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Flaccidity of geometric index for nonsingular vector fields

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1. Introduction

We consider nonsingular vector fields V on compact connected C^∞ orientable manifolds M^n without boundary. It is of interest to find criteria for detecting the occurrence of closed orbits of V. If we were to look instead at diffeomorphisms $f: M \to M$ the analogous question would be to find the periodic points of f. We may at least count the periodic points algebraically via the Lefschetz fixed point formula, if $\{x \mid f^k(x) = x\} = \operatorname{Fix}(f^k)$ is finite:

$$\sum_{f^k(x)=x} \imath_{f^k}(x) = \sum_{j=0}^{n} (-1)^j \operatorname{tr}(f^k_{*j}: H_j(M; \mathbb{R}) \circlearrowleft). \tag{1}$$

Here $\imath_{f^k}(x)$ is the local index of the fixed point x of f^k, and the right hand side is, by definition, the Lefschetz number $\Lambda(f^k)$ of f^k.

We define here an index for nonsingular Morse–Smale (NMS) vector fields [2] as follows. Let V be an NMS field on M. Let C_i $1 \le i \le r$ be all the closed orbits of V. Let $\bar{C}_i \in H_1(M; \mathbb{Z})$ be the homology class of C_i (oriented by V) and let $\varepsilon_i = \pm 1$ be the fixed point index of the Poincaré map induced by the flow of V around C_i. Then we define

$$J(V) = \sum_{i=1}^{r} \varepsilon_i \bar{C}_i \quad \text{in} \quad H_1(M; \mathbb{Z}) \tag{2}$$

to be the *geometric index of the NMS field V*.

Let $\mathcal{V}$ denote all nonsingular vector fields on M, with the C^0 topology. In Section 2 we describe some situations where J remains constant over a large open set or on distinct homotopic NMS vector fields. In particular we show that if V is the suspension of a Morse–Smale diffeomorphism, then $J(V)$ is constant through large perturbations of V.

In Section 3, however, we show by example that if the dimension of M is greater than 3, then for each $\alpha \in H_1(M; \mathbb{Z})$ and for each homotopy class $\mathfrak{D}$ of nonsingular vector fields on M, there exists a $V_\alpha \in \mathscr{D}$ such that V_α is NMS and $J(V_\alpha)=\alpha$. Fuller's Theorem 1 [8] then imposes conditions on any homotopy between V_α and V_β, $\alpha \neq \beta$. We note that NMS vector fields are structurally stable [9].

2. Examples

A. Let $p: E^n \to S^1$ be a smooth oriented fibre bundle with fibre the compact manifold F^{n-1}. Let $\{V_s\}_{0\le s\le 1}$ denote a homotopy of nonsingular vector fields on E which are never tangent to a fibre. Assume that V_0 and V_1 are NMS, with orientable stable and unstable manifolds. Then we have

PROPOSITION A.

$$J(V_0)=J(V_1). \tag{3}$$

Proof. Without loss of generality we may assume that for all $(x, s)\in E\times I$ we have

$$Dp(V_s(x)) = 2\pi \cdot d/d\theta, \tag{4}$$

where $Dp: TE \to TS^1$ is the tangent mapping of p. Let $p_*: H_1E \to H_1S^1$ be the map on integer homology. Identify H_1S^1 with Z by sending the counterclockwise generator to 1. Now set $J_k(V_0)=\sum_{\bar{C}_i\in p_*^{-1}(k)} \varepsilon_i \bar{C}_i$. Then

$$J(V_0)=\sum_{k=1}^{\infty} J_k(V_0) \tag{5}$$

and similarly

$$J(V_1)=\sum_{k=1}^{\infty} J_k(V_1). \tag{6}$$

Hence the proposition follows from showing $J_k(V_0)=J_k(V_1)$ for all $k=1, 2, \ldots$.

We notice that by (4) the time-one map φ_s^1 of the flow $\{\varphi_s^t\}_{t\in R}$ of V_s is a map $\varphi_s^1: E\to E$ which preserves fibres. Similarly for φ_s^k, k any integer. Let $p^*(E)$ denote the space $\{(x, y)\in E\times E \mid p(x)=p(y)\}$, the total space of the pullback by p

of the bundle $E \xrightarrow{p} S^1$. Then $p^*(E)$ fibres over S^1 with fibre $F \times F$, via $(x, y) \mapsto p(x) = p(y) \in S^1$. We define the subset $\Delta \subset p^*(E)$ by $\Delta = \{(x, x) \in p^*(E)\}$. Also let $\bar{p} : p^*(E) \to E$ be the projection given by $\bar{p}(x, y) = x$.

Now we may form the map $\Gamma_s^1 : E \to p^*(E)$ defined by $\Gamma_s^1(x) = (x, \varphi_s^1(x))$. This map Γ_s^1 is transverse to $\Delta \subset p^*(E)$ precisely when the map φ_s^1 restricted to any fibre F has no eigenvalues equal to 1. In particular, this is the case for $s = 0, 1$. It is clear from the definitions that as a point set, $\bar{p}(\Gamma_i^1(E) \cap \Delta)$ is the union of all the closed orbits of V_i which go around E once, i.e., those closed orbits comprising $J_1(V_i)$, for $i = 0, 1$.

Now considering homology intersection $\cdot$, we let $[\Delta] \in H_n(p^*(E))$ denote the fundamental class of the diagonal, i.e., the image of $[E] \in H_n(E)$ under $x \mapsto (x, x)$. Let $\Gamma_{s*}^1 : H_n(E) \to H_n(p^*(E))$ and $\bar{p}_* : H_1(p^*(E)) \to H_1(E)$ denote the induced maps on homology. Then

$$\Lambda_s^1 = \bar{p}_*(\Gamma_{s*}^1([E]) \cdot [\Delta]) \quad \text{in} \quad H_1(E) \tag{7}$$

is an integer homology class independent of s, and so

$$\Lambda_0^1 = \Lambda_1^1. \tag{8}$$

Let $f_i : F \to F$ denote $\varphi_i^1 \mid F$ for the fibre $F = p^{-1}(x)$ of E. Let $y \in F$ be the fixed point of f_i corresponding to the closed orbit C_y of V_i. Then C_y contributes $\varepsilon_y \bar{C}_y \in H_1(E)$ to the class Λ_i^1, where

$$\varepsilon_y = \det \left(\begin{array}{c|c} I_{n-1} & I_{n-1} \\ \hline Df(y) & I_{n-1} \end{array} \right)$$

i.e.,

$$\begin{aligned} \varepsilon_y &= \det (I_{n-1} - Df(y)) \\ &= \iota_{f_i}(y), \end{aligned} \tag{9}$$

the local fixed point index at y (since 1 is not an eigenvalue of $Df(y)$). Thus we have shown that

$$\Lambda_i^1 = J_1(V_i) \qquad i = 0, 1 \tag{10}$$

and so by (8) this shows $J_1(V_0) = J_1(V_1)$.

Now using the fact that we assumed all stable and unstable manifolds to be *orientable* for V_i, $i = 0, 1$, we may easily check that for $i = 0, 1$, we have

$$\pi_{k*} J_1(\tilde{V}_i) = k \sum_{d \mid k} J_d(V_i) \in H_1(E). \tag{11}$$

Here $\pi_k:\tilde{E}_k \to E$ denotes the canonical k-fold covering space over E (since E is a fibre bundle over S^1) and $\tilde{V}_i$ denotes the unique lift of V_i to a nonsingular vector field on $\tilde{E}_k$.

Hence $J_k(V_i)$ is expressible in terms of $\pi_{k*}J_1(\tilde{V}_i)$ and $J_d(V_i)$ for $1 \leq d < k$. Now $J_1(\tilde{V}_i)$ is independent of $i=0, 1$ by the considerations we applied above to $J_1(V_i)$ (since the homotopy V_s gives rise to a homotopy $\tilde{V}_s$, $0 \leq s \leq 1$). Hence we also have

$$\pi_{k*}J_1(\tilde{V}_0)=\pi_{k*}J_1(\tilde{V}_1) \tag{12}$$

and by induction, therefore,

$$J_k(V_0)=J_k(V_1), \qquad k \geq 1. \tag{13}$$

Hence summing over k we obtain

$$J(V_0)=J(V_1), \qquad \text{as desired.} \tag{14}$$

Remarks. 1. Proposition A could have also been obtained using Fuller's Theorem 1 of [8].

2. The assumption of orientable stable manifolds insures that $L_{f_i^k}(y)$ is independent of k. If we omitted this assumption the theorem would be false. For example, the 180° rotation of the 2-sphere S^2 may be perturbed to a Morse–Smale (M.–S.) diffeomorphism f having 2 fixed orientation-reversing saddles (of index 1) at the poles, and four alternating sources and sinks of period 2 each (and index 1) along the equator (see Figure 1). Let f_s be a homotopy of $f=f_0$ to the gradient M.–S. diffeomorphism f_1 given by $z \mapsto z^2$ for $z \in S^2$. Then if V_s are the corresponding "suspension" vector fields on $S^1 \times S^2$, we have

$$J(V_0)=6 \quad \text{but} \quad J(V_1)=2,$$

where we have identified $H_1(S^1 \times S^2)$ canonically with $\mathbb{Z}$.

Example B. For a second example, consider the 2-torus T^2. It follows from [8] that if V_0, V_1 are two NMS vector fields on T^2, then they are homotopic through nonsingular vector fields if and only if $J(V_0)=J(V_1)$.

Example C. We consider the case of a vector field V tangent to the fibres of a principal circle bundle E^n over a compact manifold M^{n-1}. Let G denote a small gradient Morse–Smale vector field on M (not necessarily nonsingular). A choice of connection on E enables us to lift G to a unique S^1-invariant vector field $\tilde{G}$ on E.

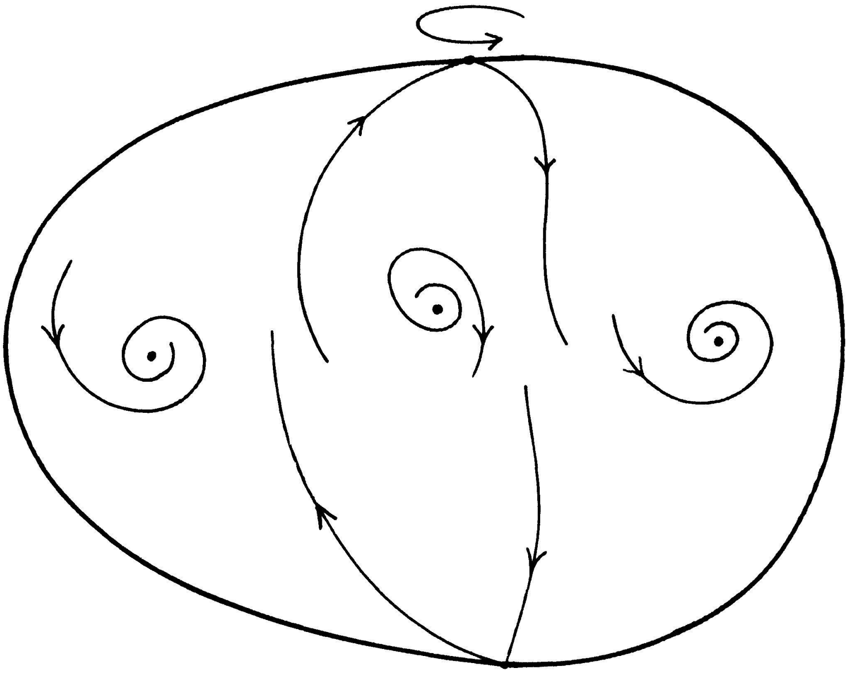


Figure 1.

It is easy to verify that $V' = V + \tilde{G}$ is a NMS vector field on E. Furthermore, considering the cases n odd and n even separately, we may check that in fact

$$J(V') = \chi(M) \cdot \bar{F} \quad \text{in} \quad H_1(E), \tag{15}$$

where $\bar{F}$ is the homology class of the oriented fibre $F \approx S^1$.

By picking topologically distinct gradient fields G one may create countably many topologically distinct NMS fields on E all homotopic to one another and all having the same geometric index.

Example D. Let M^n have the round handle decomposition

$$M \approx Q_1 + Q_2 + \cdots + Q_r \tag{16}$$

where the Q_i, $1 \le i \le r$, are round handles of various indices attached successively (see [2], [3]), and we assume $n \ge 4$. We may assume M carries a NMS vector field V compatible with the decomposition.

Then there is an operation which creates a new round handle decomposition and a corresponding compatible vector field V'. We introduce a cancelling pair of round handles in between Q_i and Q_{i+1}:

$$M \approx Q_1 + \cdots + Q_i + R^j + R^{j+1} + Q_{i+1} + \cdots + Q_r. \tag{17}$$

Here $R^j \approx S^1 \times D^j \times D^{n-j-1}$ and $R^{j+1} \approx S^1 \times D^{j+1} \times D^{n-j-2}$. The union $R^j \cup R^{j+1} \approx S^1 \times (h^j \cup h^{j+1})$ where $h^j + h^{j+1}$ represents a cancelling pair of ordinary handles of dimension $n-1$. $R^j + R^{j+1}$ is attached in such a way that

$$Q_1 + \cdots + Q_i + R^j + R^{j+1} \approx Q_1 + \cdots + Q_i. \tag{18}$$

On each of R^j, R^{j+1} we define a nonsingular vector field that is essentially the neighborhood of a single hyperbolic closed orbit whose Poincaré map has j or $j+1$ contracting dimensions, respectively. As in [2], this defines a new NMS vector field, say V', which is homotopic to V and such that also

$$J(V') = J(V), \tag{19}$$

since we have added two closed orbits which are homologous but whose Poincaré maps have fixed point indices of opposite sign, namely $(-1)^{n-1-j}$ and $(-1)^{n-2-j}$, respectively.

3.

THEOREM 1. *We assume $n = \dim M \geq 4$. Let $\alpha \in H_1(M; \mathbb{Z})$ be arbitrary, and let $\mathfrak{D}$ be any homotopy class of nonsingular vector fields on M. Then there is a* NMS *vector field $V \in \mathfrak{D}$ such that $J(V) = \alpha$. It follows from structural stability that there is in fact an entire C^1 neighborhood $N(V)$ such that*

$$V' \in N(V) \Rightarrow J(V') = \alpha.$$

Proof. We begin by constructing an example of a certain NMS vector field X_1 on the solid torus $B = S^1 \times D^2$. We construct X_1 by a round handle decomposition $B = R^0 + R^1$. As in Section 5 of [2], R^i, $i = 0$, 1 is supplied with a nonsingular vector field having exactly one closed orbit, whose Poincaré map is hyperbolic with i contracting dimensions. In coordinates the vector field on $R^0 = S^1 \times D^2$ is given by $d/d\theta + x_1\, \partial/\partial x_1 + x_2\, \partial/\partial x_2$, and the vector field on $R^1 = S^1 \times D^1 \times D^1$ is given by $d/d\varphi - y_1\, \partial/\partial y_1 + y_2\, \partial/\partial y_2$. The attaching region $\partial_-(R^1)$ is defined by $\{(\varphi, y_1, y_2) \in R^1 \mid |y_1| = 1\}$ and is the disjoint union $S^1 \times S^0 \times D^1$ of two annuli $A_{-1} = S^1 \times \{-1\} \times D^1$ and $A_1 = S^1 \times \{1\} \times D^1$.

We must specify, up to isotopy, the attaching map

$$h : S^1 \times S^0 \times D^1 \to \partial(R^0)$$

(where $\partial(R^0)=\partial(S^1\times D^2)$ is a 2-torus). Any h satisfying the following conditions will suffice for our purposes:

a) The composition $A_{-1}\xrightarrow{h|A_{-1}}\partial(R^0)\subseteq R^0$ induces an isomorphism $H_1(A_{-1})\to H_1(R^0)$.

b) $h(A_1)$ deforms to a point in $\partial(R^0)$.

c) A_{-1} and A_1 are embedded in $\partial(R^0)$ with opposite orientations.

LEMMA 1. *If* $h:\partial_+(R^1)\to\partial(R^0)$ *satisfies* a), b), c) *above, then the quotient space* $R^0+R^1=R^0\cup R^1/x\sim h(x)$ *is diffeomorphic to a solid torus.*

Proof. By the isotopy extension lemma [6] an isotopy of an embedding of the core circle of A_i $i=\pm1$ extends to an isotopy of all of $\partial_-(R^1)$. Using this it is straightforward to show that if h satisfies a), b), c) then up to isotopy $h(A_{-1})$ and $h(A_1)$ are as depicted in Figure 2 (after embedding R^0 appropriately in $\mathbb{R}^3$).

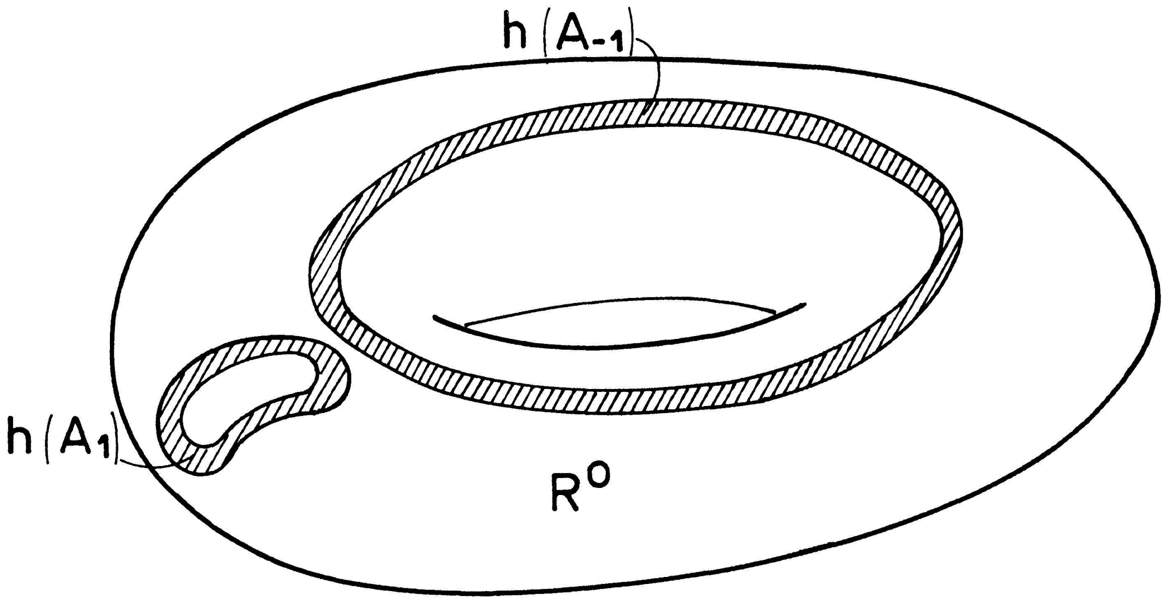


Figure 2.

Hence R^0+R^1 will resemble Figure 3.

Then performing the isotopy indicated in Figure 4a) to d) shows the lemma.

Now X_1 is defined as the NMS vector field on $B=S^1\times D^2$ induced from the round handle decomposition of B described above. Let us assume X_1 is normal to ∂B.

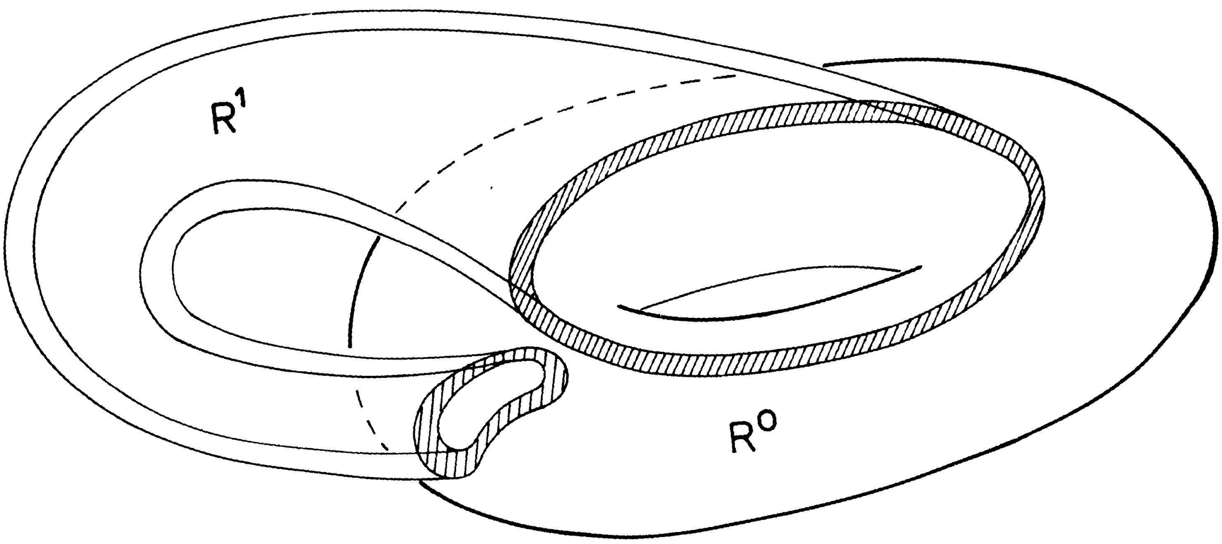
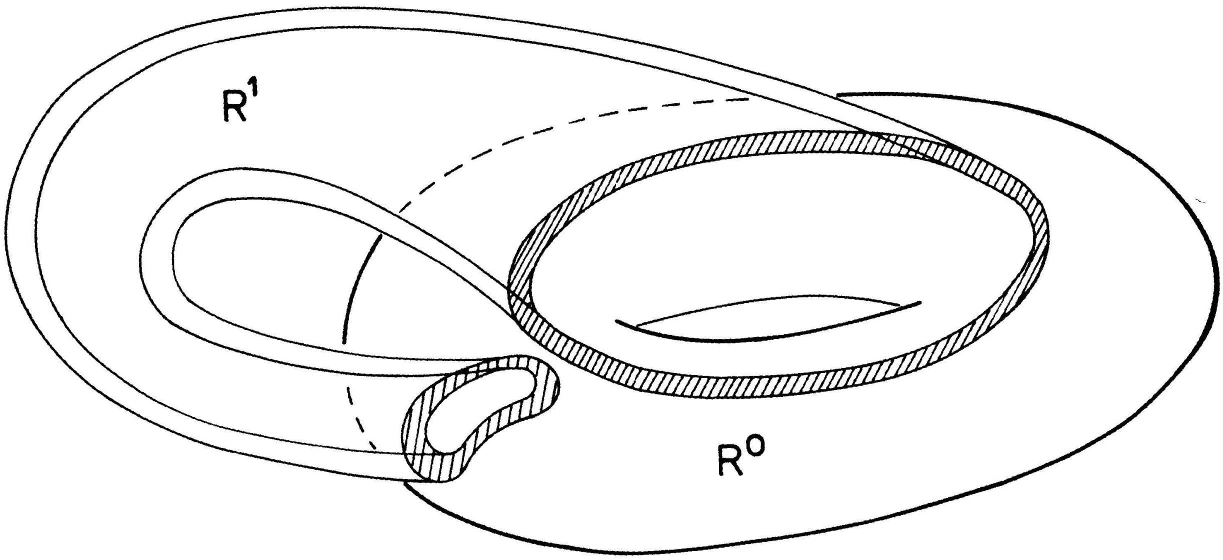
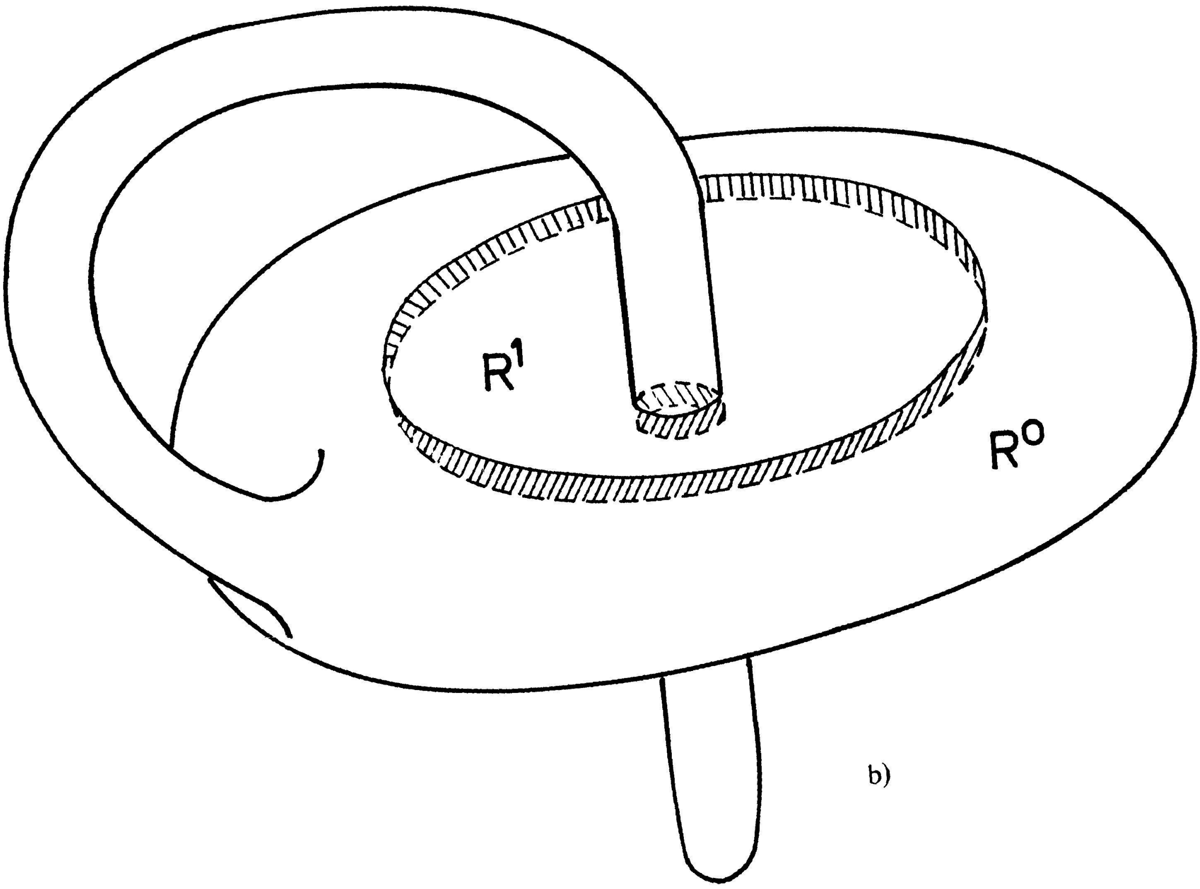


Figure 3.

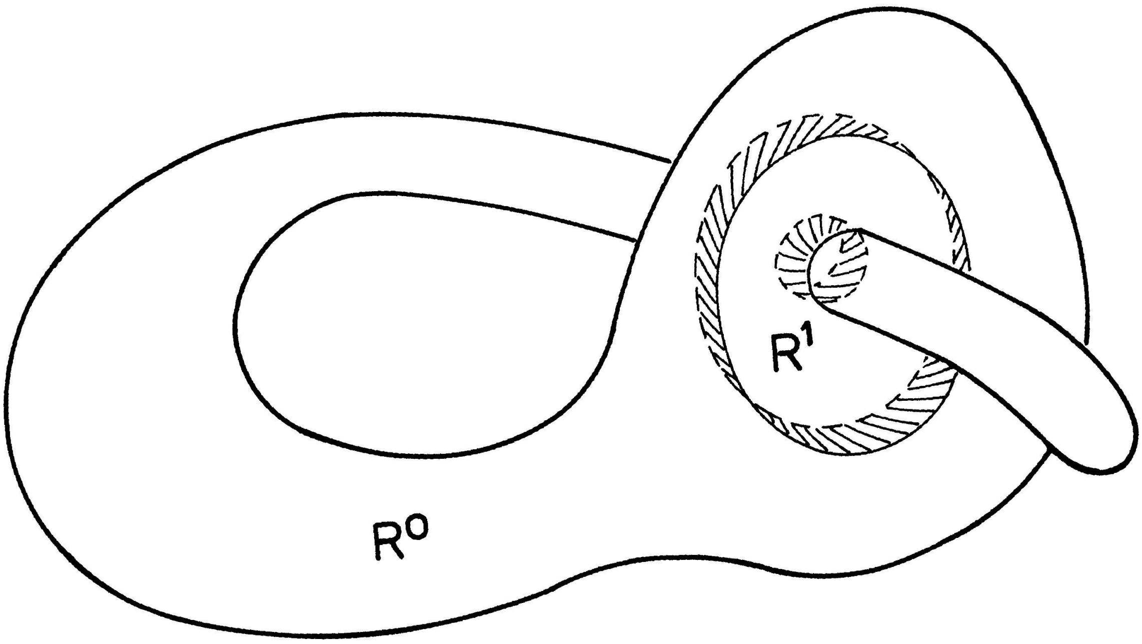


a)

Figure 4.



b)



c)

Figure 4 (*cont'd*).

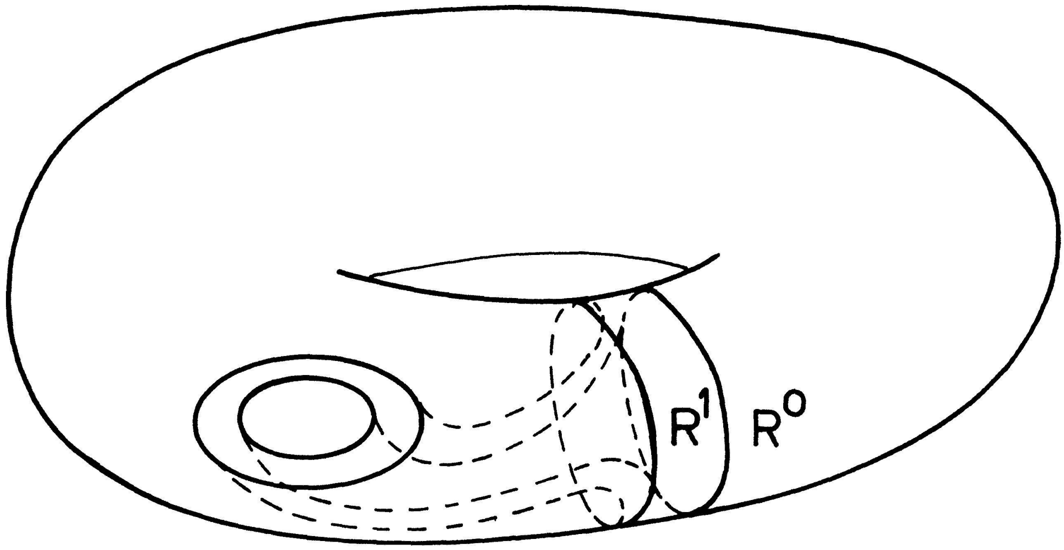


Figure 4 (*cont'd*).

LEMMA 2. *The first obstruction between X_1 and V_0 standard source vector field Y on B vanishes in*

$$H^2(B, \partial B; \pi_2(S^2)) \approx H_1(B; \mathbb{Z}) \approx \mathbb{Z}.$$

Proof. Let $\gamma = d_1(X_1, Y) \in H^2(B, \partial B; \pi_2(S^2))$. Then γ is detected by its Kronecker product $\langle \gamma, w \rangle$ with the generator w of $H_2(B, \partial B; \mathbb{Z}) \approx \mathbb{Z}$. Now w is represented by $\{x\} \times D^2 \subset S^1 \times D^2 = B$. Then $\langle \beta, w \rangle = \langle d_1(X_1, Y), w \rangle$ can be identified with an element $c \in \pi_2(S^2) \approx \mathbb{Z}$ obtained as follows. Let S^2 be the union of two copies of $\{x\} \times D^2$ identified on their boundaries. Then choosing any framing for $T(B)$ restricted to $\{x\} \times D^2$, each of the nonsingular vector fields X_1 and Y, once they have been normalized to unit length, defines a map $\{x\} \times D^2 \to S^2$. These two maps agree on $\{x\} \times \partial D^2$ and hence induce a map $S^2 \to S^2$ which is well defined up to homotopy, and its homotopy class is c.

Now we notice that a circle which surrounds A_1 (cf. Figure 2) bounds a 2-disc D unique up to homotopy rel ∂D, embedded in R^0 so that $D \cap \partial R^0 = \partial D \cap \partial R^0$ (see Figure 5). This disc D will correspond under the isotopy shown in Figure 4 to a representative of c. On D it is easy to see by inspection that up to homotopy both X_1 and Y can be represented in continuous coordinates x_1, x_2 along D, and x_3 normal to D, via

$$x_1\, \partial/\partial x_1 + x_2\, \partial/\partial x_2 + \sqrt{1 - x_1^2 - x_2^2}\, \partial/\partial x_3.$$

Thus $c = 0$ in $\pi_2(S^2)$ and the lemma is proved.

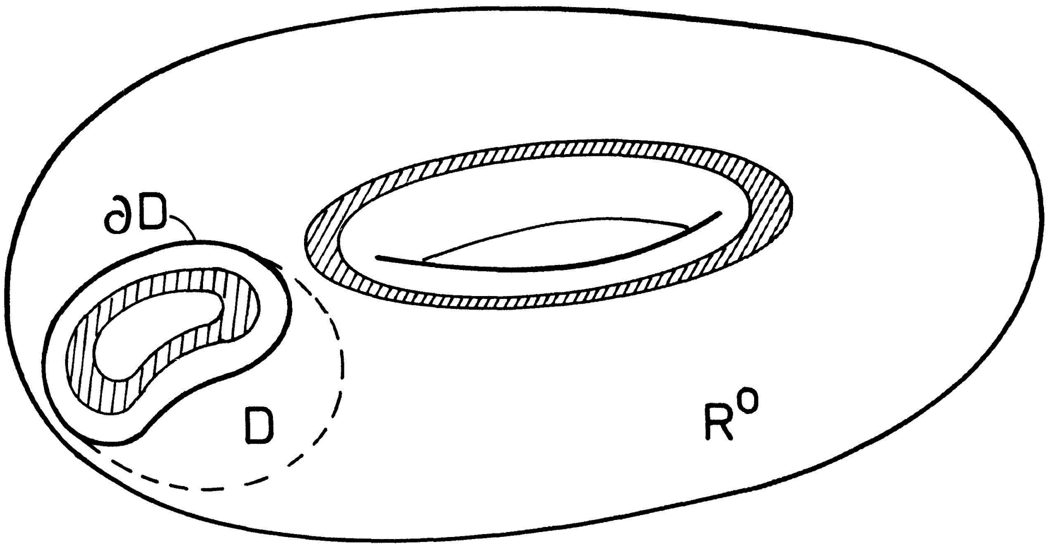


Figure 5.

LEMMA 3. *There is a NMS vector field X on B, exiting on ∂B, such that*

a) *X is homotopic to the source field Y* rel (∂B), *and*
b) *$J(X)=0$; in fact each closed orbit of X bounds a disc in B.*

Remark. Since NMS vector fields are structurally stable [9], X shows that Fuller's example [7] is not just a "pathological" phenomenon.

Proof of Lemma 3. We know from the work of Hopf and Boltyanskii [5] that if the first obstruction between two vector fields vanishes, then they are homotopic if the second obstruction vanishes (as computed using homotopic fields which agree on the codimension-one skeleton). Since X_1 and Y agree on $\{x\}\times D^2 \subset S^1\times D^2 = B$, an arbitrarily small perturbation will cause them to actually agree on $[x, y]\times D^2$. Hence they agree on the 2-skeleton of a certain cell decomposition of B, namely on $\{x\}\times D^2 \cup \{y\}\times D^2 \cup \partial B$. (We continue to denote the perturbed fields by X_1 and Y.)

Now we observe that the second obstruction $d_2(X_1, Y) = m\eta$, some positive multiple of a generator η of $H^3(B, \partial B; \pi_3(S^2)) \approx \mathbb{Z}$.

Let C_i, $1 \le i \le m$ be disjoint nullhomotopic circles embedded in ∂B. In a small tubular neighborhood of each C_i we attach a cancelling pair of round handles $\bar{R}_i^0 + \bar{R}_i^1$ (see Figure 6).

We supply $\bar{R}_i^0$ and $\bar{R}_i^1$ each with a single hyperbolic closed orbit whose Poincaré map has, respectively, 0 or 1 contracting dimension. We also require that the two closed orbits thus created have opposite sense (i.e., represent both generators of $H_1(\bar{R}^0+\bar{R}^1) \approx H_1(S^1\times D^2) \approx \mathbb{Z}$). This resulting vector field is X (NMS after a small C^1 perturbation by [1]).

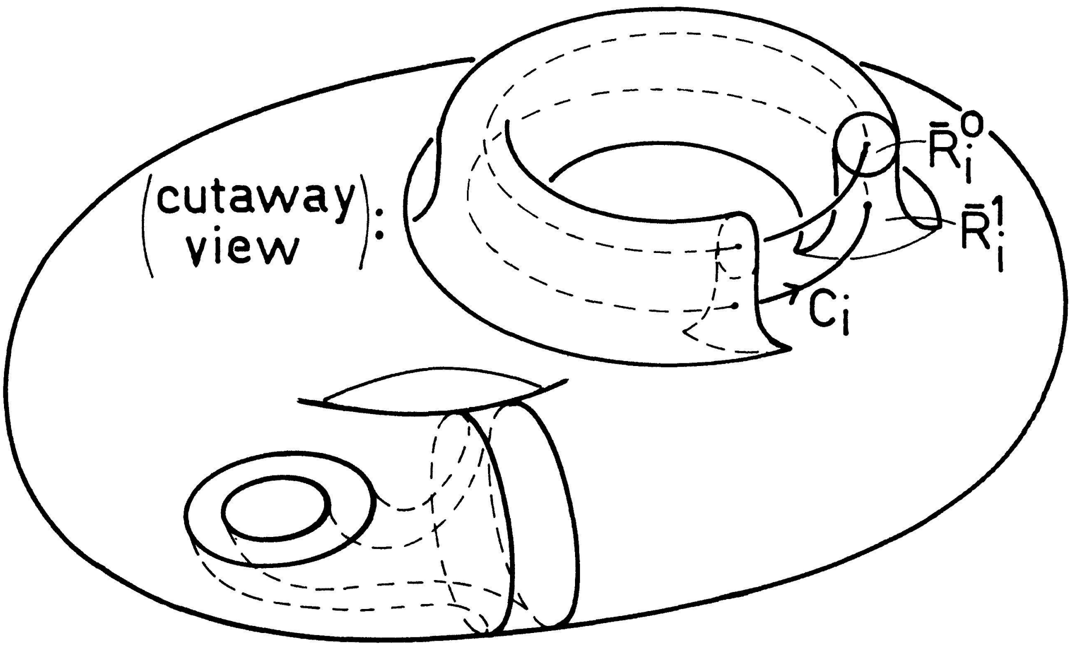


Figure 6.

Let us consider the effect of adding only *one* such cancelling pair $\bar{R}^0+\bar{R}^1$ to $B=R^0+R^1$. We extend the vector field on $B+\bar{R}^0+\bar{R}^1$ to a collar on ∂B just as in [3], and we call this field by the name X'. We also extend X_1 to the collar $(\partial B)\times I$, letting it be the vertical vector field $\partial/\partial t$ there. We may now identify $B\cup(\partial B)\times I$ with B by the usual isotopic deformation down the collar.

This gives us two vector fields on B, which we still call X_1 and X'. The construction above, done carefully, will result in their agreeing on the 2-skeleton $\{x\}\times D^2\cup\{y\}\times D^2\cup\partial B$ of B. To compute $d_2(X_1, X')\in H^3(B, \partial B; \pi_3(S^2))$ we use the same argument as in Lemmas 10 and 11 of [3], only noticing that the framed cobordism class of $Q_0^{-1}(\mathscr{G})$ is indeed the generator of Φ_1, the framed cobordism group of 1-dimensional framed submanifolds in D^3. This follows immediately from [4]. Hence $d_2(X_1, X')$ generates $H^3(B, \partial B; \pi_3(S^2))$, so is $\pm\eta$. By changing the sense of both closed orbits in $\bar{R}^0+\bar{R}^1$ if necessary, we may assume

$$d_2(X_1, X')=\eta. \tag{20}$$

Now we add all m cancelling pairs $\bar{R}_i^0+\bar{R}_i^1$ to $B=R^0+R^1$, (with the proper sense to the closed orbits of the corresponding vector fields). Just as above we identify the resulting space $B+R_1^0+R_1^1+\cdots+R_m^0+R_m^1$ with B itself. Then applying induction to the argument of the last paragraph we obtain

$$d_2(X_1, X)=m\eta. \tag{21}$$

Thus by the addition formula for difference cocycles, we have

$$\begin{aligned} d_2(X, Y) &= d_2(X, X_1) + d_2(X_1, Y) \\ &= -d_2(X_1, X) + d_2(X_1, Y) \\ &= -m\eta + m\eta = 0. \end{aligned} \tag{22}$$

Hence by [5], X is homotopic to $Y \operatorname{rel}(\partial B)$. By construction, each of the $2+2m = 2(m+1)$ closed orbits of X bounds a disc in B. Hence $J(X) = 0$.

LEMMA 4. *Let W be an* NMS *vector field on the compact manifold P. In case $\partial P \neq \emptyset$ we assume W to exit on ∂P. Let Z^k denote a simple hyperbolic source vector field on D^k, such as the positive vector field*

$$Z^k(x_1, \ldots, x_n) = \sum_{i=1}^{k} x_i \, \partial/\partial x_i \tag{23}$$

Then the direct sum vector field $W \oplus Z^k$ is exiting on $\partial(P \times D^k)$ (after appropriate smoothing of the corner $\partial P \times \partial D^k$) and by a C^1-small perturbation $W \oplus Z^k$ can be made NMS.

Proof. It is straightforward to verify that the nonwandering set of $W \oplus Z^k$ is a finite union of hyperbolic closed orbits. Transversal intersection of stable and unstable manifolds is then obtainable by a C^1-small perturbation that preserves the truth of the previous sentence [1]. To see that $W \oplus Z^k$ exits on $\partial(P \times D^k)$ it is necessary to choose a "convex" straightening of the angle [6] of the product of two half-spaces. We use charts where W and Z^k are unit normal fields to the boundaries and the result follows.

LEMMA 5. *Let $n \geq 3$. There is a* NMS *vector field X^n on the solid torus $S^1 \times D^{n-1}$ such that*

a) *X^n exits on $S^1 \times \partial D^{n-1} = \partial(S^1 \times D^{n-1})$.*

b) *X^n is homotopic to a hyperbolic source* NMS *vector field on $S^1 \times D^{n-1}$, through nonsingular vector fields which remain everywhere transverse to $\partial(S^1 \times D^{n-1})$.*

c) *$J(X^n) = 0$.*

Proof. Let $\{X_t\}_{0 \leq t \leq 1}$ denote a homotopy through nonsingular fields between $X_0 = X$ and $X_1 = Y$ on $S^1 \times D^2$, as in Lemma 3. We define

$$\bar{X}^n = X \oplus Z^{n-3} \tag{24}$$

on $S^1\times D^{n-1}$ (obtained as in Lemma 4 on $S^1\times D^3\times D^{n-3}$ by rounding the corners). Then

$$\{\bar{X}_t^n \oplus Z^{n-3}\}_{0\le t\le 1} \tag{25}$$

provides a homotopy of nonsingular vector fields on $S^1\times D^{n-1}$ which are each transverse to $\partial(S^1\times D^{n-1})$ by Lemma 4. It is easy to see that $\bar{X}_1^n = Y\oplus Z^{n-3}$ is a standard hyperbolic source vector field on $S^1\times D^{n-1}$. Finally by Lemma 4 again, $\bar{X}^n = X\oplus Z^{n-3}$ can be C^1-small perturbed to a field X^n which is NMS. The closed orbits of X^n will correspond to those of $\bar{X}^n$ and may be assumed to have the same Poincaré maps. Then by definition of Z^{n-3} we have

$$J(X^n) = (-1)^{n-3} i_* J(X)$$

where $i_*: H_1(S^1\times D^2) \xrightarrow[\approx]{} H_1(S^1\times D^{n-1})$. Hence by Lemma 3, b), we are done.

Conclusion of proof of Theorem 1. In [3] we showed the existence of a NMS vector field V_0 in any desired homotopy class $\mathfrak{D}$. As in [3], Lemma 5 (cf. Example D, Section 1 of this paper) we may homotope V_0 to another NMS field V_0' obtained from V_0 by adding a cancelling pair of new round handles of index 0 and 1. By (19), $J(V_0) = J(V_0') = \alpha_0 \in H_1(M)$, say. We may arrange that the S^1 direction of each of these new round handles represents the element $(-1)^n(\alpha-\alpha_0)$ in $H_1(M)$.

We finally define V as follows:

$$V(x) = \begin{cases} V_0'(x) & \text{if } x \text{ is outside the new round 0-handle} \\ X^n(x) & \text{if } x \text{ is inside the new round 0-handle.} \end{cases}$$

Here we are identifying $S^1\times D^{n-1}$ of Lemma 5 with the new round 0-handle. We assume a smooth interpolation near a collar of $\partial(S^1\times D^{n-1})$ if necessary, to fit V_0' and X^n together smoothly. We also assume a C^1-small perturbation if necessary to make V NMS (as in [3], Lemma 4).

Hence V is a NMS vector field in $\mathfrak{D}$, and by choice of embedding of the new round handles and by Lemma 5, c), we have

$$\begin{aligned} J(V) &= J(V_0) + (-1)^{n-2}((-1)^n(\alpha-\alpha_0)) \\ &= \alpha_0 + \alpha - \alpha_0 = \alpha \quad \text{as desired.} \end{aligned}$$

Remarks. The above technique shows that in dimension 3, the conclusion of Theorem 2 holds when there exists a round handle decomposition.

REFERENCES

[1] R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, 1967.
[2] D. Asimov, *Round handles and non-singular Morse–Smale flows*, Ann. of Math. *102* (1975), 41–54.
[3] ——, *Homotopy of non-singular vector fields to structurally stable ones*, ibid., 55–65.
[4] V. G. Boltyanskii, *Homotopy of mappings and vector fields*, A.M.S. Translations Series 2, Volume 7, 1957.
[5] ——, *Homotopy classification of vector fields*, Dokl. Akad. Nauk SSR, 1958, 731 ff.
[6] J. Cerf, $\Gamma_4 = 0$, Lecture notes in mathematics *53*, Springer-Verlag, 1968.
[7] F. B. Fuller, *Note on trajectories in a solid torus*, Ann. of Math. *56* (1952), 438–439.
[8] ——, *An index of fixed point type for periodic orbits*, Am. J. Math. *89* (1967), 133–148.
[9] J. Palis and S. Smale, *Structural stability theorems*, A.M.S. Proc. Symp. Pure Math. *14* (1970), 223–231.
[10] S. Smale, *Differentiable dynamical systems*, Bull. A.M.S. *73* (1967), 747–817.

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