *Organization of the Paper.* Section 2 gives a simplified outline of our general approach. Section 3 contains some preparatory material of a category-theoretical nature; Section 4 gives a generalization to transfinite towers of a result which is well known for towers indexed by the natural numbers; and Section 5 presents a proof of 1.2. Section 6 contains the construction of the tower $\{T_\alpha X\}_\alpha$ and the proof of its properties; Section 7 has an inductive "Artin-Mazur-like" interpretation of the functors T_α, and the final section contains some examples.

1.7. *Notation and Terminology.* Although our arguments are not usually combinatorial, the word *space* is used as a synonym for *simplical set* ([3, VIII], [6], [7]); S denotes the category of spaces. For convenience we will sometimes use the terminology of homotopical algebra [7: I, 1.1 and II, 3.14]; for instance, a *cofibration* is an injection of simplicial sets, a *fibration* is a Kan fibration, and a space X is *fibrant* if the unique map of X to the one-point space is a fibration, i.e., if X satisfies the Kan extension condition.

2. Outline of the proof

This section presents the main arguments of the paper in a schematic setting in which most of the technicalities disappear. We hope that this will help the reader to catch sight of the underlying simplicity of the basic ideas.

Warning! This section is independent of the rest of the paper in notation and terminology.

Let C be a category closed under inverse limits. An *augmented functor* (T, ϕ) on C is a functor $T: C \to C$ together with a natural transformation $\phi: 1_C \to T$.

2.1. *An Equalizer Construction.* Let (T, ϕ) be an augmented functor on C. For $X \in C$, let $T^\wedge(X)$ denote the equalizer of the two maps $\phi(TX)$, $T(\phi(X)): TX \to T^2X$, that is, let $T^\wedge(X)$ be the inverse limit of the diagram

$$TX \underset{\phi(TX)}{\overset{T(\phi(X))}{\rightrightarrows}} T^2X$$

The construction of $T^\wedge(X)$ is a functorial in X; moreover, since $(\phi T) \circ \phi = T(\phi) \circ \phi$, $T^\wedge(X)$ comes equipped with a natural map $\phi^\wedge(X): X \to T^\wedge(X)$ such that the obvious diagram

$$\begin{array}{ccc} & \overset{\phi^\wedge(X)}{\nearrow} & T^\wedge(X) \\ X & & \downarrow \\ & \underset{\phi(X)}{\searrow} & T(X) \end{array}$$

commutes.

2.2. *Collapse Lemma* (cf. 3.6). *If $\phi(X): X \to TX$ has a left inverse, then the map $\phi^\wedge(X): X \to T^\wedge(X)$ is an isomorphism.*

Proof. Let $s: TX \to X$ be a left inverse for $\phi(X)$, and let $i: T^{\wedge}(X) \to TX$ be the natural map. Then it is easy to see that $s \circ i$ is a two-sided inverse for $\phi^{\wedge}(X)$.

2.3. *Bousfield Objects.* Let E be a distinguished class of morphisms of C called *equivalences* (or *homology equivalences*). An object $Z \in \mathrm{C}$ is said to be *Bousfield* if any equivalence $f: X \to Y$ induces a bijection $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. The class of Bousfield objects is closed under inverse limits. A localization map for $X \in \mathrm{C}$ is a map $e: X \to Z$ such that e is an equivalence and Z is Bousfield. Such a Z is called a *localization* of X; if one exists, it is unique up to a canonical isomorphism.

2.4. *Assumption. Every object $X \in \mathrm{C}$ has a localization.*

Suppose that I_0 is some naturally given class of basic Bousfield objects. By induction, for each ordinal $\alpha > 0$ let I_α denote the class of all objects which can be written up to isomorphism as $\lim_{\leftarrow} D$ where D is a small diagram of objects in C each of which belongs to I_β for some $\beta < \alpha$.

2.5. *Assumption* (cf. 1.2). *For any Bousfield object $X \in \mathrm{C}$ there is an ordinal β such that $X \in I_\beta$.*

2.6. *A Long Localization Tower* (cf. §6). Suppose (R, ϕ) is an augmented functor on C such that

(i) for any $X \in \mathrm{C}$, RX is Bousfield, and

(ii) if $f: X \to Y$ is an equivalence, then $Rf: RX \to RY$ is an isomorphism. Thus R fails to be a localization functor only because $\phi(X): X \to RX$ need not be an equivalence. Suppose that (R, ϕ) satisfies the additional restriction

(iii) if $X \in I_0$, then the natural map $\phi(X): X \to RX$ has a left inverse. Define a long tower $\{(T_\alpha, \phi_\alpha)\}_\alpha$ of augmented functors by transfinite induction as follows. The pair (T_0, ϕ_0) is (R, ϕ). If $\alpha = \beta + 1$ is a successor ordinal, then (T_α, ϕ_α) is $(T_\beta^{\wedge}, \phi_\beta^{\wedge})$. If α is a limit ordinal, then (T_α, ϕ_α) is the inverse limit of (T_β, ϕ_β) over all ordinals $\beta < \alpha$.

2.7. PROPOSITION. *For any $X \in \mathrm{C}$ there is some ordinal α such that for all $\beta > \alpha$ the natural map $\phi_\beta(X): X \to T_\beta X$ is a localization map.*

Proof. Let $e: X \to Z$ be a localization map for X. It follows from 2.6(ii) that e induces isomorphisms $T_\alpha X \to T_\alpha Z$ for all ordinals α; thus it suffices to show that there is some ordinal α such that for all $\beta \geq \alpha$ the map $Z \to T_\beta Z$ is an isomorphism. This follows from 2.5 and

2.8. LEMMA. *If* $Z \in T_\alpha$, *then* $\phi_\beta(Z): Z \to T_\beta Z$ *is an isomorphism for all* $\beta \geq \alpha + 1$.

Proof of Lemma. The technique is transfinite induction on α. The case $\alpha = 0$ follows from 2.6(iii) and 2.2. Pick $X \in I_\alpha$, $\alpha > 0$, with $X = \lim_{\leftarrow} D$, where D is some diagram of objects each of which belongs to I_β for some $\beta < \alpha$. Consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & T_\alpha X \\ \| \downarrow & & \downarrow \\ \varprojlim D & \longrightarrow & \varprojlim T_\alpha D \end{array}$$

where the horizontal maps are induced by ϕ_α. The induction hypothesis shows that the lower horizontal arrow is an isomorphism, since $\phi_\alpha(Y): Y \to T_\alpha Y$ is an isomorphism for each object Y in the diagram D. Thus $\phi_\alpha(X): X \to T_\alpha X$ has a left inverse, and the inductive step follows from 2.2.

2.9. *Remark.* The above program does not lead to an inverse limit construction for homology localizations only because the homotopy category of the category S of spaces is not closed under inverse limits. This paper is guided around that obstacle by the principle that the notion of *homotopy inverse limit* [3, XI] in S provides a natural substitute for the missing notion of inverse limit in HoS.

3. Categorical preliminaries

3.1. *Restricted Cosimplicial Spaces.* A *restricted cosimplicial space* $\mathbf{X}$ is a "cosimplicial space without codegeneracies," that is, $\mathbf{X}$ consists of

(i) for each integer $n \geq 0$ a space $\mathbf{X}^n$, and
(ii) for each pair (i, n) of integers with $0 \leq i \leq n$ *coface* maps

$$d^i : X^{n-1} \to X^n$$

such that $d^j d^i = d^i d^{j-1}$ if $i < j$ [3, p. 267].

The object $\mathbf{X}$ is said to be *augmented* by $\mathbf{X}^{-1}$ if there is a map $d^0 : \mathbf{X}^{-1} \to \mathbf{X}^0$ such that $d^0 d^0 = d^1 d^0 : \mathbf{X}^{-1} \to \mathbf{X}^1$.

In the same way in which cosimplicial spaces are associated to *triples* on S [3, pp. 20, 323], restricted cosimplicial spaces are associated to

3.2. *Augmented Functors.* An *augmented functor* (T, ϕ) on S is a pair in which $T: S \to S$ is a functor and $\phi: 1_S \to T$ is a natural transformation. The fact that ϕ is a natural transformation implies that $(T(\phi)) \circ \phi = (\phi T) \circ \phi$.

Let (T, ϕ) be an augmented functor on S and let $X \in S$. The *restricted cosimplicial resolution* of X with respect to T is the augmented restricted cosimplicial space $\mathbf{T}X$ given by

$$(\mathbf{T}X)^k = T^{k+1}X$$

in codimension k, and

$$((\mathbf{T}X)^{k-1} \xrightarrow{d^i} (\mathbf{T}X)^k) = (T^k X \xrightarrow{T^i \phi T^{k-i}} T^{k+1}X).$$

3.3. *T-Completions.* Let Δ_{rest} denote the *restricted simplicial category*, that is, the category whose objects are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ and whose morphisms are strictly monotone maps. The restricted cosimplicial space $\mathbf{T}X$ (without its augmentation) can be thought of as a functor

$$\mathbf{T}X: \Delta_{\text{rest}} \to S.$$

The *T-completion* of X, denoted $T^\wedge(X)$, is defined to be the homotopy inverse limit of $\mathbf{T}X$ [3, p. 295]:

$$T^\wedge(X) = \operatorname{holim}_{\leftarrow} \mathbf{T}X.$$

The augmentation $\phi(X): X \to TX = (\mathbf{T}X)^0$ induces a natural map $\phi^\wedge(X): X \to T^\wedge(X)$. There is also a natural map $T^\wedge(X) \to T(X)$ which induces a morphism $(T^\wedge, \phi^\wedge) \to (T, \phi)$ of augmented functors.

3.4. LEMMA. *If the spaces $T^n X$ $(n \geq 1)$ are fibrant, then the natural map $T^\wedge(X) \to TX$ is a fibration.*

This is proved below in 3.11.

3.5. *A Collapse Criterion.* Let (T, ϕ) be an augmented functor on S. It is useful to have a criterion that guarantees, for a given $X \in S$, that the map $\phi^\wedge(X): X \to T^\wedge(X)$ is a homotopy equivalence.

3.6. *Collapse Lemma. Suppose that $T^n X$ is fibrant for all $n \geq 1$ and that the* *natural map $\phi(X): X \to TX$ has a left inverse. Then the completion map $\phi^{\wedge}(X): X \to T^{\wedge}(X)$ is a homotopy equivalence.*

3.7. *Relationship to Cosimplicial Constructions.* Let Δ denote the full simplicial category, that is, the category whose objects are the same as those of Δ_{rest}, but whose morphisms are all weakly monotone maps. If (T, ϕ, ψ) is a *triple* or *monad* on the category of spaces [3, p. 13], Bousfield and Kan associate to any space X a *cosimplicial resolution* with respect to T [3, pp. 20, 323]; this is a *cosimplicial space*, or, equivalently, a functor

$$\mathbf{T}^*(X): \Delta \to S.$$

Let (T, ϕ) be the underlying augmented functor of (T, ϕ, ψ), and let $J: \Delta_{\text{rest}} \to \Delta$ be the obvious inclusion functor. For any X there is a commutative diagram

$$\begin{array}{ccc} \Delta_{\text{rest}} & \xrightarrow{J} & \Delta \\ & {\scriptstyle \mathbf{T}X}\searrow \quad \swarrow{\scriptstyle \mathbf{T}^*(X)} & \\ & \mathrm{S} & \end{array}$$

which induces a natural map [3, p. 316]

$$\operatorname{holim}_{\leftarrow} \mathbf{T}^*(X) \to \operatorname{holim}_{\leftarrow} \mathbf{T}X = T^{\wedge}(X).$$

3.8. LEMMA. *If $T^n X$ is fibrant for all $n \geq 1$, then the map $\operatorname{holim}_{\leftarrow} \mathbf{T}^*(X) \to T^{\wedge}(X)$ is a homotopy equivalence.*

If (T, ϕ, ψ) is a triple, let $T_\infty X$ denote $\operatorname{Tot}(\mathbf{T}^*(X))$ [3, p. 17].

3.9. COROLLARY. *If $\mathbf{T}^*(X)$ is a fibrant cosimplicial space* [3, p. 275], *there is a natural homotopy equivalence $T_\infty X \to T^{\wedge}(X)$.*

This follows from [3, p. 300].

The rest of this section contains the proofs of 3.4, 3.6 and 3.8.

3.10 *The Over Category.* Suppose that C and D are categories, and that $J: \mathrm{C} \to \mathrm{D}$ is a functor. For each $d \in \mathrm{D}$ the *over category* J/d is defined as having one object for each pair (c, f) where $c \in \mathrm{C}$ and $f \in \operatorname{Hom}_{\mathrm{D}}(J(c), d)$, and one morphism

$(c, f) \to (c', f')$ for each $g \in \mathrm{Hom}_C(c, c')$ such that

$$\begin{array}{ccc} J(c) & \xrightarrow{J(g)} & J(c') \\ & {}_f\searrow \quad \swarrow_{f'} & \\ & d & \end{array}$$

commutes. The composition rule in J/d is induced by the rule in C.

If C is small, then J/d is small and thus has a *nerve* or *spatial realization*, which is a space also denoted by J/d [3, p. 29]. In general we will make no notational distinction between a small category and its spatial realization. For instance, if C is a small category and $c \in C$, C/c will denote (the spatial realization of) the category $1_C/c$, where 1_C is the identity functor on C. Similarly, $C/-$ will denote the functor $C \to S$ which assigns to each object $c \in C$ the space C/c.

3.11. *Proof of* 3.4. Let $V(n, k)$ denote the space formed by the boundary of the standard n-simplex with the k'th face deleted [7, II, 2.1]. There is a cofibration $V(n, k) \to \Delta[n]$, where $\Delta[n]$ is the standard n-simplex itself. To prove the lemma it is enough to show that the dotted arrow exists in any diagram of the form

$$\begin{array}{ccc} V(n, k) & \xrightarrow{f} & T^{\wedge}(X) \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & T(X). \end{array}$$

By an adjointness argument [3, p. 296] this comes down to showing that there is a map $g : \Delta[n] \times (\Delta_{\mathrm{rest}}/-) \to \mathbf{T}X$ which extends both the given map

$$\Delta[n] = \Delta[n] \times (\Delta_{\mathrm{rest}}/[0]) \to TX$$

and the map

$$f' : V(n, k) \times (\Delta_{\mathrm{rest}}/-) \to \mathbf{T}X$$

which is adjoint to f.

The map g is built up by using induction on m to construct its components

$$g_m : \Delta[n] \times (\Delta_{\mathrm{rest}}/[m]) \to T^{m+1}X.$$

The map g_0 is given. The space $\Delta_{\mathrm{rest}}/[m]$ is the first barycentric subdivision $sd\Delta[m]$ of the standard m-simplex and the prescription of g_{m-1} determines the restriction of g_m to $\Delta[n]\times sd(\dot{\Delta}[m])$, where $\dot{\Delta}[m]$ is the boundary of the m-simplex. In addition, the requirement that g extend f' determines the restriction of g_m to $V(n,k)\times sd\Delta[m]$. Thus g_m must be constructed as the dotted arrow in a diagram of the form

$$\begin{array}{ccc} (V(n,k)\times sd\Delta[m])\cup(\Delta[n]\times sd\dot{\Delta}[m]) & \longrightarrow & T^{m+1}X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]\times sd\Delta[m] & \longrightarrow & * \end{array}$$

where $*$ is a one-point space. The existence of such a dotted arrow follows from the fact that the right vertical arrow is a fibration while the left vertical arrow is a cofibration and weak equivalence.

The proofs of 3.6 and 3.8 depend on a basic property of homotopy inverse limits. Recall that a functor $J:\mathrm{C}\to\mathrm{D}$ (C small) is said to be *left cofinal* if for each $d\in\mathrm{D}$ the space J/d has the weak homotopy type of a point.

3.12. COFINALITY THEOREM [3, p. 316]. *Suppose that* C *and* D *are small categories, and that*

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{J} & \mathrm{D} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathrm{S} & \end{array}$$

is a commutative diagram such that

(i) *J is left cofinal, and*
(ii) *for each $d\in\mathrm{D}$, $G(d)$ is fibrant.*

Then the induced map $\operatorname{holim}_{\leftarrow} G\to\operatorname{holim}_{\leftarrow} F$ *is a homotopy equivalence.*

We will apply 3.12 by showing that appropriate functors with domain Δ_{rest} are left cofinal. The simplest way to do this is to interpret cofinality geometrically.

3.13. *Restricted Simplicial Sets.* A *restricted simplicial set X* is a "simplicial set without degeneracies," that is, X consists of

(i) a set X_n (of n-simplices) for each $n\geq 0$, and
(ii) maps $d_i:X_n\to X_{n-1}$, $0\leq i\leq n$, such that $d_id_j=d_{j-1}d_i$ if $i<j$ [3, p. 230].

Restricted simplicial sets can be identified in the usual way with functors

$(\Delta_{\mathrm{rest}})^{\mathrm{op}} \to$ Sets. A restricted simplicial set X has a natural enveloping space, denoted $R(X)$, defined by

$$R(X) = \left(\coprod_{n \geq 0} X_n \times \Delta[n] \right) \Big/ \sim .$$

Hence $\Delta[n]$ is the standard n-simplex and the equivalence relation $\sim$ is generated by

$$(d_i x, s) \sim (x, \delta^i s) \qquad \begin{array}{l} x \in X_{n+1} \\ s \in \Delta[n] \end{array}$$

where $\delta^i : \Delta[n] \to \Delta[n+1]$ is the i'th face inclusion.

The functor R is the left adjoint to the forgetful functor from simplicial sets to restricted simplicial sets. It is not hard to see that the nondegenerate simplices of $R(X)$ are in one-one correspondence with the simplices of X itself.

3.14. LEMMA. *Let $J : \Delta_{\mathrm{rest}} \to \mathrm{D}$ be a functor, and let $d \in \mathrm{D}$. Then J/d is weakly homotopy equivalent to $R(X)$, where X is the functor $(\Delta_{\mathrm{rest}})^{\mathrm{op}} \to$ Sets given by*

$$X([n]) = \operatorname{Hom}_{\mathrm{D}} (J([n]), d).$$

Proof. A calculation shows that for any small category C and functor $J : \mathrm{C} \to \mathrm{D}$, the space J/d is isomorphic to

$$\left(\coprod_{c \in \mathrm{C}} \operatorname{Hom}_{\mathrm{D}} (J(c), d) \times (\mathrm{C}/c) \right) \Big/ \sim .$$

Here the equivalence relation $\sim$ is generated by

$$(f \circ Jg, s) \sim (f, g_* s)$$

where $f \in \operatorname{Hom}_{\mathrm{D}} (J(c_2), d)$, $g \in \operatorname{Hom}_{\mathrm{C}} (c_1, c_2)$, and $g_* : \mathrm{C}/c_1 \to \mathrm{C}/c_2$ is induced by g. The proof consists in applying this to Δ_{rest} and using the fact that for each n the space $\Delta_{\mathrm{rest}}/[n]$ is isomorphic to the first barycentric subdivision of the standard n-simplex.

An *augmentation* for a restricted simplicial set X is a map $d_0 : X_0 \to X_{-1}$ such that $d_0 d_0 = d_0 d_1 : X_1 \to X_{-1}$. A *contracting homotopy* for an augmented restricted simplicial set X is a family of maps $s : X_n \to X_{n+1}$ such that

(i) $d_{n+1} s = \text{identity}$, $n \geq -1$, and
(ii) $d_i s = s d_i$, $0 \leq i \leq n$.

In the statement of the following lemma, the set X_{-1} is identified with the discrete space it represents.

3.15. LEMMA. *If X is an augmented restricted simplicial set with a contracting homotopy, the induced augmentation map $R(X) \to X_{-1}$ is a weak homotopy equivalence.*

Proof. Since R commutes with disjoint union, it is enough to prove the lemma when X_{-1} is a single point. In this case one computes that the fundamental group of $R(X)$ is trivial and that s induces a contracting homotopy on the normalized integral chain complex of $R(X)$.

3.16 *Proof* of 3.6. Let Δ^+_{rest} denote the augmented restricted simplicial category with a contracting homotopy, that is, Δ^+_{rest} consists of

(i) one object $[n]$ for each $n \geq -1$,
(ii) for every pair (i, n) of integers with $0 \leq i \leq n$ coface maps

$$d^i : [n-1] \to [n]$$

such that $d^j d^i = d^i d^{j-1}$ if $i < j$, and

(iii) for each $n \geq 0$ a map

$$s : [n] \to [n-1]$$

such that

$$sd^n = \text{identity}, \ n \geq -1$$
$$sd^i = d^i s, \ i < n.$$

There is an obvious inclusion functor $J : \Delta_{\text{rest}} \to \Delta^+_{\text{rest}}$. Suppose that the map $\phi(X) : X \to T(X)$ admits a left inverse $r : T(X) \to X$. The resolution functor $\mathbf{T}X : \Delta_{\text{rest}} \to \mathrm{S}$ can then be extended to a functor $\mathbf{T}^+X : \Delta^+_{\text{rest}} \to \mathrm{S}$ by setting

$$(\mathbf{T}^+X[n] \xrightarrow{s} \mathbf{T}^+X[n-1]) = (T^{n+1}X \xrightarrow{T^n(r)} T^n X)$$

and

$$(\mathbf{T}^{+}X[-1]\xrightarrow{d^0}\mathbf{T}^{+}X[0])=(X\xrightarrow{\phi(X)}TX).$$

This gives a commutative diagram

$$\begin{array}{ccc} \Delta_{\text{rest}} & \xrightarrow{J} & \Delta^{+}_{\text{rest}} \\ & \searrow \quad \swarrow & \\ & \mathbf{S} & \end{array}$$

The category Δ^{+}_{rest} has $[-1]$ as an initial object, so the canonical map

$$X=\mathbf{T}^{+}X[-1]=\varprojlim \mathbf{T}^{+}X\to \operatorname{holim}_{\leftarrow} \mathbf{T}^{+}X$$

is a homotopy equivalence [3, p. 299]. (This can also be derived from the fact that the inclusion of the singleton category $[-1]$ into Δ^{+}_{rest} is left cofinal.) Thus, by 3.12, it is enough to show that the functor J is left cofinal.

Pick $[m]\in\Delta^{+}_{\text{rest}}$. If the restricted simplicial set X given by

$$X[n]=\operatorname{Hom}_{\Delta^{+}_{\text{rest}}}(J([n]),[m])$$

is furnished with the augmentation

$$X[0]\xrightarrow{d_0}\operatorname{Hom}_{\Delta^{+}_{\text{rest}}}([-1],[0])$$

induced by composition on the right with d^0, then composition on the left with s provides maps

$$X[n]\to X[n+1]$$

which give a contracting homotopy for X. By 3.15 $R(X)$ is contractible, and the lemma thus follows from 3.14.

3.17. *Proof of* 3.8. Let $J:\Delta_{\text{rest}}\to\Delta$ be the inclusion functor. According to 3.12, it is sufficient to show that J is left cofinal. Pick $[m]\in\Delta$ and let X be the restricted simplicial set with

$$X[n]=\operatorname{Hom}_{\Delta}(J([n]),[m]).$$

It is clear that X is just the underlying restricted simplicial set of the standard m-simplex $\Delta[m]$. An easy calculation shows that $R(X)$ is simply-connected; in addition, since the normalized integral chain complex of $R(X)$ is the same as the unnormalized integral chain complex of $\Delta[m]$, the reduced integral homology of $R(X)$ vanishes. Therefore, $R(X)$ is contractible and the lemma follows from 3.14.

4. A tower lemma

4.1. *Fibrant Towers.* A *tower* of spaces $\{X_\alpha\}_{\alpha<\beta}$ of length β is a functor $\Omega_\beta \to \mathrm{S}$, where Ω_β is the full subcategory of Ω containing all ordinals less than β. Unlike long towers, towers are small diagrams of spaces and thus have both inverse limits and homotopy inverse limits.

The tower $\{X_\alpha\}_{\alpha<\beta}$ is said to be *fibrant* if

(i) X_0 is a fibrant space, and
(ii) for each $\alpha<\beta$ the natural map

$$X_\alpha \to \varprojlim \{X_\gamma\}_{\gamma<\alpha}$$

is a fibration.

4.2. FIBRANT TOWER LEMMA. *If $\{X_\alpha\}_{\alpha<\beta}$ is a fibrant tower, then the natural map*

$$\varprojlim \{X_\alpha\}_{\alpha<\beta} \to \operatorname{holim}_{\leftarrow} \{X_\alpha\}_{\alpha<\beta}$$

is a homotopy equivalence.

The *function complex* Hom $(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta})$ of maps between two towers is the space whose n-simplices $(n\geq 0)$ comprise all tower maps

$$\{A_\alpha \times \Delta[n]\}_{\alpha<\beta} \to \{X_\alpha\}_{\alpha<\beta}$$

and whose face and degeneracy operators are induced by the standard inclusion $\Delta[n]\to\Delta[n+1]$ and the standard collapses $\Delta[n]\to\Delta[n-1]$ [3, p. 295]. If $\{*\}_{\alpha<\beta}$ is the constant one-point tower, then

$$\operatorname{Hom}(\{*\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) = \varprojlim \{X_\alpha\}_{\alpha<\beta}$$

while if $\{\Omega_\beta/\alpha\}_{\alpha<\beta}$ is the tower of 3.10, then

$$\operatorname{Hom}(\{\Omega_\beta/\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) = \operatorname{holim}_{\leftarrow} \{X_\alpha\}_{\alpha<\beta}.$$

4.3. LEMMA. *Suppose that* $\{A_\alpha\}_{\alpha<\beta} \to \{B_\alpha\}_{\alpha<\beta}$ *is a tower map which induces a trivial cofibration* $A_\alpha \to B_\alpha$ *for each* $\alpha < \beta$. *Then for any fibrant tower* $\{X_\alpha\}_{\alpha<\beta}$ *the restriction map*

$$\mathrm{Hom}\,(\{B_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \to \mathrm{Hom}\,(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta})$$

is a trivial fibration.

4.4. *Remark.* A fibration or cofibration is *trivial* if it is also a weak homotopy equivalence.

Lemma 4.2 is proved by applying 4.3 twice: first to the obvious map

$$\{\Omega_\beta/\alpha\}_{\alpha<\beta} \to \{\Omega_\beta/0\}_{\alpha<\beta}$$

where the second tower is constant, and then to any inclusion

$$\{*\}_{\alpha<\beta} \to \{\Omega_\beta/0\}_{\alpha<\beta}.$$

Note that each of the spaces Ω_β/α is a contractible by [3, p. 293].

Proof of 4.3. The conclusion of 4.3 holds if and only if a dotted arrow exists in every diagram of the form [7, II, 2.1]

$$\begin{array}{ccc} \dot{\Delta}[n] & \to & \mathrm{Hom}\,(\{B_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \to & \mathrm{Hom}\,(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \end{array}$$

where $\dot{\Delta}[n] \to \Delta[n]$ is the inclusion of the boundary of the standard n-simplex. By an adjointness argument this is equivalent to showing that the dotted arrow exists in each diagram

$$\begin{array}{ccc} \{A_\alpha\}_{\alpha<\beta} & \to & \{\mathrm{Hom}\,(\Delta[n], X_\alpha)\}_{\alpha<\beta} \\ \downarrow & \nearrow & \downarrow \\ \{B_\alpha\}_{\alpha<\beta} & \to & \{\mathrm{Hom}\,(\dot{\Delta}[n], X_\alpha)\}_{\alpha<\beta} \end{array}$$

where in this case Hom denotes the standard function complex of maps between spaces [6, p. 16]. This second dotted arrow is constructed by an induction on α. The case $\alpha = 0$ is straightforward and uses the assumption that X_0 is fibrant. The induction step for $\alpha > 0$ depends on the existence of yet another dotted arrow in the diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\qquad\qquad} & \mathrm{Hom}\,(\Delta[n], X_\alpha) \\ \downarrow & \nearrow & \downarrow \\ B_\alpha \to \varprojlim\,\{\mathrm{Hom}\,(\Delta[n], X_\gamma)\}_{\gamma<\alpha} & \times_{\lim\{\mathrm{Hom}\,(\dot{\Delta}[n], X_\gamma)\}_{\gamma<\alpha}} & \mathrm{Hom}\,(\dot{\Delta}[n], X_\alpha) \end{array}$$

This dotted arrow exists because the left vertical map is a trivial cofibration and the right vertical arrow is a fibration.

5. R-Bousfield spaces

The purpose of this section is to prove 1.2. The proof is based on Bousfield's algebraic characterization of R-Bousfield spaces [1, §5].

We will use the terminology of [2] except that *HR-local* groups and *H$\mathbb{Z}$-local* π-modules will be called *R-Bousfield* groups and *$\mathbb{Z}$-Bousfield* π-modules. Recall that R is a subring of the rational numbers or a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, p prime.

5.1. PROPOSITION [2:3.10, 2.6]. *The R-Bousfield groups form the smallest class of groups such that*

(i) *the class contains the trivial group,*
(ii) *the class is closed under inverse limits of arbitrary towers,*
(iii) *if Y is in the class and $1 \to W \to X \to Y \to 1$ is a central extension with W an R-module, then X is in the class,*
(iv) *if X is in the class and $1 \to W \to X \to Y \to 1$ is a short exact sequence with Y abelian and an R-module, then W is in the class.*

Let π be a group and let M be a π-module. Then M will be called an *R-Bousfield* π-module if

(i) M is R-Bousfield as an (abelian) group, and
(ii) M is $\mathbb{Z}$-Bousfield as a π-module.

It is not hard to prove using [2:8.9, 7.3] that

5.2. LEMMA. *The R-Bousfield π-modules form the smallest class of π-modules such that*

(i) *the class contains the zero π-module,*

(ii) *the class is closed under inverse limits of arbitrary towers,*
(iii) *if Y is in the class and* $0 \to W \to X \to Y \to 0$ *is an extension of* π*-modules with W simple (= trivial* π*-action) and an R-module, then X is in the class,*
(iv) *if X is in the class and* $0 \to W \to X \to Y \to 0$ *is a short exact sequence with Y simple and an R-module, then W is in the class.*

In fact, it is clear that the class of R-Bousfield π-modules contains the class described in 5.2. If $R \subseteq \mathbb{Q}$ the opposite inclusion follows easily from the fact that by naturality the $H\mathbb{Z}$-tower of an R-Bousfield π-module M is itself a tower of R-modules. If $R = \mathbb{Z}/p\mathbb{Z}$ it is possible to use the natural action of π on the HR-tower of the underlying abelian group of M and to show by transfinite induction that each π-module in this tower belongs to the class described.

5.3. LEMMA [1, §5]. *A fibrant space X is R-Bousfield if and only if for every* $i \geq 2$ *and every choice of basepoint* $x \in X$,

(i) $\pi_1(X, x)$ *is an R-Bousfield group, and*
(ii) $\pi_i(X, x)$ *is an R-Bousfield* $\pi_1(X, x)$*-module.*

5.4. *Proof of* 1.2. Let C denote the union of the classes I_α. It is necessary to show that every R-Bousfield space X belongs to C. Note that by definition C is closed under arbitrary homotopy inverse limits.

Let π be a group and let M be a π-module. For $n \geq 1$, $L(\pi, M, n)$ denotes the split fibration over $K(\pi, 1)$ with fibre $K(M, n)$ which is determined by the action of π on M.

Every fibrant space X is homotopy equivalent to the homotopy inverse limit of its Postnikov tower $\{P_nX\}_{n<\omega}$, where ω is the first infinite ordinal. Moreover, if B runs through a selection of basepoints for X, one for each path component, there are homotopy fibre squares

$$\begin{array}{ccc} P_nX & \to & \coprod_{b \in B} K(\pi_1(X, b), 1) \\ \downarrow & & \downarrow \\ P_{n-1}X & \to & \coprod_{b \in B} L(\pi_1(X, b), \pi_n(X, b), n+1). \end{array}$$

Thus by 5.3 it suffices to show that for each fixed $n \geq 1$ every space which is a disjoint union of spaces of the form $L(\pi, M, n)$ for various R-Bousfield groups π and various R-Bousfield π-modules M belongs to C.

This is done by induction on n. We will assume $n > 1$ and prove that every (connected) space of the form $L(\pi, M, n)$ belongs to C. The general case can be proved in the same way by using the fact that homotopy inverse limits over categories with connected nerves commute with disjoint unions. The initial case $n=1$ is similar to the case $n>1$ but simpler.

Let π be an R-Bousfield group. It is easily seen that the class of R-Bousfield π-modules M such that $L(\pi, M, n)$ belongs to C satisfies parts (i), (iii) and (iv) of 5.2, so it remains to show that if $\{M_\alpha\}_{\alpha<\beta}$ is a tower of R-Bousfield π-modules such that each $L(\pi, M_\alpha, n)$ belongs to C, then $L(\pi, M, n)$ belongs to C, where $M=\lim_{\leftarrow}\{M_\alpha\}_{\alpha<\beta}$. This is done as follows. Using bar construction techniques [6, p. 83] one devises a way of constructing the spaces $L(\pi, M_\alpha, n)$ which is functorial in M_α. Thus the tower $\{M_\alpha\}_{\alpha<\beta}$ of π-modules gives rise to a tower $\{L(\pi, M_\alpha, n)\}_{\alpha<\beta}$ of spaces. Let X denote $\operatorname{holim}_{\leftarrow}\{L(\pi, M_\alpha, n)\}_{\alpha<\beta}$. The space X belongs to C and by [3, p. 309] and naturality there are π-module isomorphisms

$$\pi_n X \approx M$$
$$\pi_i X \approx 0, \qquad i>n.$$

Note that the homotopy groups of X actually are π-modules by virtue of the fact that the composite

$$f: X \to L(\pi, M_0, n) \to K(\pi, 1)$$

has a section $K(\pi, 1) \to X$.

Let $P_{n-2}(f)$ denotes the $n-2$ stage in the Moore-Postnikov factorization of f [6, p. 34]. The inductive hypothesis implies that $P_{n-2}(f)$ belongs to C, so the space Y which is defined as the homotopy inverse limit of the square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ K(\pi, 1) & \to & P_{n-2}(f) \end{array}$$

also belongs to C. Up to homotopy the space $P_{n-1}(g)$ is a split fibration over $K(\pi, 1)$ with $K(\pi_{n-1}(Y), n-1)$ as the fibre, so, by induction, $P_{n-1}(g)$ belongs to C too. The proof is finished by noting that there is a homotopy fibre square

$$\begin{array}{ccc} L(\pi, M, n) & \to & Y \\ \downarrow & & \downarrow \\ K(\pi, 1) & \to & P_{n-1}(g). \end{array}$$

6. Construction of the tower

The object of this section is to construct for each $\alpha \in \Omega$ an augmented functor (T_α, ϕ_α) on S and compatible morphisms $(T_\beta, \phi_\beta) \to (T_\alpha, \phi_\alpha)$ for $\beta > \alpha$. For $X \in$ S the augmented long tower $X \to \{T_\alpha X\}_\alpha$ is the *R-homology localization tower* of X.

The construction is by transfinite induction. The pair (T_0, ϕ_0) is the underlying augmented functor of the triple (R, ϕ, ψ) of [3, p. 13]. If $\alpha = \beta + 1$ is a successor ordinal, (T_α, ϕ_α) is $(\hat{T_\beta}, \hat{\phi_\beta})$; by 3.3 there is a natural morphism $(T_\alpha, \phi_\alpha) \to (T_\beta, \phi_\beta)$. Finally, if α is a limit ordinal the pair (T_α, ϕ_α) is $\lim_\leftarrow \{(T_\beta, \phi_\beta)\}_{\beta<\alpha}$; this evidently comes with a natural map into (T_β, ϕ_β) for each $\beta < \alpha$.

The identification of T_1 and T_2 made in 1.5 follows easily from 3.8, 3.9 and 6.1 below. The rest of this section is taken up with proving that the tower $\{T_\alpha X\}_\alpha$ has the properties listed in 1.4. Recall the

6.1. *Homotopy Invariance Lemma* [3, p. 304]. *Let* D *be a small category, let* $F, G: \mathrm{D} \to \mathrm{S}$ *be functors, and let* $\tau: F \to G$ *be a natural transformation. Suppose that for all* $d \in \mathrm{D}$

(i) *the spaces* $F(d)$ *and* $G(d)$ *are fibrant, and*
(ii) *the map* $\tau(d): F(d) \to G(d)$ *is a homotopy equivalence.*

Then τ *induces a homotopy equivalence* $\mathrm{Holim}_\leftarrow F \to \mathrm{Holim}_\leftarrow G$.

6.2. *Proof of* 1.4(i). The space $T_0X = RX$ is always fibrant, since choice of a basepoint for X makes RX into a simplicial R-module [3, p. 14]. Using 3.4 it is easy to show by induction that $T_\alpha X$ is fibrant for all α.

For any space X, $\pi_* RX$ is naturally isomorphic to $\tilde{H}_*(X; R)$ (reduced homology). This implies that a map $f: X \to Y$ induces a homotopy equivalence $T_0X \to T_0Y$ iff it induces an isomorphism $\tilde{H}_*(X; R) \to \tilde{H}_*(Y; R)$. Thus 1.4(i) follows inductively from 6.1 and, in the limit ordinal case, 4.2.

6.3. *Proof of* 1.4(ii). This follows inductively from the definitions and, in the limit ordinal case, 4.2.

6.4. *Proof of* 1.4(iii). We will show by induction on α that if $X \in I_\alpha$ the natural map $\phi_\alpha(X): X \to T_\alpha X$ has a left inverse $r: T_\alpha X \to X$. The desired result then follows from 3.6.

In the case $\alpha = 0$, it is possible to assume that each component of X has the structure of a simplicial R-module. Thus if X is connected there is an obvious canonical retraction $RX \to X$ given by evaluating formal sums. A retraction in the disconnected case can be constructed by using the fact that the map $\pi_0(\phi(X)) = \pi_0 X \to \pi_0 RX$ is injective, since it is essentially the Hurewicz homomorphism $\pi_0 X \to \tilde{H}_0(X; R)$.

Suppose $\alpha > 0$. It is enough to show that there is a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\phi_\alpha(X)} & T_\alpha X \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

in which the map $X \to W$ is a weak homotopy equivalence. In fact it is clear that given such a triangle there exists perhaps another one in which $T_\alpha X \to W$ is a cofibration. The map $\phi_\alpha(X)$ is a cofibration (since $X \to RX = T_0 X$ is) so it follows that $X \to W$ is a cofibration too. The fact that X is fibrant then implies that the map $X \to W$ has a left inverse.

Note that the induction hypothesis implies that if $\beta < \alpha$ and $Y \in I_\beta$, then the map $\phi_\alpha(Y): Y \to T_\alpha Y$ is a homotopy equivalence. This is immediate if α is a successor ordinal and follows from 4.2 and a tower cofinality argument ([3, p. 317] and 3.12) if α is a limit ordinal.

It is possible to assume that there is some small category C and functor $F: \mathrm{C} \to \mathrm{S}$ such that

(i) $X = \underleftarrow{\mathrm{holim}}\, F$, and

(ii) for each $c \in \mathrm{C}$ there is a $\beta < \alpha$ such that $F(c) \in I_\beta$.

Consider the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & \underleftarrow{\mathrm{holim}}\,\mathrm{con}\,(X) & \xrightarrow{s} & \underleftarrow{\mathrm{holim}}\, X \times \mathrm{C}/- & \longrightarrow & \underleftarrow{\mathrm{holim}}\, F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_\alpha X & \longrightarrow & \underleftarrow{\mathrm{holim}}\,\mathrm{con}\,(T_\alpha X) & \xleftarrow{t} & \underleftarrow{\mathrm{holim}}\, T_\alpha(X \times \mathrm{C}/-) & \longrightarrow & \underleftarrow{\mathrm{holim}}\, T_\alpha F. \end{array}$$

Here con (X) and con $(T_\alpha X)$ denote the obvious constant functors $\mathrm{C} \to \mathrm{S}$ and C/– is as in 3.10. The vertical maps are induced by ϕ_α, the left-hand horizontal maps by the natural transformation $\lim_{\leftarrow} \to \mathrm{holim}_{\leftarrow}$ [3, p. 298] and the right-hand horizontal maps by the morphism $X \times \mathrm{C}/- \to F$ which is adjoint to the identity map $X \to \mathrm{holim}_{\leftarrow} F$ [3, p. 296]. The map s takes $f \in \mathrm{Hom}\,(\mathrm{C}/-, \mathrm{con}\,(X)) = \mathrm{holim}_{\leftarrow}\,\mathrm{con}\,(X)$ to $f \times id \in \mathrm{Hom}\,(\mathrm{C}/-, \mathrm{con}\,(X)) \times \mathrm{Hom}\,(\mathrm{C}/-, \mathrm{C}/-) = \mathrm{holim}_{\leftarrow} X \times \mathrm{C}/-$. Finally, t is induced by the projection

$$T_\alpha(X \times \mathrm{C}/-) \to \mathrm{con}\,(T_\alpha X).$$

The composite of the maps on the top line is the identity map, and t is a homotopy equivalence by 6.1. Furthermore, the induction hypothesis shows that the map $\operatorname{holim}_{\leftarrow} F \to \operatorname{holim}_{\leftarrow} T_\alpha F$ is a homotopy equivalence.

Factor the map t as the composite of a trivial cofibration (4.4)

$$\underleftarrow{\operatorname{holim}}\; T_\alpha(X \times \mathrm{C}/-) \to \mathrm{Y}$$

and a trivial fibration

$$Y \to \underleftarrow{\operatorname{holim}}\; \operatorname{con}(T_\alpha X).$$

Let Y' be the pushout of the diagram

$$\begin{array}{ccc} \operatorname{holim} T_\alpha(X \times \mathrm{C}/-) & \to & \underleftarrow{\operatorname{holim}}\; T_\alpha F \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

so that the map $\operatorname{holim}_{\leftarrow} T_\alpha F \to Y'$ is a weak homotopy equivalence [7:I, §1, M4]. There results a commutative diagram of solid arrows

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Y' \\ \downarrow & \nearrow & \downarrow & & \\ T_\alpha X & \to & \underleftarrow{\operatorname{holim}}\; \operatorname{con}(T_\alpha X) & & \end{array}$$

in which the composite $X \to Y'$ is a weak homotopy equivalence. The dotted arrow can then be found because the left vertical arrow is a cofibration and the right vertical arrow is a trivial fibration.

7. An interpretation of the functors T_α

The purpose of this section is to show that the spaces $T_\alpha X$ of §6 can be identified, up to homotopy, with the homotopy inverse limits of Artin-Mazur-like large diagrams of spaces. This is a natural extension of the identification of $R_\infty X(\sim T_1 X)$ made in [3, p. 324].

Let (T, ϕ) be an augmented functor on S. A space Y is said to admit a *T-structure* if the natural map $\phi(Y): Y \to TY$ has a left inverse $r: TY \to Y$. For any space X and ordinal $\beta > 0$, let $T_{\alpha<\beta} \backslash X$ be the category consisting of

(i) one object for each map $X \to Y$ of S such that Y admits a T_α-structure for some $\alpha < \beta$, and

(ii) one morphism $(X \to Y) \to (X \to Y')$ for each $f: Y \to Y'$ in S such that

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & \xrightarrow{f} & Y' \end{array}$$

commutes.

There is an Artin-Mazur functor

$$AM_\beta(X): T_{\alpha<\beta}\backslash X \to \mathrm{S}$$

which sends $(X \to Y)$ to the target space Y.

7.1. PROPOSITION. *For any ordinal $\beta > 0$ the space $T_\beta X$ has the homotopy type of the homotopy inverse limit of $AM_\beta(X)$.*

From a qualitative point of view the proposition says that the map $X \to T_\beta X$ comes as close as homotopy theory allows to being universal for all maps $X \to Y$ with the property that Y admits a T_α-structure for some $\alpha < \beta$.

Part of the work in proving 7.1 is to show that the homotopy inverse limit of the large diagram $AM_\beta(X)$ is well defined, up to homotopy. Recall that a (large) category D is said to be *left small* if there is a left cofinal (3.12) functor $J: \mathrm{C} \to \mathrm{D}$. (Note that C, as the domain of a left cofinal functor, is necessarily a small category.)

7.2. PROPOSITION [3, p. 321–322]. *If* D *is a left small category and $F: \mathrm{D} \to \mathrm{S}$ is a functor, then the homotopy inverse limit of F is well defined, up to homotopy. Moreover, if $J: \mathrm{C} \to \mathrm{D}$ is left cofinal and $F(J(c))$ is fibrant for each $c \in \mathrm{C}$, then the homotopy inverse limit of F has the homotopy type of* $\mathrm{holim}_{\leftarrow} F \circ J$.

The proof of 7.1 breaks up into two cases.

7.3. *The Successor Case.* Suppose that $\beta = \gamma + 1$ is a successor ordinal. The argument of 6.4 shows that for any space Y the space $T_\gamma Y$ admits a T_γ-structure; in particular, the spaces $T_\gamma^n X$ $(n \geq 1)$ admit T_γ-structures. Thus the restricted cosimplicial space $\mathbf{T}_\gamma X$ together with its augmentation determines a functor

$$\mathbf{T}'_\gamma X: \Delta_{\mathrm{rest}} \to T_{\alpha<\beta}\backslash X.$$

Since each of the spaces $T_\gamma^n X$ $(n \geq 1)$ is fibrant (6.2), it suffices to prove that $T'_\gamma X$ is left cofinal.

Pick an object $X \to Y$ of $T_{\alpha<\beta}\backslash X$. By 3.14 it is enough to show that $R(W)$ is contractible, where W is the restricted simplicial set given in dimension n by

$$W_n = \operatorname*{Hom}_{T_{\alpha<\beta}\backslash X} (X \to T_\gamma^{n+1}X, X \to Y).$$

The space Y admits a T_α-structure for some $\alpha \leq \gamma$; this easily implies that Y admits a T_γ-structure. Let $r: T_\gamma Y \to Y$ be a left inverse for $\phi_\gamma(Y): Y \to T_\gamma Y$. Define maps $s: W_n \to W_{n+1}$ by

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ T_\gamma^{n+1}X & \xrightarrow{\ f\ } & Y \end{array} \quad \xrightarrow{\ s\ } \quad \begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ T_\gamma^{n+2}X & \xrightarrow{\ r \circ T_\gamma(f)\ } & Y \end{array}$$

If W is augmented in the natural way by letting W_{-1} be the one-point set representing the commutative diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ X & \longrightarrow & Y \end{array}$$

then the maps s provide a contracting homotopy for W. The desired result then follows from 3.15.

7.4. *The Limit Ordinal Case.* Suppose that β is a limit ordinal. Let

$$\{\mathbf{T}_\alpha X\}_{\alpha<\beta} : \Omega_\beta \times \Delta_{\text{rest}} \to \mathrm{S}$$

be the functor which assigns to each space X the tower $\{\mathbf{T}_\alpha X\}_{\alpha<\beta}$ of restricted cosimplicial spaces. As in 7.3 it is easy to see this lifts to a functor

$$\{\mathbf{T}'_\alpha X\}_{\alpha<\beta} : \Omega_\beta \times \Delta_{\text{rest}} \to T_{\alpha<\beta}\backslash X.$$

Since $\operatorname{holim}_{\leftarrow} \{\mathbf{T}_\alpha X\}_{\alpha<\beta}$ is homotopy equivalent to $T_\beta X$ [3, p. 300, 4.3] it is enough to show that $\{\mathbf{T}'_\alpha X\}_{\alpha<\beta}$ is left cofinal.

Pick $X \to Y$ in $T_{\alpha<\beta}X$. We will use the language of *homotopy direct limits* [3, p. 325] to sketch a proof that $\{\mathbf{T}'_\alpha X\}_{\alpha<\beta}/(X \to Y)$ is contractible.

First there is a general observation. Let C be a small category and let $J: \mathrm{C} \to \mathrm{D}$ be a functor. For any element $d \in \mathrm{D}$ there is a functor $H_d : \mathrm{C}^{\mathrm{op}} \to \mathrm{SETS}$ sending $c \in \mathrm{C}$ to the set $\mathrm{Hom}_{\mathrm{D}}(J(c), d)$. Since any set can be identified with a discrete space, H_d can be thought of as a functor $\mathrm{C}^{\mathrm{op}} \to \mathrm{S}$. The following calculation was implicitly referred to in the proof of 3.14.

7.5. LEMMA. *For each* $d \in \mathrm{D}$ *there is an isomorphism of spaces*

$$J/d \approx \underrightarrow{\mathrm{holim}}\, H_d.$$

According to the properties of homotopy direct limits over product categories [3, p. 331], this implies that

$$\{T'_\alpha X\}_{\alpha<\beta}/(X \to Y) = \underrightarrow{\mathrm{holim}}\, F$$

where $F: \Omega_\beta^{\mathrm{op}} \to \mathrm{S}$ is the functor which sends $\alpha \in \Omega_\beta^{\mathrm{op}}$ to $\mathbf{T}'_\alpha X/(X \to Y)$.

If Y admits a T_γ-structure, the argument of 7.3 shows that $F(\alpha)$ is contractible for all $\alpha \in \Omega_\beta^{\mathrm{op}}$, $\alpha \geq \gamma$. The desired result then follows from the fact that since $\Omega_\beta^{\mathrm{op}}$ is right filtering, $\mathrm{holim}_{\to} F$ is weakly homotopy equivalent to $\lim_{\to} F$ [3, p. 332].

8. Examples

The purpose of this section is to extract some information about the behaviour of the long homology localization tower $\{T_\alpha X\}_\alpha$ for certain special classes of spaces X. In particular, we are interested in how rapidly the tower converges to X_R. The main tool for studying this is 1.4(iii).

8.1. *Nilpotent Spaces.* It follows from 1.5 that $X_R \sim T_1 X$ ($\sim$ = homotopy equivalence) iff X is R-*good* in the sense of Bousfield and Kan. In particular,

8.2. PROPOSITION [3, V, VI]. *If X is a nilpotent space and R is any of the admissible rings, then $X_R \sim T_1 X$.*

If $R \subseteq \mathbb{Q}$ we know of no spaces X for which $X_R \sim T_1 X$ and X_R is not nilpotent. If $R = \mathbb{Z}/p\mathbb{Z}$, however, there are many such examples ([3, VII], [4]).

8.3. *Virtually Nilpotent Spaces.* A connected space X is said to be *virtually nilpotent* if each Postnikov stage $P_n X$ can be finitely covered by a nilpotent space.

If $R = \mathbb{Z}/p\mathbb{Z}$, then all such spaces are R-good [4]. The main result of [4] shows that if X is virtually nilpotent and $R \subseteq \mathbb{Q}$ there is a homotopy fibre square

$$\begin{array}{ccc} X_R & \longrightarrow & W_1 \\ \downarrow & & \downarrow \\ W_3 & \longrightarrow & W_2 \end{array}$$

in which the spaces W_1, W_2 and W_3 have the homotopy type of homotopy inverse limits of (cosimplicial) diagrams of simplicial R-modules; that is, $W_1, W_2, W_3 \in I_1$. It follows immediately that $X_R \in I_2$, so

8.4. PROPOSITION. *If X is a virtually nilpotent space and $R \subseteq \mathbb{Q}$, then $X_R \sim T_3 X$.*

This result may not be best possible. In fact, it is not hard to show that if $\pi_1 X$ is *finite* and $R \subseteq \mathbb{Q}$, then $X_R \sim T_2 X$. The argument for this uses [4] and the fibre lemma of [3, p. 62].

8.5. *Pre-nilpotent Fundamental Groups.* A group π is said to be *pre-nilpotent* [5, 3.1] if the lower central series of π stabilizes, not necessarily at the trivial group, after a finite number of steps. Let ω be the first infinite ordinal.

8.6. PROPOSITION. *Suppose that $R = \mathbb{Z}$ and that X is a connected space with a finitely generated pre-nilpotent fundamental group. Then $X_R \sim T_{\omega+1} X$.*

8.7. *Remark.* Analogous results almost certainly hold for other rings. At least over $\mathbb{Z}$, the finite generation condition can be replaced by the assumption that $H_1(X; \mathbb{Z})$ is finitely generated.

We will only sketch the proof of 8.6, since the main point is purely algebraic. Assume $R = \mathbb{Z}$. The hypothesis on X implies that $\pi_1 X_R$ is a finitely generated nilpotent group [1, 7.3, 7.5] so, by 1.4(i)–(iii) it is enough to show that if Y is a connected R-Bousfield space with a finitely generated nilpotent fundamental group, then $Y \in I_\omega$. By the Postnikov argument of 5.4 it is enough to show that whenever π is a finitely generated nilpotent group and M is an R-Bousfield π-module, then $L(\pi, M, n) \in I_m$ for some integer m.

Let E denote the $H\mathbb{Z}$-localization functor on the category of π-modules and let $F \to M \to 0$ be an epimorphism from the free π-module F to the R-Bousfield π-module M. Since E is right exact [1, 8.11] there is a short exact sequence

$$0 \to K \to E(F) \to M \to 0$$

where K is some R-Bousfield [1, 8.5] submodule of $E(F)$. In view of the homotopy fibre square

$$\begin{array}{ccc} L(\pi, M, n-1) & \longrightarrow & K \\ \downarrow & & \downarrow \\ L(\pi, K, n) & \longrightarrow & L(\pi, E(F), n) \end{array}$$

it suffices to prove that both $L(\pi, K, n)$ and $L(\pi, E(F), n)$ belong to some I_m.

Let J be the augmentation ideal inside the integral group ring $\mathbb{Z}[\pi]$ of π. Then [5, 3.1] asserts that there is an isomorphism $E(F) \approx \lim_{\leftarrow} \{F/J^s \cdot F\}_{s<\omega}$. Since K is a submodule of $E(F)$ it is clear that K injects into $\lim_{\leftarrow} \{K/J^s \cdot K\}_{s<\omega}$. However,

$$H_0(\pi; \varprojlim \{K/J^s \cdot K\}_{s<\omega}) = \varprojlim \{H_0(\pi; K/J^s \cdot K)\}_{s<\omega} = H_0(\pi; K)$$

since π is finitely generated and $\lim^1_{\leftarrow} \{H_1(\pi; K/J^s \cdot K)\}_{s<\omega}$, being a quotient of $\lim^1_{\leftarrow} \{H_1(\pi; K)\}_{s<\omega}$, vanishes (compare [5, Proof of 3.7]). It follows from [2, 7.8] that $K \approx \lim_{\leftarrow} \{K/J^s \cdot K\}_{s<\omega}$.

By 8.2 the spaces $L(\pi, K/J^s \cdot K, n)$ and $L(\pi, F/J^s \cdot F, n)$ belong to I_1, since they are nilpotent. The groups

$$\varprojlim{}^1 \{K/J^s \cdot K\}_{s<\omega} \quad \text{and} \quad \varprojlim{}^1 \{F/J^s \cdot F\}_{s<\omega}$$

vanish, since both of these module towers are towers of epimorphisms [3, p. 252]. Thus [3, pp. 287, 254],

$$L(\pi, K, n) \sim \operatorname{holim}_{\leftarrow} \{L(\pi, K/J^s \cdot K, n)\}_{s<\omega} \in I_2$$

$$L(\pi, F, n) \sim \operatorname{holim}_{\leftarrow} \{L(\pi, F/J^s \cdot F, n)\}_{s<\omega} \in I_2.$$

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Added note

An alternative approach to constructing the homology localization as an inverse limit is given in

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