Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 52 (1977)

Artikel: Flaccidity of geometric index for nonsingular vector fields.

Autor: Asimov, Daniel

DOI: https://doi.org/10.5169/seals-39992

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 04.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Flaccidity of geometric index for nonsingular vector fields

DANIEL ASIMOV

1. Introduction

We consider nonsingular vector fields V on compact connected C^{∞} orientable manifolds M^n without boundary. It is of interest to find criteria for detecting the occurrence of closed orbits of V. If we were to look instead at diffeomorphisms $f: M \to M$ the analogous question would be to find the periodic points of f. We may at least count the periodic points algebraically via the Lefschetz fixed point formula, if $\{x \mid f^k(x) = x\} = \operatorname{Fix}(f^k)$ is finite:

$$\sum_{f^{k}(x)=x} i_{f^{k}}(x) = \sum_{j=0}^{n} (-1)^{j} \operatorname{tr}(f_{*j}^{k}: H_{j}(M; \mathbb{R}) \leq). \tag{1}$$

Here $i_{f^k}(x)$ is the local index of the fixed point x of f^k , and the right hand side is, by definition, the Lefschetz number $\Lambda(f^k)$ of f^k .

We define here an index for nonsingular Morse-Smale (NMS) vector fields [2] as follows. Let V be an NMS field on M. Let C_i $1 \le i \le r$ be all the closed orbits of V. Let $\bar{C}_i \in H_1(M; \mathbb{Z})$ be the homology class of C_i (oriented by V) and let $\varepsilon_i = \pm 1$ be the fixed point index of the Poincaré map induced by the flow of V around C_i . Then we define

$$J(V) = \sum_{i=1}^{r} \varepsilon_{i} \bar{C}_{i} \quad \text{in} \quad H_{1}(M; \mathbb{Z})$$
 (2)

to be the geometric index of the NMS field V.

Let \mathcal{V} denote all nonsingular vector fields on M, with the C^0 topology. In Section 2 we describe some situations where J remains constant over a large open set or on distinct homotopic NMS vector fields. In particular we show that if V is the suspension of a Morse-Smale diffeomorphism, then J(V) is constant through large perturbations of V.

In Section 3, however, we show by example that if the dimension of M is greater than 3, then for each $\alpha \in H_1(M; \mathbb{Z})$ and for each homotopy class \mathfrak{D} of

nonsingular vector fields on M, there exists a $V_{\alpha} \in \mathcal{D}$ such that V_{α} is NMS and $J(V_{\alpha}) = \alpha$. Fuller's Theorem 1 [8] then imposes conditions on any homotopy between V_{α} and V_{β} , $\alpha \neq \beta$. We note that NMS vector fields are structurally stable [9].

2. Examples

A. Let $p: E^n \to S^1$ be a smooth oriented fibre bundle with fibre the compact manifold F^{n-1} . Let $\{V_s\}_{0 \le s \le 1}$ denote a homotopy of nonsingular vector fields on E which are never tangent to a fibre. Assume that V_0 and V_1 are NMS, with orientable stable and unstable manifolds. Then we have

PROPOSITION A.

$$J(V_0) = J(V_1). \tag{3}$$

Proof. Without loss of generality we may assume that for all $(x, s) \in E \times I$ we have

$$Dp(V_s(x)) = 2\pi \cdot d/d\theta, \tag{4}$$

where $Dp: TE \to TS^1$ is the tangent mapping of p. Let $p_*: H_1E \to H_1S^1$ be the map on integer homology. Identify H_1S^1 with Z by sending the counterclockwise generator to 1. Now set $J_k(V_0) = \sum_{\bar{C}_i \in p_*^{-1}(k)} \varepsilon_i \bar{C}_i$. Then

$$J(V_0) = \sum_{k=1}^{\infty} J_k(V_0)$$
 (5)

and similarly

$$J(V_1) = \sum_{k=1}^{\infty} J_k(V_1).$$
 (6)

Hence the proposition follows from showing $J_k(V_0) = J_k(V_1)$ for all $k = 1, 2, \ldots$

We notice that by (4) the time-one map φ_s^1 of the flow $\{\varphi_s^t\}_{t\in\mathbb{R}}$ of V_s is a map $\varphi_s^1:E\to E$ which preserves fibres. Similarly for φ_s^k , k any integer. Let $p^*(E)$ denote the space $\{(x,y)\in E\times E\mid p(x)=p(y)\}$, the total space of the pullback by p

of the bundle $E \stackrel{p}{\to} S^1$. Then $p^*(E)$ fibres over S^1 with fibre $F \times F$, via $(x, y) \mapsto p(x) = p(y) \in S^1$. We define the subset $\Delta \subset p^*(E)$ by $\Delta = \{(x, x) \in p^*(E)\}$. Also let $\bar{p}: p^*(E) \to E$ be the projection given by $\bar{p}(x, y) = x$.

Now we may form the map $\Gamma_s^1: E \to p^*(E)$ defined by $\Gamma_s^1(x) = (x, \varphi_s^1(x))$. This map Γ_s^1 is transverse to $\Delta \subset p^*(E)$ precisely when the map φ_s^1 restricted to any fibre F has no eigenvalues equal to 1. In particular, this is the case for s = 0, 1. It is clear from the definitions that as a point set, $\bar{p}(\Gamma_i^1(E) \cap \Delta)$ is the union of all the closed orbits of V_i which go around E once, i.e., those closed orbits comprising $J_1(V_i)$, for i = 0, 1.

Now considering homology intersection \cdot , we let $[\Delta] \in H_n(p^*(E))$ denote the fundamental class of the diagonal, i.e., the image of $[E] \in H_n(E)$ under $x \mapsto (x, x)$. Let $\Gamma_{s^*}^1 : H_n(E) \to H_n(p^*(E))$ and $\bar{p}_* : H_1(p^*(E)) \to H_1(E)$ denote the induced maps on homology. Then

$$\Lambda_s^1 = \bar{p}_*(\Gamma_{s^*}^1([E]) \cdot [\Delta]) \quad \text{in} \quad H_1(E)$$
 (7)

is an integer homology class independent of s, and so

$$\Lambda_0^1 = \Lambda_1^1. \tag{8}$$

Let $f_i: F \to F$ denote $\varphi_i^1 \mid F$ for the fibre $F = p^{-1}(x)$ of E. Let $y \in F$ be the fixed point of f_i corresponding to the closed orbit C_y of V_i . Then C_y contributes $\varepsilon_y \bar{C}_y \in H_1(E)$ to the class Λ_i^1 , where

$$\varepsilon_{y} = \det \left(\frac{I_{n-1} | I_{n-1}}{Df(y) | I_{n-1}} \right)$$

i.e.,

$$\varepsilon_{y} = \det (I_{n-1} - Df(y))$$

$$= i_{f_{i}}(y),$$
(9)

the local fixed point index at y (since 1 is not an eigenvalue of Df(y)). Thus we have shown that

$$\Lambda_i^1 = J_1(V_i) \qquad i = 0, 1 \tag{10}$$

and so by (8) this shows $J_1(V_0) = J_1(V_1)$.

Now using the fact that we assumed all stable and unstable manifolds to be orientable for V_i , i = 0, 1, we may easily check that for i = 0, 1, we have

$$\pi_{k} * J_1(\tilde{V}_i) = k \sum_{d \mid k} J_d(V_i) \in H_1(E). \tag{11}$$

Here $\pi_k : \tilde{E}_k \to E$ denotes the canonical k-fold covering space over E (since E is a fibre bundle over S^1) and \tilde{V}_i denotes the unique lift of V_i to a nonsingular vector field on \tilde{E}_k .

Hence $J_k(V_i)$ is expressible in terms of $\pi_{k*}J_1(\tilde{V}_i)$ and $J_d(V_i)$ for $1 \le d < k$. Now $J_1(\tilde{V}_i)$ is independent of i = 0, 1 by the considerations we applied above to $J_1(V_i)$ (since the homotopy V_s gives rise to a homotopy \tilde{V}_s , $0 \le s \le 1$). Hence we also have

$$\pi_{k} J_{1}(\tilde{V}_{0}) = \pi_{k} J_{1}(\tilde{V}_{1}) \tag{12}$$

and by induction, therefore,

$$J_k(V_0) = J_k(V_1), \qquad k \ge 1.$$
 (13)

Hence summing over k we obtain

$$J(V_0) = J(V_1), \qquad \text{as desired.} \tag{14}$$

Remarks. 1. Proposition A could have also been obtained using Fuller's Theorem 1 of [8].

2. The assumption of orientable stable manifolds insures that $L_{f,k}(y)$ is independent of k. If we omitted this assumption the theorem would be false. For example, the 180° rotation of the 2-sphere S^2 may be perturbed to a Morse-Smale (M.-S.) diffeomorphism f having 2 fixed orientation-reversing saddles (of index 1) at the poles, and four alternating sources and sinks of period 2 each (and index 1) along the equator (see Figure 1). Let f_s be a homotopy of $f = f_0$ to the gradient M.-S. diffeomorphism f_1 given by $z \mapsto z^2$ for $z \in S^2$. Then if V_s are the corresponding "suspension" vector fields on $S^1 \times S^2$, we have

$$J(V_0) = 6$$
 but $J(V_1) = 2$,

where we have identified $H_1(S^1 \times S^2)$ canonically with \mathbb{Z} .

Example B. For a second example, consider the 2-torus T^2 . It follows from [8] that if V_0 , V_1 are two NMS vector fields on T^2 , then they are homotopic through nonsingular vector fields if and only if $J(V_0) = J(V_1)$.

Example C. We consider the case of a vector field V tangent to the fibres of a principal circle bundle E^n over a compact manifold M^{n-1} . Let G denote a small gradient Morse-Smale vector field on M (not necessarily nonsingular). A choice of connection on E enables us to lift G to a unique S^1 -invariant vector field \tilde{G} on E.

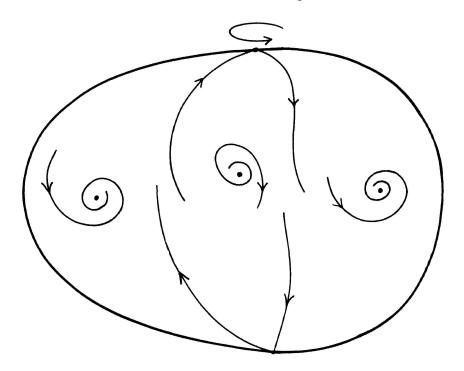


Figure 1.

It is easy to verify that $V' = V + \tilde{G}$ is a NMS vector field on E. Furthermore, considering the cases n odd and n even separately, we may check that in fact

$$J(V') = \chi(M) \cdot \bar{F} \quad \text{in} \quad H_1(E), \tag{15}$$

where \bar{F} is the homology class of the oriented fibre $F \approx S^1$.

By picking topologically distinct gradient fields G one may create countably many topologically distinct NMS fields on E all homotopic to one another and all having the same geometric index.

Example D. Let M^n have the round handle decomposition

$$M \approx Q_1 + Q_2 + \dots + Q_r \tag{16}$$

where the Q_i , $1 \le i \le r$, are round handles of various indices attached successively (see [2], [3]), and we assume $n \ge 4$. We may assume M carries a NMS vector field V compatible with the decomposition.

Then there is an operation which creates a new round handle decomposition and a corresponding compatible vector field V'. We introduce a cancelling pair of round handles in between Q_i and Q_{i+1} :

$$M \approx Q_1 + \dots + Q_i + R^j + R^{j+1} + Q_{i+1} + \dots + Q_r.$$
 (17)

Here $R^j \approx S^1 \times D^j \times D^{n-j-1}$ and $R^{j+1} \approx S^1 \times D^{j+1} \times D^{n-j-2}$. The union $R^j \cup R^{j+1} \approx S^1 \times (h^j \cup h^{j+1})$ where $h^j + h^{j+1}$ represents a cancelling pair of ordinary handles of dimension n-1. $R^j + R^{j+1}$ is attached in such a way that

$$Q_1 + \cdots + Q_i + R^j + R^{j+1} \approx Q_1 + \cdots + Q_i.$$
 (18)

On each of R^j , R^{j+1} we define a nonsingular vector field that is essentially the neighborhood of a single hyperbolic closed orbit whose Poincaré map has j or j+1 contracting dimensions, respectively. As in [2], this defines a new NMS vector field, say V', which is homotopic to V and such that also

$$J(V') = J(V), \tag{19}$$

since we have added two closed orbits which are homologous but whose Poincaré maps have fixed point indices of opposite sign, namely $(-1)^{n-1-j}$ and $(-1)^{n-2-j}$, respectively.

3.

THEOREM 1. We assume $n = \dim M \ge 4$. Let $\alpha \in H_1(M; \mathbb{Z})$ be arbitrary, and let \mathfrak{D} be any homotopy class of nonsingular vector fields on M. Then there is a NMS vector field $V \in \mathfrak{D}$ such that $J(V) = \alpha$. It follows from structural stability that there is in fact an entire C^1 neighborhood N(V) such that

$$V' \in N(V) \Rightarrow J(V') = \alpha$$
.

Proof. We begin by constructing an example of a certain NMS vector field X_1 on the solid torus $B = S^1 \times D^2$. We construct X_1 by a round handle decomposition $B = R^0 + R^1$. As in Section 5 of [2], R^i , i = 0, 1 is supplied with a nonsingular vector field having exactly one closed orbit, whose Poincaré map is hyperbolic with i contracting dimensions. In coordinates the vector field on $R^0 = S^1 \times D^2$ is given by $d/d\theta + x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2$, and the vector field on $R^1 = S^1 \times D^1 \times D^1$ is given by $d/d\varphi - y_1 \partial/\partial y_1 + y_2 \partial/\partial y_2$. The attaching region $\partial_-(R^1)$ is defined by $\{(\varphi, y_1, y_2) \in R^1 \mid |y_1| = 1\}$ and is the disjoint union $S^1 \times S^0 \times D^1$ of two annuli $A_{-1} = S^1 \times \{-1\} \times D^1$ and $A_1 = S^1 \times \{1\} \times D^1$.

We must specify, up to isotopy, the attaching map

$$h: S^1 \times S^0 \times D^1 \to \partial(R^0)$$

(where $\partial(R^0) = \partial(S^1 \times D^2)$ is a 2-torus). Any h satisfying the following conditions will suffice for our purposes:

- a) The composition $A_{-1} \xrightarrow{h|A_{-1}|} \partial(R^0) \subseteq R^0$ induces an isomorphism $H_1(A_{-1}) \to H_1(R^0)$.
 - b) $h(A_1)$ deforms to a point in $\partial(R^0)$.
 - c) A_{-1} and A_1 are embedded in $\partial(R^0)$ with opposite orientations.

LEMMA 1. If $h: \partial_+(R^1) \to \partial(R^0)$ satisfies a), b), c) above, then the quotient space $R^0 + R^1 = R^0 \cup R^1/x \sim h(x)$ is diffeomorphic to a solid torus.

Proof. By the isotopy extension lemma [6] an isotopy of an embedding of the core circle of A_i $i=\pm 1$ extends to an isotopy of all of $\partial_-(R^1)$. Using this it is straightforward to show that if h satisfies a), b), c) then up to isotopy $h(A_{-1})$ and $h(A_1)$ are as depicted in Figure 2 (after embedding R^0 appropriately in \mathbb{R}^3).

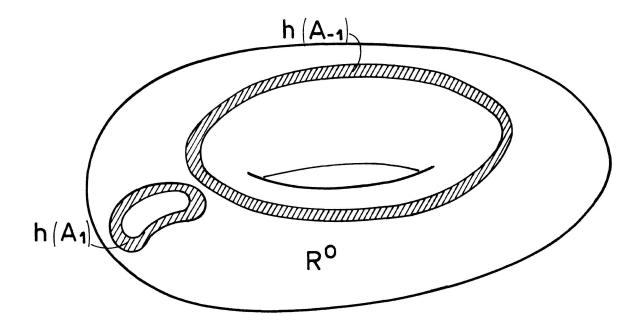


Figure 2.

Hence $R^0 + R^1$ will resemble Figure 3.

Then performing the isotopy indicated in Figure 4a) to d) shows the lemma. Now X_1 is defined as the NMS vector field on $B = S^1 \times D^2$ induced from the round handle decomposition of B described above. Let us assume X_1 is normal to ∂B .

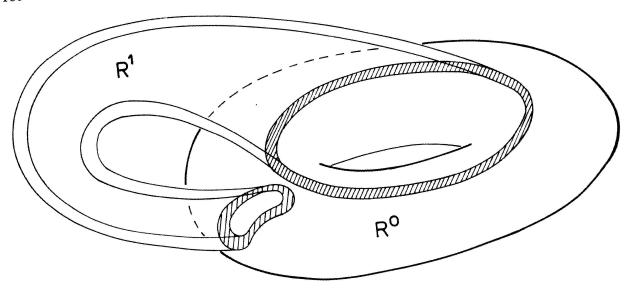


Figure 3.

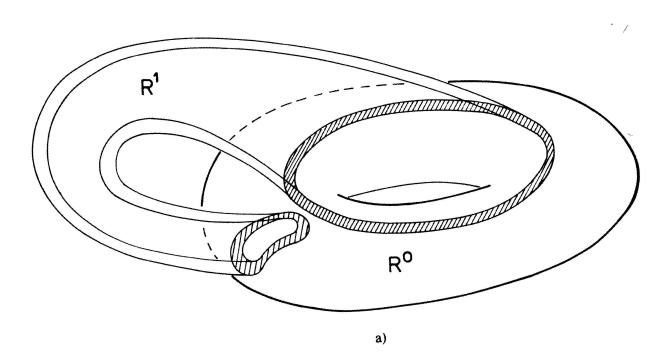


Figure 4.

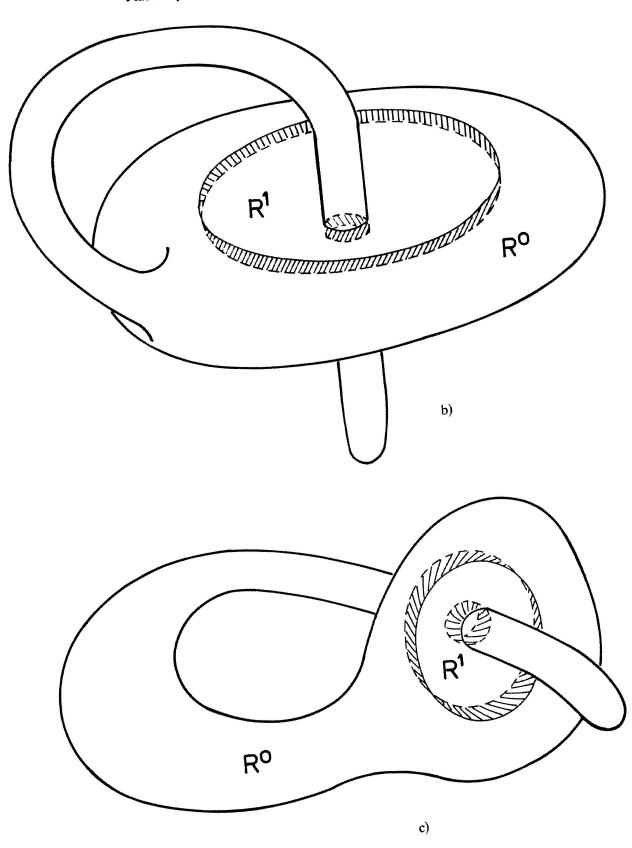
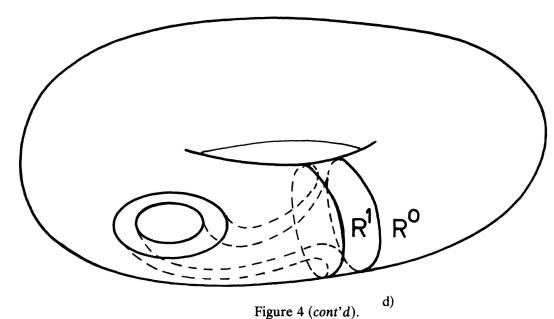


Figure 4 (cont'd).



LEMMA 2. The first obstruction between X_1 and V_0 standard source vector

$$H^2(B, \partial B; \pi_2(S^2)) \approx H_1(B; \mathbb{Z}) \approx \mathbb{Z}.$$

field Y on B vanishes in

Proof. Let $\gamma = d_1(X_1, Y) \in H^2(B, \partial B; \pi_2(S^2))$. Then γ is detected by its Kronecker product $\langle \gamma, w \rangle$ with the generator w of $H_2(B, \partial B; \mathbb{Z}) \approx \mathbb{Z}$. Now w is represented by $\{x\} \times D^2 \subset S^1 \times D^2 = B$. Then $\langle \beta, w \rangle = \langle d_1(X_1, Y), w \rangle$ can be identified with an element $c \in \pi_2(S^2) \approx \mathbb{Z}$ obtained as follows. Let S^2 be the union of two copies of $\{x\} \times D^2$ identified on their boundaries. Then choosing any framing for T(B) restricted to $\{x\} \times D^2$, each of the nonsingular vector fields X_1 and Y, once they have been normalized to unit length, defines a map $\{x\} \times D^2 \to S^2$. These two maps agree on $\{x\} \times \partial D^2$ and hence induce a map $S^2 \to S^2$ which is well defined up to homotopy, and its homotopy class is c.

Now we notice that a circle which surrounds A_1 (cf. Figure 2) bounds a 2-disc D unique up to homotopy rel ∂D , embedded in R^0 so that $D \cap \partial R^0 = \partial D \cap \partial R^0$ (see Figure 5). This disc D will correspond under the isotopy shown in Figure 4 to a representative of c. On D it is easy to see by inspection that up to homotopy both X_1 and Y can be represented in continuous coordinates x_1 , x_2 along D, and x_3 normal to D, via

$$x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2 + \sqrt{1 - x_1^2 - x_2^2} \partial/\partial x_3$$
.

Thus c = 0 in $\pi_2(S^2)$ and the lemma is proved.

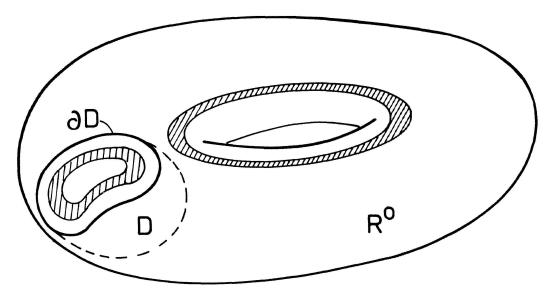


Figure 5.

LEMMA 3. There is a NMS vector field X on B, exiting on ∂B , such that

- a) X is homotopic to the source field Y rel (∂B) , and
- b) J(X) = 0; in fact each closed orbit of X bounds a disc in B.

Remark. Since NMS vector fields are structurally stable [9], X shows that Fuller's example [7] is not just a "pathological" phenomenon.

Proof of Lemma 3. We know from the work of Hopf and Boltyanskii [5] that if the first obstruction between two vector fields vanishes, then they are homotopic if the second obstruction vanishes (as computed using homotopic fields which agree on the codimension-one skeleton). Since X_1 and Y agree on $\{x\} \times D^2 \subset S^1 \times D^2 = B$, an arbitrarily small perturbation will cause them to actually agree on $[x, y] \times D^2$. Hence they agree on the 2-skeleton of a certain cell decomposition of B, namely on $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$. (We continue to denote the perturbed fields by X_1 and Y.)

Now we observe that the second obstruction $d_2(X_1, Y) = m\eta$, some positive multiple of a generator η of $H^3(B, \partial B; \pi_3(S^2)) \approx \mathbb{Z}$.

Let C_i , $1 \le i \le m$ be disjoint nullhomotopic circles embedded in ∂B . In a small tubular neighborhood of each C_i we attach a cancelling pair of round handles $\bar{R}_i^0 + \bar{R}_i^1$ (see Figure 6).

We supply \bar{R}_i^0 and \bar{R}_i^1 each with a single hyperbolic closed orbit whose Poincaré map has, respectively, 0 or 1 contracting dimension. We also require that the two closed orbits thus created have opposite sense (i.e., represent both generators of $H_1(\bar{R}^0 + \bar{R}^1) \approx H_1(S^1 \times D^2) \approx \mathbb{Z}$). This resulting vector field is X (NMS after a small C^1 perturbation by [1]).

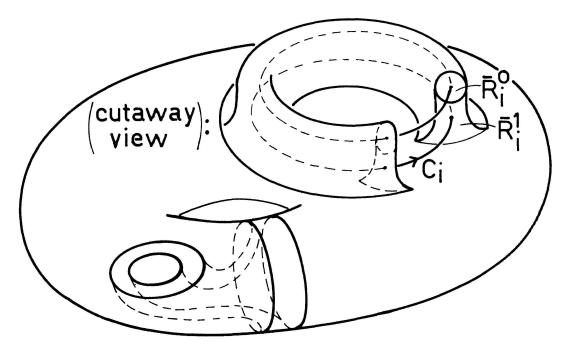


Figure 6.

Let us consider the effect of adding only one such cancelling pair $\bar{R}^0 + \bar{R}^1$ to $B = R^0 + R^1$. We extend the vector field on $B + \bar{R}^0 + \bar{R}^1$ to a collar on ∂B just as in [3], and we call this field by the name X'. We also extend X_1 to the collar $(\partial B) \times I$, letting it be the vertical vector field $\partial/\partial t$ there. We may now identify $B \cup (\partial B) \times I$ with B by the usual isotopic deformation down the collar.

This gives us two vector fields on B, which we still call X_1 and X'. The construction above, done carefully, will result in their agreeing on the 2-skeleton $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$ of B. To compute $d_2(X_1, X') \in H^3(B, \partial B; \pi_3(S^2))$ we use the same argument as in Lemmas 10 and 11 of [3], only noticing that the framed cobordism class of $Q_0^{-1}(\mathcal{G})$ is indeed the generator of Φ_1 , the framed cobordism group of 1-dimensional framed submanifolds in D^3 . This follows immediately from [4]. Hence $d_2(X_1, X')$ generates $H^3(B, \partial B; \pi_3(S^2))$, so is $\pm \eta$. By changing the sense of both closed orbits in $\bar{R}^0 + \bar{R}^1$ if necessary, we may assume

$$d_2(X_1, X') = \eta. \tag{20}$$

Now we add all m cancelling pairs $\bar{R}_i^0 + \bar{R}_i^1$ to $B = R^0 + R^1$, (with the proper sense to the closed orbits of the corresponding vector fields). Just as above we identify the resulting space $B + R_1^0 + R_1^1 + \cdots + R_m^0 + R_m^1$ with B itself. Then applying induction to the argument of the last paragraph we obtain

$$d_2(X_1, X) = m\eta. \tag{21}$$

Thus by the addition formula for difference cocycles, we have

$$d_2(X, Y) = d_2(X, X_1) + d_2(X_1, Y)$$

$$= -d_2(X_1, X) + d_2(X_1, Y)$$

$$= -m\eta + m\eta = 0.$$
(22)

Hence by [5], X is homotopic to $Y \operatorname{rel}(\partial B)$. By construction, each of the 2+2m=2(m+1) closed orbits of X bounds a disc in B. Hence J(X)=0.

LEMMA 4. Let W be an NMS vector field on the compact manifold P. In case $\partial P \neq \emptyset$ we assume W to exit on ∂P . Let Z^k denote a simple hyperbolic source vector field on D^k , such as the positive vector field

$$Z^{k}(x_{1},\ldots,x_{n})=\sum_{i=1}^{k}x_{i}\,\partial/\partial x_{i}$$
(23)

Then the direct sum vector field $W \oplus Z^k$ is exiting on $\partial(P \times D^k)$ (after appropriate smoothing of the corner $\partial P \times \partial D^k$) and by a C^1 -small perturbation $W \oplus Z^k$ can be made NMS.

Proof. It is straightforward to verify that the nonwandering set of $W \oplus Z^k$ is a finite union of hyperbolic closed orbits. Transversal intersection of stable and unstable manifolds is then obtainable by a C^1 -small perturbation that preserves the truth of the previous sentence [1]. To see that $W \oplus Z^k$ exits on $\partial(P \times D^k)$ it is necessary to choose a "convex" straightening of the angle [6] of the product of two half-spaces. We use charts where W and Z^k are unit normal fields to the boundaries and the result follows.

LEMMA 5. Let $n \ge 3$. There is a NMS vector field X^n on the solid torus $S^1 \times D^{n-1}$ such that

- a) X^n exits on $S^1 \times \partial D^{n-1} = \partial (S^1 \times D^{n-1})$.
- b) X^n is homotopic to a hyperbolic source NMS vector field on $S^1 \times D^{n-1}$, through nonsingular vector fields which remain everywhere transverse to $\partial (S^1 \times D^{n-1})$.
 - c) $J(X^n) = 0$.

Proof. Let $\{X_t\}_{0 \le t \le 1}$ denote a homotopy through nonsingular fields between $X_0 = X$ and $X_1 = Y$ on $S^1 \times D^2$, as in Lemma 3. We define

$$\bar{X}^n = X \oplus Z^{n-3} \tag{24}$$

on $S^1 \times D^{n-1}$ (obtained as in Lemma 4 on $S^1 \times D^3 \times D^{n-3}$ by rounding the corners). Then

$$\{\bar{X}_t^n \oplus Z^{n-3}\}_{0 \le t \le 1} \tag{25}$$

provides a homotopy of nonsingular vector fields on $S^1 \times D^{n-1}$ which are each transverse to $\partial (S^1 \times D^{n-1})$ by Lemma 4. It is easy to see that $\bar{X}_1^n = Y \oplus Z^{n-3}$ is a standard hyperbolic source vector field on $S^1 \times D^{n-1}$. Finally by Lemma 4 again, $\bar{X}^n = X \oplus Z^{n-3}$ can be C^1 -small perturbed to a field X^n which is NMS. The closed orbits of X^n will correspond to those of \bar{X}^n and may be assumed to have the same Poincaré maps. Then by definition of Z^{n-3} we have

$$J(X^n) = (-1)^{n-3} i_* J(X)$$

where $i_*: H_1(S^1 \times D^2) \xrightarrow{\approx} H_1(S^1 \times D^{n-1})$. Hence by Lemma 3, b), we are done.

Conclusion of proof of Theorem 1. In [3] we showed the existence of a NMS vector field V_0 in any desired homotopy class $\mathfrak D$. As in [3], Lemma 5 (cf. Example D, Section 1 of this paper) we may homotope V_0 to another NMS field V_0' obtained from V_0 by adding a cancelling pair of new round handles of index 0 and 1. By (19), $J(V_0) = J(V_0') = \alpha_0 \in H_1(M)$, say. We may arrange that the S^1 direction of each of these new round handles represents the element $(-1)^n(\alpha - \alpha_0)$ in $H_1(M)$.

We finally define V as follows:

$$V(x) = \begin{cases} V'_0(x) & \text{if } x \text{ is outside the new round 0-handle} \\ X^n(x) & \text{if } x \text{ is inside the new round 0-handle.} \end{cases}$$

Here we are identifying $S^1 \times D^{n-1}$ of Lemma 5 with the new round 0-handle. We assume a smooth interpolation near a collar of $\partial (S^1 \times D^{n-1})$ if necessary, to fit V'_0 and X^n together smoothly. We also assume a C^1 -small perturbation if necessary to make V NMS (as in [3], Lemma 4).

Hence V is a NMS vector field in \mathfrak{D} , and by choice of embedding of the new round handles and by Lemma 5, c), we have

$$J(V) = J(V_0) + (-1)^{n-2}((-1)^n(\alpha - \alpha_0))$$

= $\alpha_0 + \alpha - \alpha_0 = \alpha$ as desired.

Remarks. The above technique shows that in dimension 3, the conclusion of Theorem 2 holds when there exists a round handle decomposition.

REFERENCES

- [1] R. ABRAHAM and J. ROBBIN, Transversal mappings and flows, Benjamin, 1967.
- [2] D. ASIMOV, Round handles and non-singular Morse-Smale flows, Ann. of Math. 102 (1975), 41-54.
- [3] —, Homotopy of non-singular vector fields to structurally stable ones, ibid., 55-65.
- [4] V. G. BOLTYANSKII, Homotopy of mappings and vector fields, A.M.S. Translations Series 2, Volume 7, 1957.
- [5] —, Homotopy classification of vector fields, Dokl. Akad. Nauk SSR, 1958, 731 ff.
- [6] J. Cerf, $\Gamma_4 = 0$, Lecture notes in mathematics 53, Springer-Verlag, 1968.
- [7] F. B. Fuller, Note on trajectories in a solid torus, Ann. of Math. 56 (1952), 438-439.
- [8] —, An index of fixed point type for periodic orbits, Am. J. Math. 89 (1967), 133-148.
- [9] J. Palis and S. Smale, Structural stability theorems, A.M.S. Proc. Symp. Pure Math. 14 (1970), 223-231.
- [10] S. SMALE, Differentiable dynamical systems, Bull. A.M.S. 73 (1967), 747-817.

School of Mathematics, Institute for Advanced Study, Princeton, N.J. 08540, U.S.A.

and

School of Mathematics, University of Minnesota, Minneapolis, Minn. 55455 U.S.A.

Current Address: Department of Mathematics, University of North Carolina,

Chapel Hill, N.C. 27514

U.S.A.

Received January 12, 1976.