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# A long homology localization tower

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## 1. Introduction

Let $R$ be fixed as a subring of the rational numbers or a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, $p$ prime. The purpose of this paper is to give a new description of the *$R$-homology localization* $X_R$ of a space $X$ [1]. The main ingredient is an *inverse limit* construction for $X_R$ (complementary to Bousfield's *direct limit* construction [1, 11.5]) which is obtained by transfinitely iterating the *$R$-nilpotent completion* process of [3]. Thus one immediate benefit is a clearer understanding of the relationship between $X_R$ and the $R$-nilpotent completion $R_\infty X$ of $X$.

A space $X$ is said to be *$R$-Bousfield* if $X_R$ is homotopy equivalent to $X$. The possibility of two constructions for $X_R$ is suggested by the fact that the natural map $X \to X_R$ has two universal properties:

(i) $X \to X_R$ is terminal, up to homotopy, in the category of all maps $X \to Y$ which induce isomorphisms $H_*(X; R) \approx H_*(Y; R)$.

(ii) $X \to X_R$ is initial, up to homotopy, in the category of all maps $X \to Y$ which have an $R$-Bousfield target space $Y$.

In order to exploit property (ii) effectively, it is necessary to study the

1.1. *Structure of $R$-Bousfield Spaces.* For each ordinal $\alpha \geq 0$, let $I_\alpha$ be the class of $R$-Bousfield spaces defined inductively as follows.

(i) $I_0$ contains all fibrant spaces with the property that each connected component has the homotopy type of a simplicial $R$-module.

(ii) $I_\alpha$ $(\alpha > 0)$ contains all fibrant spaces which are of the homotopy type of $\mathrm{holim}_{\leftarrow} D$ [3, p. 295], where $D$ is a small diagram of spaces, each of which belongs to $I_\beta$ for some $\beta < \alpha$.

The spaces in $I_0$ are $R$-Bousfield by [1, §4] and, inductively, the spaces in $I_\alpha$ $(\alpha > 0)$ are $R$-Bousfield by [1, §12]. Using [2] it is not hard to prove (see §5)

1.2. PROPOSITION. *If $X$ is an $R$-Bousfield space, then $X \in I_\alpha$ for some ordinal $\alpha$.*

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1.3. *The Long Tower.* Let $\Omega$ be the opposite category of the category of all ordinals, that is, $\Omega$ is the category with one object for each ordinal $\alpha$ and one morphism $\beta \to \alpha$ for each $\beta \geq \alpha$. A *long tower* in a category C is a functor $F: \Omega \to C$, usually written $\{F(\alpha)\}_\alpha$. The long tower is said to be *augmented* by the object $X$ of C if there are compatible maps $X \to F(\alpha)$, $\alpha \in \Omega$.

For any space $X$ we will construct a natural *long $R$-homology localization tower* $\{T_\alpha X\}_\alpha$ of spaces, naturally augmented by $X$, such that

1.4. (i) *if $f: X \to Y$ induces an isomorphism $H_*(X; R) \to H_*(Y; R)$ then $f$ induces homotopy equivalences $T_\alpha X \sim T_\alpha Y$,*
(ii) *for any $X$ or $\alpha$, $T_\alpha X \in I_\alpha$, and*
(iii) *if $X \in I_\alpha$, then the natural map $X \to T_{\alpha+1}X$ is a homotopy equivalence.*

In view of 1.2, these properties imply that

(iv) *for each space $X$ there is some ordinal $\alpha$ such that for all $\beta > \alpha$, the map $X \to T_\beta X$ is up to homotopy the $R$-homology localization map $X \to X_R$.*

In fact, $\alpha$ can be chosen to be any ordinal such that $X_R \in I_\alpha$. Thus the point at which the localization tower finally stabilizes for a given $X$ depends explicitly on the minimal number of homotopy inverse limits needed to construct $X_R$ from the spaces in $I_0$ (that is, from disjoint unions of products of $R$-module Eilenberg-MacLane spaces [6, 24.5]). This ordinal is an intrinsic measure of the homotopical complexity of $X_R$ or of the homological complexity of $X$ itself.

1.5. *Relationship to the $R$-completion.* The first few spaces in the tower $\{T_\alpha X\}_\alpha$ appear at least implicitly in [3]. The space $T_0 X$ is exactly $RX$ [3, p. 14], $T_1 X$ is homotopy equivalent to $R_\infty X$, and $T_2 X$ is homotopy equivalent to the homotopy inverse limit of the cosimplicial resolution of $X$ [3, p. 20] constructed using the triple structure of $R_\infty$ [3, p. 26]. The whole tower $\{T_\alpha X\}_\alpha$ is obtained by imitating the process of passing from $RX$ to $R_\infty X$ at successor ordinals and taking inverse limits at limit ordinals (see §6). The main technical innovation is the substitution of *augmented functors* (§3) for *triples* [3, p. 13].

This paper was inspired by Bousfield's algebraic work in [2], but, although we use his results heavily, our constructions do not seem to be related in a simple way to his.

1.6. *Organization of the Paper.* Section 2 gives a simplified outline of our general approach. Section 3 contains some preparatory material of a category-theoretical nature; Section 4 gives a generalization to transfinite towers of a result which is well known for towers indexed by the natural numbers; and Section 5 presents a proof of 1.2. Section 6 contains the construction of the tower $\{T_\alpha X\}_\alpha$ and the proof of its properties; Section 7 has an inductive "Artin-Mazur-like" interpretation of the functors $T_\alpha$, and the final section contains some examples.

1.7. *Notation and Terminology.* Although our arguments are not usually combinatorial, the word *space* is used as a synonym for *simplical set* ([3, VIII], [6], [7]); S denotes the category of spaces. For convenience we will sometimes use the terminology of homotopical algebra [7: I, 1.1 and II, 3.14]; for instance, a *cofibration* is an injection of simplicial sets, a *fibration* is a Kan fibration, and a space $X$ is *fibrant* if the unique map of $X$ to the one-point space is a fibration, i.e., if $X$ satisfies the Kan extension condition.

## 2. Outline of the proof

This section presents the main arguments of the paper in a schematic setting in which most of the technicalities disappear. We hope that this will help the reader to catch sight of the underlying simplicity of the basic ideas.

*Warning! This section is independent of the rest of the paper in notation and terminology.*

Let C be a category closed under inverse limits. An *augmented functor* $(T, \phi)$ on C is a functor $T: C \to C$ together with a natural transformation $\phi: 1_C \to T$.

2.1. *An Equalizer Construction.* Let $(T, \phi)$ be an augmented functor on C. For $X \in C$, let $T^\wedge(X)$ denote the equalizer of the two maps $\phi(TX)$, $T(\phi(X)): TX \to T^2X$, that is, let $T^\wedge(X)$ be the inverse limit of the diagram

$$TX \underset{\phi(TX)}{\overset{T(\phi(X))}{\rightrightarrows}} T^2X$$

The construction of $T^\wedge(X)$ is a functorial in $X$; moreover, since $(\phi T) \circ \phi = T(\phi) \circ \phi$, $T^\wedge(X)$ comes equipped with a natural map $\phi^\wedge(X): X \to T^\wedge(X)$ such that the obvious diagram

$$\begin{tikzcd} & T^\wedge(X) \arrow[d] \\ X \arrow[ur, "\phi^\wedge(x)"] \arrow[r, "\phi(X)"'] & T(X) \end{tikzcd}$$

commutes.

2.2. *Collapse Lemma* (cf. 3.6). *If $\phi(X): X \to TX$ has a left inverse, then the map $\phi^\wedge(X): X \to T^\wedge(X)$ is an isomorphism.*

*Proof.* Let $s: TX \to X$ be a left inverse for $\phi(X)$, and let $i: T^{\wedge}(X) \to TX$ be the natural map. Then it is easy to see that $s \circ i$ is a two-sided inverse for $\phi^{\wedge}(X)$.

2.3. *Bousfield Objects.* Let $E$ be a distinguished class of morphisms of C called *equivalences* (or *homology equivalences*). An object $Z \in \mathrm{C}$ is said to be *Bousfield* if any equivalence $f: X \to Y$ induces a bijection $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. The class of Bousfield objects is closed under inverse limits. A localization map for $X \in \mathrm{C}$ is a map $e: X \to Z$ such that $e$ is an equivalence and $Z$ is Bousfield. Such a $Z$ is called a *localization* of $X$; if one exists, it is unique up to a canonical isomorphism.

2.4. *Assumption. Every object $X \in \mathrm{C}$ has a localization.*

Suppose that $I_0$ is some naturally given class of basic Bousfield objects. By induction, for each ordinal $\alpha > 0$ let $I_\alpha$ denote the class of all objects which can be written up to isomorphism as $\lim_{\leftarrow} D$ where $D$ is a small diagram of objects in C each of which belongs to $I_\beta$ for some $\beta < \alpha$.

2.5. *Assumption* (cf. 1.2). *For any Bousfield object $X \in \mathrm{C}$ there is an ordinal $\beta$ such that $X \in I_\beta$.*

2.6. *A Long Localization Tower* (cf. §6). Suppose $(R, \phi)$ is an augmented functor on C such that

(i) for any $X \in \mathrm{C}$, $RX$ is Bousfield, and

(ii) if $f: X \to Y$ is an equivalence, then $Rf: RX \to RY$ is an isomorphism. Thus $R$ fails to be a localization functor only because $\phi(X): X \to RX$ need not be an equivalence. Suppose that $(R, \phi)$ satisfies the additional restriction

(iii) if $X \in I_0$, then the natural map $\phi(X): X \to RX$ has a left inverse. Define a long tower $\{(T_\alpha, \phi_\alpha)\}_\alpha$ of augmented functors by transfinite induction as follows. The pair $(T_0, \phi_0)$ is $(R, \phi)$. If $\alpha = \beta + 1$ is a successor ordinal, then $(T_\alpha, \phi_\alpha)$ is $(T_\beta^{\wedge}, \phi_\beta^{\wedge})$. If $\alpha$ is a limit ordinal, then $(T_\alpha, \phi_\alpha)$ is the inverse limit of $(T_\beta, \phi_\beta)$ over all ordinals $\beta < \alpha$.

2.7. PROPOSITION. *For any $X \in \mathrm{C}$ there is some ordinal $\alpha$ such that for all $\beta > \alpha$ the natural map $\phi_\beta(X): X \to T_\beta X$ is a localization map.*

*Proof.* Let $e: X \to Z$ be a localization map for $X$. It follows from 2.6(ii) that $e$ induces isomorphisms $T_\alpha X \to T_\alpha Z$ for all ordinals $\alpha$; thus it suffices to show that there is some ordinal $\alpha$ such that for all $\beta \geq \alpha$ the map $Z \to T_\beta Z$ is an isomorphism. This follows from 2.5 and

2.8. LEMMA. *If* $Z \in T_\alpha$, *then* $\phi_\beta(Z): Z \to T_\beta Z$ *is an isomorphism for all* $\beta \geq \alpha + 1$.

*Proof of Lemma.* The technique is transfinite induction on $\alpha$. The case $\alpha = 0$ follows from 2.6(iii) and 2.2. Pick $X \in I_\alpha$, $\alpha > 0$, with $X = \lim_{\leftarrow} D$, where $D$ is some diagram of objects each of which belongs to $I_\beta$ for some $\beta < \alpha$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\parallel"'] & T_\alpha X \arrow[d] \\ \varprojlim D \arrow[r] & \varprojlim T_\alpha D \end{tikzcd}$$

where the horizontal maps are induced by $\phi_\alpha$. The induction hypothesis shows that the lower horizontal arrow is an isomorphism, since $\phi_\alpha(Y): Y \to T_\alpha Y$ is an isomorphism for each object $Y$ in the diagram $D$. Thus $\phi_\alpha(X): X \to T_\alpha X$ has a left inverse, and the inductive step follows from 2.2.

2.9. *Remark.* The above program does not lead to an inverse limit construction for homology localizations only because the homotopy category of the category S of spaces is not closed under inverse limits. This paper is guided around that obstacle by the principle that the notion of *homotopy inverse limit* [3, XI] in S provides a natural substitute for the missing notion of inverse limit in HoS.

## 3. Categorical preliminaries

3.1. *Restricted Cosimplicial Spaces.* A *restricted cosimplicial space* $\mathbf{X}$ is a "cosimplicial space without codegeneracies," that is, $\mathbf{X}$ consists of

(i) for each integer $n \geq 0$ a space $\mathbf{X}^n$, and
(ii) for each pair $(i, n)$ of integers with $0 \leq i \leq n$ *coface* maps

$$d^i : X^{n-1} \to X^n$$

such that $d^j d^i = d^i d^{j-1}$ if $i < j$ [3, p. 267].

The object $\mathbf{X}$ is said to be *augmented* by $\mathbf{X}^{-1}$ if there is a map $d^0 : \mathbf{X}^{-1} \to \mathbf{X}^0$ such that $d^0 d^0 = d^1 d^0 : \mathbf{X}^{-1} \to \mathbf{X}^1$.

In the same way in which cosimplicial spaces are associated to *triples* on S [3, pp. 20, 323], restricted cosimplicial spaces are associated to

3.2. *Augmented Functors.* An *augmented functor* $(T, \phi)$ on S is a pair in which $T: \mathrm{S} \to \mathrm{S}$ is a functor and $\phi: 1_{\mathrm{S}} \to T$ is a natural transformation. The fact that $\phi$ is a natural transformation implies that $(T(\phi)) \circ \phi = (\phi T) \circ \phi$.

Let $(T, \phi)$ be an augmented functor on S and let $X \in \mathrm{S}$. The *restricted cosimplicial resolution* of $X$ with respect to $T$ is the augmented restricted cosimplicial space $\mathbf{T}X$ given by

$$(\mathbf{T}X)^k = T^{k+1}X$$

in codimension $k$, and

$$((\mathbf{T}X)^{k-1} \xrightarrow{d^i} (\mathbf{T}X)^k) = (T^k X \xrightarrow{T^i \phi T^{k-i}} T^{k+1}X).$$

3.3. *T-Completions.* Let $\Delta_{\mathrm{rest}}$ denote the *restricted simplicial category*, that is, the category whose objects are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ and whose morphisms are strictly monotone maps. The restricted cosimplicial space $\mathbf{T}X$ (without its augmentation) can be thought of as a functor

$$\mathbf{T}X: \Delta_{\mathrm{rest}} \to \mathrm{S}.$$

The *T-completion* of $X$, denoted $T^\wedge(X)$, is defined to be the homotopy inverse limit of $\mathbf{T}X$ [3, p. 295]:

$$T^\wedge(X) = \underleftarrow{\mathrm{holim}}\, \mathbf{T}X.$$

The augmentation $\phi(X): X \to TX = (\mathbf{T}X)^0$ induces a natural map $\phi^\wedge(X): X \to T^\wedge(X)$. There is also a natural map $T^\wedge(X) \to T(X)$ which induces a morphism $(T^\wedge, \phi^\wedge) \to (T, \phi)$ of augmented functors.

3.4. LEMMA. *If the spaces $T^n X$ $(n \geq 1)$ are fibrant, then the natural map $T^\wedge(X) \to TX$ is a fibration.*

This is proved below in 3.11.

3.5. *A Collapse Criterion.* Let $(T, \phi)$ be an augmented functor on S. It is useful to have a criterion that guarantees, for a given $X \in \mathrm{S}$, that the map $\phi^\wedge(X): X \to T^\wedge(X)$ is a homotopy equivalence.

3.6. *Collapse Lemma. Suppose that $T^n X$ is fibrant for all $n \geq 1$ and that the* *natural map* $\phi(X): X \to TX$ *has a left inverse. Then the completion map* $\phi^\wedge(X): X \to T^\wedge(X)$ *is a homotopy equivalence.*

3.7. *Relationship to Cosimplicial Constructions.* Let $\Delta$ denote the full simplicial category, that is, the category whose objects are the same as those of $\Delta_{\text{rest}}$, but whose morphisms are all weakly monotone maps. If $(T, \phi, \psi)$ is a *triple* or *monad* on the category of spaces [3, p. 13], Bousfield and Kan associate to any space $X$ a *cosimplicial resolution* with respect to $T$ [3, pp. 20, 323]; this is a *cosimplicial space*, or, equivalently, a functor

$$\mathbf{T}^*(X): \Delta \to S.$$

Let $(T, \phi)$ be the underlying augmented functor of $(T, \phi, \psi)$, and let $J: \Delta_{\text{rest}} \to \Delta$ be the obvious inclusion functor. For any $X$ there is a commutative diagram

$$\begin{tikzcd} \Delta_{\text{rest}} \arrow[rr, "J"] \arrow[dr, "\mathbf{T}X"'] & & \Delta \arrow[dl, "\mathbf{T}^*(X)"] \\ & S & \end{tikzcd}$$

which induces a natural map [3, p. 316]

$$\underleftarrow{\text{holim}}\, \mathbf{T}^*(X) \to \underleftarrow{\text{holim}}\, \mathbf{T}X = T^\wedge(X).$$

3.8. LEMMA. *If* $T^n X$ *is fibrant for all* $n \geq 1$, *then the map* $\text{holim}_{\leftarrow}\, \mathbf{T}^*(X) \to T^\wedge(X)$ *is a homotopy equivalence.*

If $(T, \phi, \psi)$ is a triple, let $T_\infty X$ denote $\text{Tot}\,(\mathbf{T}^*(X))$ [3, p. 17].

3.9. COROLLARY. *If* $\mathbf{T}^*(X)$ *is a fibrant cosimplicial space* [3, p. 275], *there is a natural homotopy equivalence* $T_\infty X \to T^\wedge(X)$.

This follows from [3, p. 300].

The rest of this section contains the proofs of 3.4, 3.6 and 3.8.

3.10 *The Over Category.* Suppose that C and D are categories, and that $J: \mathrm{C} \to \mathrm{D}$ is a functor. For each $d \in \mathrm{D}$ the *over category* $J/d$ is defined as having one object for each pair $(c, f)$ where $c \in \mathrm{C}$ and $f \in \mathrm{Hom}_{\mathrm{D}}(J(c), d)$, and one morphism

$(c, f) \to (c', f')$ for each $g \in \mathrm{Hom}_C(c, c')$ such that

$$\begin{tikzcd} J(c) \arrow[rr, "J(g)"] \arrow[dr, "f"'] & & J(c') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

commutes. The composition rule in $J/d$ is induced by the rule in C.

If C is small, then $J/d$ is small and thus has a *nerve* or *spatial realization*, which is a space also denoted by $J/d$ [3, p. 29]. In general we will make no notational distinction between a small category and its spatial realization. For instance, if C is a small category and $c \in C$, $C/c$ will denote (the spatial realization of) the category $1_C/c$, where $1_C$ is the identity functor on C. Similarly, $C/-$ will denote the functor $C \to S$ which assigns to each object $c \in C$ the space $C/c$.

3.11. *Proof of* 3.4. Let $V(n, k)$ denote the space formed by the boundary of the standard $n$-simplex with the $k$'th face deleted [7, II, 2.1]. There is a cofibration $V(n, k) \to \Delta[n]$, where $\Delta[n]$ is the standard $n$-simplex itself. To prove the lemma it is enough to show that the dotted arrow exists in any diagram of the form

$$\begin{tikzcd} V(n, k) \arrow[r, "f"] \arrow[d] & T^{\wedge}(X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & T(X). \end{tikzcd}$$

By an adjointness argument [3, p. 296] this comes down to showing that there is a map $g : \Delta[n] \times (\Delta_{\mathrm{rest}}/-) \to \mathbf{T}X$ which extends both the given map

$$\Delta[n] = \Delta[n] \times (\Delta_{\mathrm{rest}}/[0]) \to TX$$

and the map

$$f' : V(n, k) \times (\Delta_{\mathrm{rest}}/-) \to \mathbf{T}X$$

which is adjoint to $f$.

The map $g$ is built up by using induction on $m$ to construct its components

$$g_m : \Delta[n] \times (\Delta_{\mathrm{rest}}/[m]) \to T^{m+1}X.$$

The map $g_0$ is given. The space $\Delta_{\mathrm{rest}}/[m]$ is the first barycentric subdivision $sd\Delta[m]$ of the standard $m$-simplex and the prescription of $g_{m-1}$ determines the restriction of $g_m$ to $\Delta[n]\times sd(\dot{\Delta}[m])$, where $\dot{\Delta}[m]$ is the boundary of the $m$-simplex. In addition, the requirement that $g$ extend $f'$ determines the restriction of $g_m$ to $V(n, k)\times sd\Delta[m]$. Thus $g_m$ must be constructed as the dotted arrow in a diagram of the form

$$\begin{tikzcd} (V(n,k)\times sd\Delta[m])\cup(\Delta[n]\times sd\dot{\Delta}[m]) \arrow[r] \arrow[d] & T^{m+1}X \arrow[d] \\ \Delta[n]\times sd\Delta[m] \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

where $*$ is a one-point space. The existence of such a dotted arrow follows from the fact that the right vertical arrow is a fibration while the left vertical arrow is a cofibration and weak equivalence.

The proofs of 3.6 and 3.8 depend on a basic property of homotopy inverse limits. Recall that a functor $J: \mathrm{C}\to \mathrm{D}$ (C small) is said to be *left cofinal* if for each $d\in \mathrm{D}$ the space $J/d$ has the weak homotopy type of a point.

3.12. COFINALITY THEOREM [3, p. 316]. *Suppose that* C *and* D *are small categories, and that*

$$\begin{tikzcd} \mathrm{C} \arrow[rr, "J"] \arrow[dr, "F"'] & & \mathrm{D} \arrow[dl, "G"] \\ & \mathrm{S} & \end{tikzcd}$$

*is a commutative diagram such that*

(i) *$J$ is left cofinal, and*
(ii) *for each $d\in \mathrm{D}$, $G(d)$ is fibrant.*

*Then the induced map* $\operatorname{holim}_{\leftarrow} G\to \operatorname{holim}_{\leftarrow} F$ *is a homotopy equivalence.*

We will apply 3.12 by showing that appropriate functors with domain $\Delta_{\text{rest}}$ are left cofinal. The simplest way to do this is to interpret cofinality geometrically.

3.13. *Restricted Simplicial Sets.* A *restricted simplicial set* $X$ is a "simplicial set without degeneracies," that is, $X$ consists of

(i) a set $X_n$ (of $n$-simplices) for each $n\geq 0$, and
(ii) maps $d_i: X_n\to X_{n-1}$, $0\leq i\leq n$, such that $d_i d_j = d_{j-1} d_i$ if $i<j$ [3, p. 230].

Restricted simplicial sets can be identified in the usual way with functors

$(\Delta_{\text{rest}})^{\text{op}} \to$ Sets. A restricted simplicial set $X$ has a natural enveloping space, denoted $R(X)$, defined by

$$R(X) = \left( \coprod_{n \geq 0} X_n \times \Delta[n] \right) \Big/ \sim .$$

Hence $\Delta[n]$ is the standard $n$-simplex and the equivalence relation $\sim$ is generated by

$$(d_i x, s) \sim (x, \delta^i s) \qquad \begin{array}{l} x \in X_{n+1} \\ s \in \Delta[n] \end{array}$$

where $\delta^i : \Delta[n] \to \Delta[n+1]$ is the $i$'th face inclusion.

The functor $R$ is the left adjoint to the forgetful functor from simplicial sets to restricted simplicial sets. It is not hard to see that the nondegenerate simplices of $R(X)$ are in one-one correspondence with the simplices of $X$ itself.

3.14. LEMMA. *Let $J : \Delta_{\text{rest}} \to \mathrm{D}$ be a functor, and let $d \in \mathrm{D}$. Then $J/d$ is weakly homotopy equivalent to $R(X)$, where $X$ is the functor $(\Delta_{\text{rest}})^{\text{op}} \to$ Sets given by*

$$X([n]) = \operatorname{Hom}_{\mathrm{D}} (J([n]), d).$$

*Proof.* A calculation shows that for any small category C and functor $J : \mathrm{C} \to \mathrm{D}$, the space $J/d$ is isomorphic to

$$\left( \coprod_{c \in \mathrm{C}} \operatorname{Hom}_{\mathrm{D}} (J(c), d) \times (\mathrm{C}/c) \right) \Big/ \sim .$$

Here the equivalence relation $\sim$ is generated by

$$(f \circ Jg, s) \sim (f, g_* s)$$

where $f \in \operatorname{Hom}_{\mathrm{D}} (J(c_2), d)$, $g \in \operatorname{Hom}_{\mathrm{C}} (c_1, c_2)$, and $g_* : \mathrm{C}/c_1 \to \mathrm{C}/c_2$ is induced by $g$. The proof consists in applying this to $\Delta_{\text{rest}}$ and using the fact that for each $n$ the space $\Delta_{\text{rest}}/[n]$ is isomorphic to the first barycentric subdivision of the standard $n$-simplex.

An *augmentation* for a restricted simplicial set $X$ is a map $d_0 : X_0 \to X_{-1}$ such that $d_0d_0 = d_0d_1 : X_1 \to X_{-1}$. A *contracting homotopy* for an augmented restricted simplicial set $X$ is a family of maps $s: X_n \to X_{n+1}$ such that

(i) $d_{n+1}s = \text{identity}$, $n \geq -1$, and
(ii) $d_is = sd_i$, $0 \leq i \leq n$.

In the statement of the following lemma, the set $X_{-1}$ is identified with the discrete space it represents.

3.15. LEMMA. *If $X$ is an augmented restricted simplicial set with a contracting homotopy, the induced augmentation map $R(X) \to X_{-1}$ is a weak homotopy equivalence.*

*Proof.* Since $R$ commutes with disjoint union, it is enough to prove the lemma when $X_{-1}$ is a single point. In this case one computes that the fundamental group of $R(X)$ is trivial and that $s$ induces a contracting homotopy on the normalized integral chain complex of $R(X)$.

3.16 *Proof* of 3.6. Let $\Delta^+_{\text{rest}}$ denote the augmented restricted simplicial category with a contracting homotopy, that is, $\Delta^+_{\text{rest}}$ consists of

(i) one object $[n]$ for each $n \geq -1$,
(ii) for every pair $(i, n)$ of integers with $0 \leq i \leq n$ coface maps

$$d^i : [n-1] \to [n]$$

such that $d^jd^i = d^id^{j-1}$ if $i < j$, and

(iii) for each $n \geq 0$ a map

$$s : [n] \to [n-1]$$

such that

$$sd^n = \text{identity}, \ n \geq -1$$
$$sd^i = d^is, \ i < n.$$

There is an obvious inclusion functor $J: \Delta_{\text{rest}} \to \Delta^+_{\text{rest}}$. Suppose that the map $\phi(X): X \to T(X)$ admits a left inverse $r: T(X) \to X$. The resolution functor $\mathbf{T}X: \Delta_{\text{rest}} \to \mathrm{S}$ can then be extended to a functor $\mathbf{T}^+X: \Delta^+_{\text{rest}} \to \mathrm{S}$ by setting

$$(\mathbf{T}^+X[n] \xrightarrow{s} \mathbf{T}^+X[n-1]) = (T^{n+1}X \xrightarrow{T^n(r)} T^nX)$$

and

$$(\mathbf{T}^+X[-1] \xrightarrow{d^0} \mathbf{T}^+X[0]) = (X \xrightarrow{\phi(X)} TX).$$

This gives a commutative diagram

$$\begin{tikzcd} \Delta_{\mathrm{rest}} \arrow[rr, "J"] \arrow[dr] & & \Delta^+_{\mathrm{rest}} \arrow[dl] \\ & \mathbf{S} & \end{tikzcd}$$

The category $\Delta^+_{\mathrm{rest}}$ has $[-1]$ as an initial object, so the canonical map

$$X = \mathbf{T}^+X[-1] = \varprojlim \mathbf{T}^+X \to \operatorname{holim}_{\leftarrow} \mathbf{T}^+X$$

is a homotopy equivalence [3, p. 299]. (This can also be derived from the fact that the inclusion of the singleton category $[-1]$ into $\Delta^+_{\mathrm{rest}}$ is left cofinal.) Thus, by 3.12, it is enough to show that the functor $J$ is left cofinal.

Pick $[m] \in \Delta^+_{\mathrm{rest}}$. If the restricted simplicial set $X$ given by

$$X[n] = \operatorname{Hom}_{\Delta^+_{\mathrm{rest}}} (J([n]), [m])$$

is furnished with the augmentation

$$X[0] \xrightarrow{d_0} \operatorname{Hom}_{\Delta^+_{\mathrm{rest}}} ([-1], [0])$$

induced by composition on the right with $d^0$, then composition on the left with $s$ provides maps

$$X[n] \to X[n+1]$$

which give a contracting homotopy for $X$. By 3.15 $R(X)$ is contractible, and the lemma thus follows from 3.14.

3.17. *Proof of* 3.8. Let $J: \Delta_{\mathrm{rest}} \to \Delta$ be the inclusion functor. According to 3.12, it is sufficient to show that $J$ is left cofinal. Pick $[m] \in \Delta$ and let $X$ be the restricted simplicial set with

$$X[n] = \operatorname{Hom}_{\Delta} (J([n]), [m]).$$

It is clear that $X$ is just the underlying restricted simplicial set of the standard $m$-simplex $\Delta[m]$. An easy calculation shows that $R(X)$ is simply-connected; in addition, since the normalized integral chain complex of $R(X)$ is the same as the unnormalized integral chain complex of $\Delta[m]$, the reduced integral homology of $R(X)$ vanishes. Therefore, $R(X)$ is contractible and the lemma follows from 3.14.

## 4. A tower lemma

4.1. *Fibrant Towers.* A *tower* of spaces $\{X_\alpha\}_{\alpha<\beta}$ of length $\beta$ is a functor $\Omega_\beta \to \mathrm{S}$, where $\Omega_\beta$ is the full subcategory of $\Omega$ containing all ordinals less than $\beta$. Unlike long towers, towers are small diagrams of spaces and thus have both inverse limits and homotopy inverse limits.

The tower $\{X_\alpha\}_{\alpha<\beta}$ is said to be *fibrant* if

(i) $X_0$ is a fibrant space, and
(ii) for each $\alpha<\beta$ the natural map

$$X_\alpha \to \varprojlim \{X_\gamma\}_{\gamma<\alpha}$$

is a fibration.

4.2. FIBRANT TOWER LEMMA. *If $\{X_\alpha\}_{\alpha<\beta}$ is a fibrant tower, then the natural map*

$$\varprojlim \{X_\alpha\}_{\alpha<\beta} \to \operatorname{ho}\varprojlim \{X_\alpha\}_{\alpha<\beta}$$

*is a homotopy equivalence.*

The *function complex* Hom $(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta})$ of maps between two towers is the space whose $n$-simplices ($n\geq 0$) comprise all tower maps

$$\{A_\alpha \times \Delta[n]\}_{\alpha<\beta} \to \{X_\alpha\}_{\alpha<\beta}$$

and whose face and degeneracy operators are induced by the standard inclusion $\Delta[n]\to\Delta[n+1]$ and the standard collapses $\Delta[n]\to\Delta[n-1]$ [3, p. 295]. If $\{*\}_{\alpha<\beta}$ is the constant one-point tower, then

$$\operatorname{Hom}(\{*\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) = \varprojlim \{X_\alpha\}_{\alpha<\beta}$$

while if $\{\Omega_\beta/\alpha\}_{\alpha<\beta}$ is the tower of 3.10, then

$$\operatorname{Hom}(\{\Omega_\beta/\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) = \operatorname{ho}\varprojlim \{X_\alpha\}_{\alpha<\beta}.$$

4.3. LEMMA. *Suppose that* $\{A_\alpha\}_{\alpha<\beta} \to \{B_\alpha\}_{\alpha<\beta}$ *is a tower map which induces a trivial cofibration* $A_\alpha \to B_\alpha$ *for each* $\alpha < \beta$. *Then for any fibrant tower* $\{X_\alpha\}_{\alpha<\beta}$ *the restriction map*

$$\operatorname{Hom}(\{B_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \to \operatorname{Hom}(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta})$$

*is a trivial fibration.*

4.4. *Remark.* A fibration or cofibration is *trivial* if it is also a weak homotopy equivalence.

Lemma 4.2 is proved by applying 4.3 twice: first to the obvious map

$$\{\Omega_\beta/\alpha\}_{\alpha<\beta} \to \{\Omega_\beta/0\}_{\alpha<\beta}$$

where the second tower is constant, and then to any inclusion

$$\{*\}_{\alpha<\beta} \to \{\Omega_\beta/0\}_{\alpha<\beta}.$$

Note that each of the spaces $\Omega_\beta/\alpha$ is a contractible by [3, p. 293].

*Proof of* 4.3. The conclusion of 4.3 holds if and only if a dotted arrow exists in every diagram of the form [7, II, 2.1]

$$\begin{tikzcd} \dot\Delta[n] \arrow[r] \arrow[d] & \operatorname{Hom}(\{B_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \operatorname{Hom}(\{A_\alpha\}_{\alpha<\beta}, \{X_\alpha\}_{\alpha<\beta}) \end{tikzcd}$$

where $\dot\Delta[n] \to \Delta[n]$ is the inclusion of the boundary of the standard $n$-simplex. By an adjointness argument this is equivalent to showing that the dotted arrow exists in each diagram

$$\begin{tikzcd} \{A_\alpha\}_{\alpha<\beta} \arrow[r] \arrow[d] & \{\operatorname{Hom}(\Delta[n], X_\alpha)\}_{\alpha<\beta} \arrow[d] \\ \{B_\alpha\}_{\alpha<\beta} \arrow[r] \arrow[ur, dashed] & \{\operatorname{Hom}(\dot\Delta[n], X_\alpha)\}_{\alpha<\beta} \end{tikzcd}$$

where in this case Hom denotes the standard function complex of maps between spaces [6, p. 16]. This second dotted arrow is constructed by an induction on $\alpha$. The case $\alpha = 0$ is straightforward and uses the assumption that $X_0$ is fibrant. The induction step for $\alpha > 0$ depends on the existence of yet another dotted arrow in the diagram

$$\begin{tikzcd}
A_\alpha \arrow[r] \arrow[d] & \operatorname{Hom}(\Delta[n], X_\alpha) \arrow[d] \\
B_\alpha \arrow[r] \arrow[ur, dashed] & \varprojlim \{\operatorname{Hom}(\Delta[n], X_\gamma)\}_{\gamma<\alpha} \times_{\lim \{\operatorname{Hom}(\dot{\Delta}[n], X_\gamma)\}_{\gamma<\alpha}} \operatorname{Hom}(\dot{\Delta}[n], X_\alpha)
\end{tikzcd}$$

This dotted arrow exists because the left vertical map is a trivial cofibration and the right vertical arrow is a fibration.

## 5. $R$-Bousfield spaces

The purpose of this section is to prove 1.2. The proof is based on Bousfield's algebraic characterization of $R$-Bousfield spaces [1, §5].

We will use the terminology of [2] except that *$HR$-local* groups and *$H\mathbb{Z}$-local* $\pi$-modules will be called *$R$-Bousfield* groups and *$\mathbb{Z}$-Bousfield* $\pi$-modules. Recall that $R$ is a subring of the rational numbers or a finite field of the form $\mathbb{Z}/p\mathbb{Z}$, $p$ prime.

5.1. PROPOSITION [2:3.10, 2.6]. *The $R$-Bousfield groups form the smallest class of groups such that*

(i) *the class contains the trivial group,*
(ii) *the class is closed under inverse limits of arbitrary towers,*
(iii) *if $Y$ is in the class and $1 \to W \to X \to Y \to 1$ is a central extension with $W$ an $R$-module, then $X$ is in the class,*
(iv) *if $X$ is in the class and $1 \to W \to X \to Y \to 1$ is a short exact sequence with $Y$ abelian and an $R$-module, then $W$ is in the class.*

Let $\pi$ be a group and let $M$ be a $\pi$-module. Then $M$ will be called an *$R$-Bousfield* $\pi$-module if

(i) $M$ is $R$-Bousfield as an (abelian) group, and
(ii) $M$ is $\mathbb{Z}$-Bousfield as a $\pi$-module.

It is not hard to prove using [2:8.9, 7.3] that

5.2. LEMMA. *The $R$-Bousfield $\pi$-modules form the smallest class of $\pi$-modules such that*

(i) *the class contains the zero $\pi$-module,*

(ii) *the class is closed under inverse limits of arbitrary towers,*
(iii) *if $Y$ is in the class and $0 \to W \to X \to Y \to 0$ is an extension of $\pi$-modules with $W$ simple ($=$ trivial $\pi$-action) and an $R$-module, then $X$ is in the class,*
(iv) *if $X$ is in the class and $0 \to W \to X \to Y \to 0$ is a short exact sequence with $Y$ simple and an $R$-module, then $W$ is in the class.*

In fact, it is clear that the class of $R$-Bousfield $\pi$-modules contains the class described in 5.2. If $R \subseteq \mathbb{Q}$ the opposite inclusion follows easily from the fact that by naturality the $H\mathbb{Z}$-tower of an $R$-Bousfield $\pi$-module $M$ is itself a tower of $R$-modules. If $R = \mathbb{Z}/p\mathbb{Z}$ it is possible to use the natural action of $\pi$ on the $HR$-tower of the underlying abelian group of $M$ and to show by transfinite induction that each $\pi$-module in this tower belongs to the class described.

5.3. LEMMA [1, §5]. *A fibrant space $X$ is $R$-Bousfield if and only if for every $i \geq 2$ and every choice of basepoint $x \in X$,*

(i) *$\pi_1(X, x)$ is an $R$-Bousfield group, and*
(ii) *$\pi_i(X, x)$ is an $R$-Bousfield $\pi_1(X, x)$-module.*

5.4. *Proof of* 1.2. Let $C$ denote the union of the classes $I_\alpha$. It is necessary to show that every $R$-Bousfield space $X$ belongs to $C$. Note that by definition $C$ is closed under arbitrary homotopy inverse limits.

Let $\pi$ be a group and let $M$ be a $\pi$-module. For $n \geq 1$, $L(\pi, M, n)$ denotes the split fibration over $K(\pi, 1)$ with fibre $K(M, n)$ which is determined by the action of $\pi$ on $M$.

Every fibrant space $X$ is homotopy equivalent to the homotopy inverse limit of its Postnikov tower $\{P_nX\}_{n<\omega}$, where $\omega$ is the first infinite ordinal. Moreover, if $B$ runs through a selection of basepoints for $X$, one for each path component, there are homotopy fibre squares

$$\begin{tikzcd} P_nX \arrow[r] \arrow[d] & \coprod_{b\in B} K(\pi_1(X,b),1) \arrow[d] \\ P_{n-1}X \arrow[r] & \coprod_{b\in B} L(\pi_1(X,b), \pi_n(X,b), n+1). \end{tikzcd}$$

Thus by 5.3 it suffices to show that for each fixed $n \geq 1$ every space which is a disjoint union of spaces of the form $L(\pi, M, n)$ for various $R$-Bousfield groups $\pi$ and various $R$-Bousfield $\pi$-modules $M$ belongs to $C$.

This is done by induction on $n$. We will assume $n > 1$ and prove that every (connected) space of the form $L(\pi, M, n)$ belongs to $C$. The general case can be proved in the same way by using the fact that homotopy inverse limits over categories with connected nerves commute with disjoint unions. The initial case $n=1$ is similar to the case $n>1$ but simpler.

Let $\pi$ be an $R$-Bousfield group. It is easily seen that the class of $R$-Bousfield $\pi$-modules $M$ such that $L(\pi, M, n)$ belongs to $C$ satisfies parts (i), (iii) and (iv) of 5.2, so it remains to show that if $\{M_\alpha\}_{\alpha<\beta}$ is a tower of $R$-Bousfield $\pi$-modules such that each $L(\pi, M_\alpha, n)$ belongs to $C$, then $L(\pi, M, n)$ belongs to $C$, where $M=\lim_{\leftarrow}\{M_\alpha\}_{\alpha<\beta}$. This is done as follows. Using bar construction techniques [6, p. 83] one devises a way of constructing the spaces $L(\pi, M_\alpha, n)$ which is functorial in $M_\alpha$. Thus the tower $\{M_\alpha\}_{\alpha<\beta}$ of $\pi$-modules gives rise to a tower $\{L(\pi, M_\alpha, n)\}_{\alpha<\beta}$ of spaces. Let $X$ denote $\operatorname{holim}_{\leftarrow}\{L(\pi, M_\alpha, n)\}_{\alpha<\beta}$. The space $X$ belongs to $C$ and by [3, p. 309] and naturality there are $\pi$-module isomorphisms

$$\pi_n X \approx M$$
$$\pi_i X \approx 0, \qquad i>n.$$

Note that the homotopy groups of $X$ actually are $\pi$-modules by virtue of the fact that the composite

$$f: X \to L(\pi, M_0, n) \to K(\pi, 1)$$

has a section $K(\pi, 1) \to X$.

Let $P_{n-2}(f)$ denotes the $n-2$ stage in the Moore-Postnikov factorization of $f$ [6, p. 34]. The inductive hypothesis implies that $P_{n-2}(f)$ belongs to $C$, so the space $Y$ which is defined as the homotopy inverse limit of the square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & X \arrow[d] \\ K(\pi, 1) \arrow[r] & P_{n-2}(f) \end{tikzcd}$$

also belongs to $C$. Up to homotopy the space $P_{n-1}(g)$ is a split fibration over $K(\pi, 1)$ with $K(\pi_{n-1}(Y), n-1)$ as the fibre, so, by induction, $P_{n-1}(g)$ belongs to $C$ too. The proof is finished by noting that there is a homotopy fibre square

$$\begin{tikzcd} L(\pi, M, n) \arrow[r] \arrow[d] & Y \arrow[d] \\ K(\pi, 1) \arrow[r] & P_{n-1}(g). \end{tikzcd}$$

## 6. Construction of the tower

The object of this section is to construct for each $\alpha \in \Omega$ an augmented functor $(T_\alpha, \phi_\alpha)$ on S and compatible morphisms $(T_\beta, \phi_\beta) \to (T_\alpha, \phi_\alpha)$ for $\beta > \alpha$. For $X \in$ S the augmented long tower $X \to \{T_\alpha X\}_\alpha$ is the *R-homology localization tower* of $X$.

The construction is by transfinite induction. The pair $(T_0, \phi_0)$ is the underlying augmented functor of the triple $(R, \phi, \psi)$ of [3, p. 13]. If $\alpha = \beta + 1$ is a successor ordinal, $(T_\alpha, \phi_\alpha)$ is $(\hat{T_\beta}, \hat{\phi_\beta})$; by 3.3 there is a natural morphism $(T_\alpha, \phi_\alpha) \to (T_\beta, \phi_\beta)$. Finally, if $\alpha$ is a limit ordinal the pair $(T_\alpha, \phi_\alpha)$ is $\lim_{\leftarrow} \{(T_\beta, \phi_\beta)\}_{\beta<\alpha}$; this evidently comes with a natural map into $(T_\beta, \phi_\beta)$ for each $\beta < \alpha$.

The identification of $T_1$ and $T_2$ made in 1.5 follows easily from 3.8, 3.9 and 6.1 below. The rest of this section is taken up with proving that the tower $\{T_\alpha X\}_\alpha$ has the properties listed in 1.4. Recall the

6.1. *Homotopy Invariance Lemma* [3, p. 304]. *Let* D *be a small category, let* $F, G: \mathrm{D} \to \mathrm{S}$ *be functors, and let* $\tau: F \to G$ *be a natural transformation. Suppose that for all* $d \in \mathrm{D}$

(i) *the spaces* $F(d)$ *and* $G(d)$ *are fibrant, and*
(ii) *the map* $\tau(d): F(d) \to G(d)$ *is a homotopy equivalence.*

*Then* $\tau$ *induces a homotopy equivalence* $\mathrm{Holim}_{\leftarrow} F \to \mathrm{Holim}_{\leftarrow} G$.

6.2. *Proof of* 1.4(i). The space $T_0X = RX$ is always fibrant, since choice of a basepoint for $X$ makes $RX$ into a simplicial $R$-module [3, p. 14]. Using 3.4 it is easy to show by induction that $T_\alpha X$ is fibrant for all $\alpha$.

For any space $X$, $\pi_* RX$ is naturally isomorphic to $\tilde{H}_*(X; R)$ (reduced homology). This implies that a map $f: X \to Y$ induces a homotopy equivalence $T_0X \to T_0Y$ iff it induces an isomorphism $\tilde{H}_*(X; R) \to \tilde{H}_*(Y; R)$. Thus 1.4(i) follows inductively from 6.1 and, in the limit ordinal case, 4.2.

6.3. *Proof of* 1.4(ii). This follows inductively from the definitions and, in the limit ordinal case, 4.2.

6.4. *Proof of* 1.4(iii). We will show by induction on $\alpha$ that if $X \in I_\alpha$ the natural map $\phi_\alpha(X): X \to T_\alpha X$ has a left inverse $r: T_\alpha X \to X$. The desired result then follows from 3.6.

In the case $\alpha = 0$, it is possible to assume that each component of $X$ has the structure of a simplicial $R$-module. Thus if $X$ is connected there is an obvious canonical retraction $RX \to X$ given by evaluating formal sums. A retraction in the disconnected case can be constructed by using the fact that the map $\pi_0(\phi(X)) = \pi_0 X \to \pi_0 RX$ is injective, since it is essentially the Hurewicz homomorphism $\pi_0 X \to \tilde{H}_0(X; R)$.

Suppose $\alpha > 0$. It is enough to show that there is a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\phi_\alpha(X)"] \arrow[dr] & & T_\alpha X \arrow[dl] \\ & W & \end{tikzcd}$$

in which the map $X \to W$ is a weak homotopy equivalence. In fact it is clear that given such a triangle there exists perhaps another one in which $T_\alpha X \to W$ is a cofibration. The map $\phi_\alpha(X)$ is a cofibration (since $X \to RX = T_0X$ is) so it follows that $X \to W$ is a cofibration too. The fact that $X$ is fibrant then implies that the map $X \to W$ has a left inverse.

Note that the induction hypothesis implies that if $\beta < \alpha$ and $Y \in I_\beta$, then the map $\phi_\alpha(Y): Y \to T_\alpha Y$ is a homotopy equivalence. This is immediate if $\alpha$ is a successor ordinal and follows from 4.2 and a tower cofinality argument ([3, p. 317] and 3.12) if $\alpha$ is a limit ordinal.

It is possible to assume that there is some small category C and functor $F: \mathrm{C} \to \mathrm{S}$ such that

(i) $X = \operatorname{holim}_{\leftarrow} F$, and

(ii) for each $c \in \mathrm{C}$ there is a $\beta < \alpha$ such that $F(c) \in I_\beta$.

Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{holim}_{\leftarrow} \operatorname{con}(X) \arrow[r, "s"] \arrow[d] & \operatorname{holim}_{\leftarrow} X \times \mathrm{C}/- \arrow[r] \arrow[d] & \operatorname{holim}_{\leftarrow} F \arrow[d] \\ T_\alpha X \arrow[r] & \operatorname{holim}_{\leftarrow} \operatorname{con}(T_\alpha X) & \operatorname{holim}_{\leftarrow} T_\alpha(X \times \mathrm{C}/-) \arrow[l, "t"'] \arrow[r] & \operatorname{holim}_{\leftarrow} T_\alpha F. \end{tikzcd}$$

Here $\operatorname{con}(X)$ and $\operatorname{con}(T_\alpha X)$ denote the obvious constant functors $\mathrm{C} \to \mathrm{S}$ and $\mathrm{C}/-$ is as in 3.10. The vertical maps are induced by $\phi_\alpha$, the left-hand horizontal maps by the natural transformation $\lim_{\leftarrow} \to \operatorname{holim}_{\leftarrow}$ [3, p. 298] and the right-hand horizontal maps by the morphism $X \times \mathrm{C}/- \to F$ which is adjoint to the identity map $X \to \operatorname{holim}_{\leftarrow} F$ [3, p. 296]. The map $s$ takes $f \in \operatorname{Hom}(\mathrm{C}/-, \operatorname{con}(X)) = \operatorname{holim}_{\leftarrow} \operatorname{con}(X)$ to $f \times id \in \operatorname{Hom}(\mathrm{C}/-, \operatorname{con}(X)) \times \operatorname{Hom}(\mathrm{C}/-, \mathrm{C}/-) = \operatorname{holim}_{\leftarrow} X \times \mathrm{C}/-$. Finally, $t$ is induced by the projection

$$T_\alpha(X \times \mathrm{C}/-) \to \operatorname{con}(T_\alpha X).$$

The composite of the maps on the top line is the identity map, and $t$ is a homotopy equivalence by 6.1. Furthermore, the induction hypothesis shows that the map $\operatorname{holim}_{\leftarrow} F \to \operatorname{holim}_{\leftarrow} T_\alpha F$ is a homotopy equivalence.

Factor the map $t$ as the composite of a trivial cofibration (4.4)

$$\underleftarrow{\operatorname{holim}}\, T_\alpha(X \times C/-) \to Y$$

and a trivial fibration

$$Y \to \underleftarrow{\operatorname{holim}} \operatorname{con}(T_\alpha X).$$

Let $Y'$ be the pushout of the diagram

$$\begin{tikzcd} \operatorname{holim} T_\alpha(X \times C/-) \arrow[r] \arrow[d] & \underleftarrow{\operatorname{holim}}\, T_\alpha F \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

so that the map $\operatorname{holim}_{\leftarrow} T_\alpha F \to Y'$ is a weak homotopy equivalence [7:I, §1, M4]. There results a commutative diagram of solid arrows

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y' \\ T_\alpha X \arrow[r] \arrow[ur, dotted] & \underleftarrow{\operatorname{holim}} \operatorname{con}(T_\alpha X) & \end{tikzcd}$$

in which the composite $X \to Y'$ is a weak homotopy equivalence. The dotted arrow can then be found because the left vertical arrow is a cofibration and the right vertical arrow is a trivial fibration.

## 7. An interpretation of the functors $T_\alpha$

The purpose of this section is to show that the spaces $T_\alpha X$ of §6 can be identified, up to homotopy, with the homotopy inverse limits of Artin-Mazur-like large diagrams of spaces. This is a natural extension of the identification of $R_\infty X(\sim T_1 X)$ made in [3, p. 324].

Let $(T, \phi)$ be an augmented functor on S. A space $Y$ is said to admit a *T-structure* if the natural map $\phi(Y): Y \to TY$ has a left inverse $r: TY \to Y$. For any space $X$ and ordinal $\beta > 0$, let $T_{\alpha<\beta} \backslash X$ be the category consisting of

(i) one object for each map $X \to Y$ of S such that $Y$ admits a $T_\alpha$-structure for some $\alpha < \beta$, and

(ii) one morphism $(X \to Y) \to (X \to Y')$ for each $f: Y \to Y'$ in S such that

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, "f"] & & Y' \end{tikzcd}$$

commutes.

There is an Artin-Mazur functor

$$AM_\beta(X): T_{\alpha<\beta}\backslash X \to \mathrm{S}$$

which sends $(X \to Y)$ to the target space $Y$.

7.1. PROPOSITION. *For any ordinal $\beta > 0$ the space $T_\beta X$ has the homotopy type of the homotopy inverse limit of $AM_\beta(X)$.*

From a qualitative point of view the proposition says that the map $X \to T_\beta X$ comes as close as homotopy theory allows to being universal for all maps $X \to Y$ with the property that $Y$ admits a $T_\alpha$-structure for some $\alpha < \beta$.

Part of the work in proving 7.1 is to show that the homotopy inverse limit of the large diagram $AM_\beta(X)$ is well defined, up to homotopy. Recall that a (large) category D is said to be *left small* if there is a left cofinal (3.12) functor $J: \mathrm{C} \to \mathrm{D}$. (Note that C, as the domain of a left cofinal functor, is necessarily a small category.)

7.2. PROPOSITION [3, p. 321–322]. *If* D *is a left small category and* $F: \mathrm{D} \to \mathrm{S}$ *is a functor, then the homotopy inverse limit of $F$ is well defined, up to homotopy. Moreover, if $J: \mathrm{C} \to \mathrm{D}$ is left cofinal and $F(J(c))$ is fibrant for each $c \in \mathrm{C}$, then the homotopy inverse limit of $F$ has the homotopy type of* $\operatorname{holim}_{\leftarrow} F \circ J$.

The proof of 7.1 breaks up into two cases.

7.3. *The Successor Case.* Suppose that $\beta = \gamma + 1$ is a successor ordinal. The argument of 6.4 shows that for any space $Y$ the space $T_\gamma Y$ admits a $T_\gamma$-structure; in particular, the spaces $T_\gamma^n X$ $(n \geq 1)$ admit $T_\gamma$-structures. Thus the restricted cosimplicial space $\mathbf{T}_\gamma X$ together with its augmentation determines a functor

$$\mathbf{T}'_\gamma X: \Delta_{\text{rest}} \to T_{\alpha<\beta}\backslash X.$$

Since each of the spaces $T_\gamma^n X$ ($n \geq 1$) is fibrant (6.2), it suffices to prove that $T'_\gamma X$ is left cofinal.

Pick an object $X \to Y$ of $T_{\alpha<\beta}\backslash X$. By 3.14 it is enough to show that $R(W)$ is contractible, where $W$ is the restricted simplicial set given in dimension $n$ by

$$W_n = \operatorname*{Hom}_{T_{\alpha<\beta}\backslash X} (X \to T_\gamma^{n+1}X, X \to Y).$$

The space $Y$ admits a $T_\alpha$-structure for some $\alpha \leq \gamma$; this easily implies that $Y$ admits a $T_\gamma$-structure. Let $r: T_\gamma Y \to Y$ be a left inverse for $\phi_\gamma(Y): Y \to T_\gamma Y$. Define maps $s: W_n \to W_{n+1}$ by

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & & & & X \arrow[dl] \arrow[dr] & \\ T_\gamma^{n+1}X \arrow[rr, "f"] & & Y \arrow[r, phantom, "\xrightarrow{s}"] & & T_\gamma^{n+2}X \arrow[rr, "r\circ T_\gamma(f)"] & & Y \end{tikzcd}$$

If $W$ is augmented in the natural way by letting $W_{-1}$ be the one-point set representing the commutative diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

then the maps $s$ provide a contracting homotopy for $W$. The desired result then follows from 3.15.

7.4. *The Limit Ordinal Case.* Suppose that $\beta$ is a limit ordinal. Let

$$\{\mathbf{T}_\alpha X\}_{\alpha<\beta}: \Omega_\beta \times \Delta_{\text{rest}} \to \mathrm{S}$$

be the functor which assigns to each space $X$ the tower $\{\mathbf{T}_\alpha X\}_{\alpha<\beta}$ of restricted cosimplicial spaces. As in 7.3 it is easy to see this lifts to a functor

$$\{\mathbf{T}'_\alpha X\}_{\alpha<\beta}: \Omega_\beta \times \Delta_{\text{rest}} \to T_{\alpha<\beta}\backslash X.$$

Since $\operatorname{holim}_{\leftarrow} \{\mathbf{T}_\alpha X\}_{\alpha<\beta}$ is homotopy equivalent to $T_\beta X$ [3, p. 300, 4.3] it is enough to show that $\{\mathbf{T}'_\alpha X\}_{\alpha<\beta}$ is left cofinal.

Pick $X \to Y$ in $T_{\alpha<\beta}X$. We will use the language of *homotopy direct limits* [3, p. 325] to sketch a proof that $\{\mathbf{T}'_\alpha X\}_{\alpha<\beta}/(X \to Y)$ is contractible.

First there is a general observation. Let C be a small category and let $J: \mathrm{C} \to \mathrm{D}$ be a functor. For any element $d \in \mathrm{D}$ there is a functor $H_d: \mathrm{C}^{\mathrm{op}} \to \mathrm{SETS}$ sending $c \in \mathrm{C}$ to the set $\mathrm{Hom}_{\mathrm{D}}(J(c), d)$. Since any set can be identified with a discrete space, $H_d$ can be thought of as a functor $\mathrm{C}^{\mathrm{op}} \to \mathrm{S}$. The following calculation was implicitly referred to in the proof of 3.14.

7.5. LEMMA. *For each $d \in \mathrm{D}$ there is an isomorphism of spaces*

$$J/d \approx \underrightarrow{\mathrm{holim}}\, H_d.$$

According to the properties of homotopy direct limits over product categories [3, p. 331], this implies that

$$\{T'_\alpha X\}_{\alpha<\beta}/(X \to Y) = \underrightarrow{\mathrm{holim}}\, F$$

where $F: \Omega_\beta^{\mathrm{op}} \to \mathrm{S}$ is the functor which sends $\alpha \in \Omega_\beta^{\mathrm{op}}$ to $\mathbf{T}'_\alpha X/(X \to Y)$.

If $Y$ admits a $T_\gamma$-structure, the argument of 7.3 shows that $F(\alpha)$ is contractible for all $\alpha \in \Omega_\beta^{\mathrm{op}}$, $\alpha \geq \gamma$. The desired result then follows from the fact that since $\Omega_\beta^{\mathrm{op}}$ is right filtering, $\mathrm{holim}_\to F$ is weakly homotopy equivalent to $\lim_\to F$ [3, p. 332].

## 8. Examples

The purpose of this section is to extract some information about the behaviour of the long homology localization tower $\{T_\alpha X\}_\alpha$ for certain special classes of spaces $X$. In particular, we are interested in how rapidly the tower converges to $X_R$. The main tool for studying this is 1.4(iii).

8.1. *Nilpotent Spaces.* It follows from 1.5 that $X_R \sim T_1 X$ ($\sim$ = homotopy equivalence) iff $X$ is $R$-*good* in the sense of Bousfield and Kan. In particular,

8.2. PROPOSITION [3, V, VI]. *If $X$ is a nilpotent space and $R$ is any of the admissible rings, then $X_R \sim T_1 X$.*

If $R \subseteq \mathbb{Q}$ we know of no spaces $X$ for which $X_R \sim T_1 X$ and $X_R$ is not nilpotent. If $R = \mathbb{Z}/p\mathbb{Z}$, however, there are many such examples ([3, VII], [4]).

8.3. *Virtually Nilpotent Spaces.* A connected space $X$ is said to be *virtually nilpotent* if each Postnikov stage $P_n X$ can be finitely covered by a nilpotent space.

If $R=\mathbb{Z}/p\mathbb{Z}$, then all such spaces are $R$-good [4]. The main result of [4] shows that if $X$ is virtually nilpotent and $R\subseteq\mathbb{Q}$ there is a homotopy fibre square

$$\begin{array}{ccc} X_R & \longrightarrow & W_1 \\ \downarrow & & \downarrow \\ W_3 & \longrightarrow & W_2 \end{array}$$

in which the spaces $W_1$, $W_2$ and $W_3$ have the homotopy type of homotopy inverse limits of (cosimplicial) diagrams of simplicial $R$-modules; that is, $W_1$, $W_2$, $W_3\in I_1$. It follows immediately that $X_R\in I_2$, so

8.4. PROPOSITION. *If $X$ is a virtually nilpotent space and $R\subseteq\mathbb{Q}$, then $X_R\sim T_3X$.*

This result may not be best possible. In fact, it is not hard to show that if $\pi_1X$ is *finite* and $R\subseteq\mathbb{Q}$, then $X_R\sim T_2X$. The argument for this uses [4] and the fibre lemma of [3, p. 62].

8.5. *Pre-nilpotent Fundamental Groups.* A group $\pi$ is said to be *pre-nilpotent* [5, 3.1] if the lower central series of $\pi$ stabilizes, not necessarily at the trivial group, after a finite number of steps. Let $\omega$ be the first infinite ordinal.

8.6. PROPOSITION. *Suppose that $R=\mathbb{Z}$ and that $X$ is a connected space with a finitely generated pre-nilpotent fundamental group. Then $X_R\sim T_{\omega+1}X$.*

8.7. *Remark.* Analogous results almost certainly hold for other rings. At least over $\mathbb{Z}$, the finite generation condition can be replaced by the assumption that $H_1(X;\mathbb{Z})$ is finitely generated.

We will only sketch the proof of 8.6, since the main point is purely algebraic. Assume $R=\mathbb{Z}$. The hypothesis on $X$ implies that $\pi_1X_R$ is a finitely generated nilpotent group [1, 7.3, 7.5] so, by 1.4(i)–(iii) it is enough to show that if $Y$ is a connected $R$-Bousfield space with a finitely generated nilpotent fundamental group, then $Y\in I_\omega$. By the Postnikov argument of 5.4 it is enough to show that whenever $\pi$ is a finitely generated nilpotent group and $M$ is an $R$-Bousfield $\pi$-module, then $L(\pi, M, n)\in I_m$ for some integer $m$.

Let $E$ denote the $H\mathbb{Z}$-localization functor on the category of $\pi$-modules and let $F\to M\to 0$ be an epimorphism from the free $\pi$-module $F$ to the $R$-Bousfield $\pi$-module $M$. Since $E$ is right exact [1, 8.11] there is a short exact sequence

$$0\to K\to E(F)\to M\to 0$$

where $K$ is some $R$-Bousfield [1, 8.5] submodule of $E(F)$. In view of the homotopy fibre square

$$\begin{tikzcd} L(\pi, M, n-1) \arrow[r] \arrow[d] & K \arrow[d] \\ L(\pi, K, n) \arrow[r] & L(\pi, E(F), n) \end{tikzcd}$$

it suffices to prove that both $L(\pi, K, n)$ and $L(\pi, E(F), n)$ belong to some $I_m$.

Let $J$ be the augmentation ideal inside the integral group ring $\mathbb{Z}[\pi]$ of $\pi$. Then [5, 3.1] asserts that there is an isomorphism $E(F) \approx \lim_{\leftarrow} \{F/J^s \cdot F\}_{s<\omega}$. Since $K$ is a submodule of $E(F)$ it is clear that $K$ injects into $\lim_{\leftarrow} \{K/J^s \cdot K\}_{s<\omega}$. However,

$$H_0(\pi; \varprojlim \{K/J^s \cdot K\}_{s<\omega}) = \varprojlim \{H_0(\pi; K/J^s \cdot K)\}_{s<\omega} = H_0(\pi; K)$$

since $\pi$ is finitely generated and $\lim^1_{\leftarrow} \{H_1(\pi; K/J^s \cdot K)\}_{s<\omega}$, being a quotient of $\lim^1_{\leftarrow} \{H_1(\pi; K)\}_{s<\omega}$, vanishes (compare [5, Proof of 3.7]). It follows from [2, 7.8] that $K \approx \lim_{\leftarrow} \{K/J^s \cdot K\}_{s<\omega}$.

By 8.2 the spaces $L(\pi, K/J^s \cdot K, n)$ and $L(\pi, F/J^s \cdot F, n)$ belong to $I_1$, since they are nilpotent. The groups

$$\varprojlim{}^1 \{K/J^s \cdot K\}_{s<\omega} \quad \text{and} \quad \varprojlim{}^1 \{F/J^s \cdot F\}_{s<\omega}$$

vanish, since both of these module towers are towers of epimorphisms [3, p. 252]. Thus [3, pp. 287, 254],

$$L(\pi, K, n) \sim \operatorname{holim}_{\leftarrow} \{L(\pi, K/J^s \cdot K, n)\}_{s<\omega} \in I_2$$

$$L(\pi, F, n) \sim \operatorname{holim}_{\leftarrow} \{L(\pi, F/J^s \cdot F, n)\}_{s<\omega} \in I_2.$$

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## *Added note*

An alternative approach to constructing the homology localization as an inverse limit is given in

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