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Autor: Asimov, Daniel
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Flaccidity of geometric index for nonsingular vector fields

DANIEL ASIMOV

1. Introduction

We consider nonsingular vector fields V on compact connected C^∞ orientable manifolds M^n without boundary. It is of interest to find criteria for detecting the occurrence of closed orbits of V . If we were to look instead at diffeomorphisms $f: M \rightarrow M$ the analogous question would be to find the periodic points of f . We may at least count the periodic points algebraically via the Lefschetz fixed point formula, if $\{x \mid f^k(x) = x\} = \text{Fix}(f^k)$ is finite:

$$\sum_{f^k(x)=x} i_{f^k}(x) = \sum_{j=0}^n (-1)^j \text{tr}(f_{*j}^k: H_j(M; \mathbb{R}) \hookrightarrow). \quad (1)$$

Here $i_{f^k}(x)$ is the local index of the fixed point x of f^k , and the right hand side is, by definition, the Lefschetz number $\Lambda(f^k)$ of f^k .

We define here an index for nonsingular Morse–Smale (NMS) vector fields [2] as follows. Let V be an NMS field on M . Let C_i $1 \leq i \leq r$ be all the closed orbits of V . Let $\bar{C}_i \in H_1(M; \mathbb{Z})$ be the homology class of C_i (oriented by V) and let $\varepsilon_i = \pm 1$ be the fixed point index of the Poincaré map induced by the flow of V around C_i . Then we define

$$J(V) = \sum_{i=1}^r \varepsilon_i \bar{C}_i \quad \text{in } H_1(M; \mathbb{Z}) \quad (2)$$

to be the *geometric index of the NMS field V* .

Let \mathcal{V} denote all nonsingular vector fields on M , with the C^0 topology. In Section 2 we describe some situations where J remains constant over a large open set or on distinct homotopic NMS vector fields. In particular we show that if V is the suspension of a Morse–Smale diffeomorphism, then $J(V)$ is constant through large perturbations of V .

In Section 3, however, we show by example that if the dimension of M is greater than 3, then for each $\alpha \in H_1(M; \mathbb{Z})$ and for each homotopy class \mathcal{D} of

nonsingular vector fields on M , there exists a $V_\alpha \in \mathcal{D}$ such that V_α is NMS and $J(V_\alpha) = \alpha$. Fuller's Theorem 1 [8] then imposes conditions on any homotopy between V_α and V_β , $\alpha \neq \beta$. We note that NMS vector fields are structurally stable [9].

2. Examples

A. Let $p: E^n \rightarrow S^1$ be a smooth oriented fibre bundle with fibre the compact manifold F^{n-1} . Let $\{V_s\}_{0 \leq s \leq 1}$ denote a homotopy of nonsingular vector fields on E which are never tangent to a fibre. Assume that V_0 and V_1 are NMS, with orientable stable and unstable manifolds. Then we have

PROPOSITION A.

$$J(V_0) = J(V_1). \quad (3)$$

Proof. Without loss of generality we may assume that for all $(x, s) \in E \times I$ we have

$$Dp(V_s(x)) = 2\pi \cdot d/d\theta, \quad (4)$$

where $Dp: TE \rightarrow TS^1$ is the tangent mapping of p . Let $p_*: H_1 E \rightarrow H_1 S^1$ be the map on integer homology. Identify $H_1 S^1$ with \mathbb{Z} by sending the counterclockwise generator to 1. Now set $J_k(V_0) = \sum_{\bar{C}_i \in p_*^{-1}(k)} \varepsilon_i \bar{C}_i$. Then

$$J(V_0) = \sum_{k=1}^{\infty} J_k(V_0) \quad (5)$$

and similarly

$$J(V_1) = \sum_{k=1}^{\infty} J_k(V_1). \quad (6)$$

Hence the proposition follows from showing $J_k(V_0) = J_k(V_1)$ for all $k = 1, 2, \dots$.

We notice that by (4) the time-one map φ_s^1 of the flow $\{\varphi_s^1\}_{t \in \mathbb{R}}$ of V_s is a map $\varphi_s^1: E \rightarrow E$ which preserves fibres. Similarly for φ_s^k , k any integer. Let $p^*(E)$ denote the space $\{(x, y) \in E \times E \mid p(x) = p(y)\}$, the total space of the pullback by p

of the bundle $E \xrightarrow{p} S^1$. Then $p^*(E)$ fibres over S^1 with fibre $F \times F$, via $(x, y) \mapsto p(x) = p(y) \in S^1$. We define the subset $\Delta \subset p^*(E)$ by $\Delta = \{(x, x) \in p^*(E)\}$. Also let $\bar{p}: p^*(E) \rightarrow E$ be the projection given by $\bar{p}(x, y) = x$.

Now we may form the map $\Gamma_s^1: E \rightarrow p^*(E)$ defined by $\Gamma_s^1(x) = (x, \varphi_s^1(x))$. This map Γ_s^1 is transverse to $\Delta \subset p^*(E)$ precisely when the map φ_s^1 restricted to any fibre F has no eigenvalues equal to 1. In particular, this is the case for $s = 0, 1$. It is clear from the definitions that as a point set, $\bar{p}(\Gamma_i^1(E) \cap \Delta)$ is the union of all the closed orbits of V_i which go around E once, i.e., those closed orbits comprising $J_1(V_i)$, for $i = 0, 1$.

Now considering homology intersection \cdot , we let $[\Delta] \in H_n(p^*(E))$ denote the fundamental class of the diagonal, i.e., the image of $[E] \in H_n(E)$ under $x \mapsto (x, x)$. Let $\Gamma_s^1*: H_n(E) \rightarrow H_n(p^*(E))$ and $\bar{p}_*: H_1(p^*(E)) \rightarrow H_1(E)$ denote the induced maps on homology. Then

$$\Lambda_s^1 = \bar{p}_*(\Gamma_s^1*([E]) \cdot [\Delta]) \quad \text{in } H_1(E) \quad (7)$$

is an integer homology class independent of s , and so

$$\Lambda_0^1 = \Lambda_1^1. \quad (8)$$

Let $f_i: F \rightarrow F$ denote $\varphi_i^1|_F$ for the fibre $F = p^{-1}(x)$ of E . Let $y \in F$ be the fixed point of f_i corresponding to the closed orbit C_y of V_i . Then C_y contributes $\varepsilon_y \bar{C}_y \in H_1(E)$ to the class Λ_i^1 , where

$$\varepsilon_y = \det \left(\begin{array}{c|c} I_{n-1} & I_{n-1} \\ \hline Df(y) & I_{n-1} \end{array} \right)$$

i.e.,

$$\begin{aligned} \varepsilon_y &= \det(I_{n-1} - Df(y)) \\ &= i_{f_i}(y), \end{aligned} \quad (9)$$

the local fixed point index at y (since 1 is not an eigenvalue of $Df(y)$). Thus we have shown that

$$\Lambda_i^1 = J_1(V_i) \quad i = 0, 1 \quad (10)$$

and so by (8) this shows $J_1(V_0) = J_1(V_1)$.

Now using the fact that we assumed all stable and unstable manifolds to be *orientable* for V_i , $i = 0, 1$, we may easily check that for $i = 0, 1$, we have

$$\pi_{k*} J_1(\tilde{V}_i) = k \sum_{d|k} J_d(V_i) \in H_1(E). \quad (11)$$

Here $\pi_k: \tilde{E}_k \rightarrow E$ denotes the canonical k -fold covering space over E (since E is a fibre bundle over S^1) and \tilde{V}_i denotes the unique lift of V_i to a nonsingular vector field on \tilde{E}_k .

Hence $J_k(V_i)$ is expressible in terms of $\pi_k^* J_1(\tilde{V}_i)$ and $J_d(V_i)$ for $1 \leq d < k$. Now $J_1(\tilde{V}_i)$ is independent of $i = 0, 1$ by the considerations we applied above to $J_1(V_i)$ (since the homotopy V_s gives rise to a homotopy \tilde{V}_s , $0 \leq s \leq 1$). Hence we also have

$$\pi_k^* J_1(\tilde{V}_0) = \pi_k^* J_1(\tilde{V}_1) \quad (12)$$

and by induction, therefore,

$$J_k(V_0) = J_k(V_1), \quad k \geq 1. \quad (13)$$

Hence summing over k we obtain

$$J(V_0) = J(V_1), \quad \text{as desired.} \quad (14)$$

Remarks. 1. Proposition A could have also been obtained using Fuller's Theorem 1 of [8].

2. The assumption of orientable stable manifolds insures that $L_{f_t^*}(y)$ is independent of k . If we omitted this assumption the theorem would be false. For example, the 180° rotation of the 2-sphere S^2 may be perturbed to a Morse–Smale (M.–S.) diffeomorphism f having 2 fixed orientation-reversing saddles (of index 1) at the poles, and four alternating sources and sinks of period 2 each (and index 1) along the equator (see Figure 1). Let f_s be a homotopy of $f = f_0$ to the gradient M.–S. diffeomorphism f_1 given by $z \mapsto z^2$ for $z \in S^2$. Then if V_s are the corresponding “suspension” vector fields on $S^1 \times S^2$, we have

$$J(V_0) = 6 \quad \text{but} \quad J(V_1) = 2,$$

where we have identified $H_1(S^1 \times S^2)$ canonically with \mathbb{Z} .

Example B. For a second example, consider the 2-torus T^2 . It follows from [8] that if V_0, V_1 are two NMS vector fields on T^2 , then they are homotopic through nonsingular vector fields if and only if $J(V_0) = J(V_1)$.

Example C. We consider the case of a vector field V tangent to the fibres of a principal circle bundle E^n over a compact manifold M^{n-1} . Let G denote a small gradient Morse–Smale vector field on M (not necessarily nonsingular). A choice of connection on E enables us to lift G to a unique S^1 -invariant vector field \tilde{G} on E .

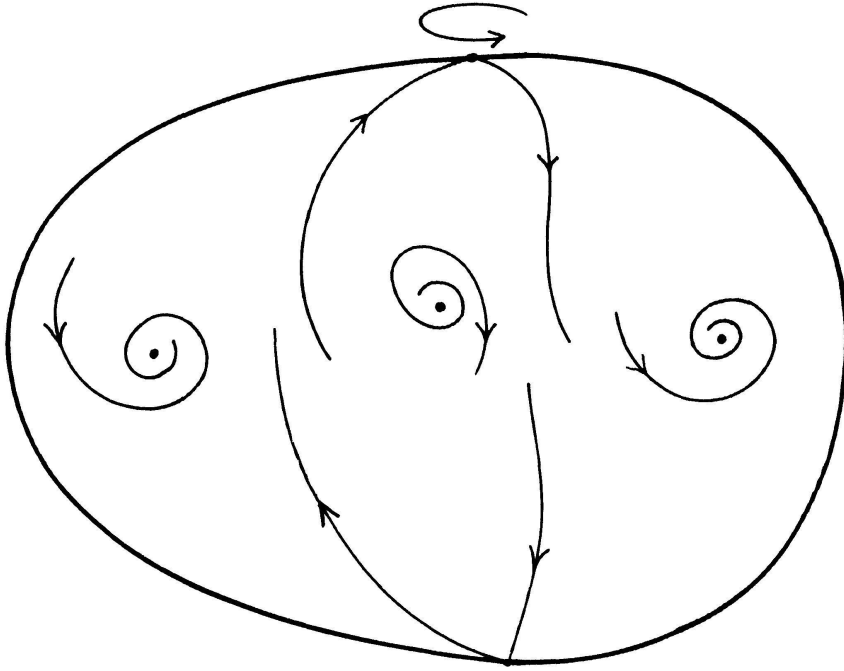


Figure 1.

It is easy to verify that $V' = V + \tilde{G}$ is a NMS vector field on E . Furthermore, considering the cases n odd and n even separately, we may check that in fact

$$J(V') = \chi(M) \cdot \bar{F} \quad \text{in } H_1(E), \quad (15)$$

where \bar{F} is the homology class of the oriented fibre $F \approx S^1$.

By picking topologically distinct gradient fields G one may create countably many topologically distinct NMS fields on E all homotopic to one another and all having the same geometric index.

Example D. Let M^n have the round handle decomposition

$$M \approx Q_1 + Q_2 + \cdots + Q_r \quad (16)$$

where the Q_i , $1 \leq i \leq r$, are round handles of various indices attached successively (see [2], [3]), and we assume $n \geq 4$. We may assume M carries a NMS vector field V compatible with the decomposition.

Then there is an operation which creates a new round handle decomposition and a corresponding compatible vector field V' . We introduce a cancelling pair of round handles in between Q_i and Q_{i+1} :

$$M \approx Q_1 + \cdots + Q_i + R^j + R^{j+1} + Q_{i+1} + \cdots + Q_r. \quad (17)$$

Here $R^j \approx S^1 \times D^j \times D^{n-j-1}$ and $R^{j+1} \approx S^1 \times D^{j+1} \times D^{n-j-2}$. The union $R^j \cup R^{j+1} \approx S^1 \times (h^j \cup h^{j+1})$ where $h^j + h^{j+1}$ represents a cancelling pair of ordinary handles of dimension $n-1$. $R^j + R^{j+1}$ is attached in such a way that

$$Q_1 + \cdots + Q_i + R^j + R^{j+1} \approx Q_1 + \cdots + Q_i. \quad (18)$$

On each of R^j, R^{j+1} we define a nonsingular vector field that is essentially the neighborhood of a single hyperbolic closed orbit whose Poincaré map has j or $j+1$ contracting dimensions, respectively. As in [2], this defines a new NMS vector field, say V' , which is homotopic to V and such that also

$$J(V') = J(V), \quad (19)$$

since we have added two closed orbits which are homologous but whose Poincaré maps have fixed point indices of opposite sign, namely $(-1)^{n-1-j}$ and $(-1)^{n-2-j}$, respectively.

3.

THEOREM 1. *We assume $n = \dim M \geq 4$. Let $\alpha \in H_1(M; \mathbb{Z})$ be arbitrary, and let \mathcal{D} be any homotopy class of nonsingular vector fields on M . Then there is a NMS vector field $V \in \mathcal{D}$ such that $J(V) = \alpha$. It follows from structural stability that there is in fact an entire C^1 neighborhood $N(V)$ such that*

$$V' \in N(V) \Rightarrow J(V') = \alpha.$$

Proof. We begin by constructing an example of a certain NMS vector field X_1 on the solid torus $B = S^1 \times D^2$. We construct X_1 by a round handle decomposition $B = R^0 + R^1$. As in Section 5 of [2], R^i , $i = 0, 1$ is supplied with a nonsingular vector field having exactly one closed orbit, whose Poincaré map is hyperbolic with i contracting dimensions. In coordinates the vector field on $R^0 = S^1 \times D^2$ is given by $d/d\theta + x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2$, and the vector field on $R^1 = S^1 \times D^1 \times D^1$ is given by $d/d\varphi - y_1 \partial/\partial y_1 + y_2 \partial/\partial y_2$. The attaching region $\partial_-(R^1)$ is defined by $\{(\varphi, y_1, y_2) \in R^1 \mid |y_1| = 1\}$ and is the disjoint union $S^1 \times S^0 \times D^1$ of two annuli $A_{-1} = S^1 \times \{-1\} \times D^1$ and $A_1 = S^1 \times \{1\} \times D^1$.

We must specify, up to isotopy, the attaching map

$$h: S^1 \times S^0 \times D^1 \rightarrow \partial(R^0)$$

(where $\partial(R^0) = \partial(S^1 \times D^2)$ is a 2-torus). Any h satisfying the following conditions will suffice for our purposes:

- a) The composition $A_{-1} \xrightarrow{h|_{A_{-1}}} \partial(R^0) \subseteq R^0$ induces an isomorphism $H_1(A_{-1}) \rightarrow H_1(R^0)$.
- b) $h(A_1)$ deforms to a point in $\partial(R^0)$.
- c) A_{-1} and A_1 are embedded in $\partial(R^0)$ with opposite orientations.

LEMMA 1. If $h: \partial_+(R^1) \rightarrow \partial(R^0)$ satisfies a), b), c) above, then the quotient space $R^0 + R^1 = R^0 \cup R^1 / x \sim h(x)$ is diffeomorphic to a solid torus.

Proof. By the isotopy extension lemma [6] an isotopy of an embedding of the core circle of A_i $i = \pm 1$ extends to an isotopy of all of $\partial_-(R^1)$. Using this it is straightforward to show that if h satisfies a), b), c) then up to isotopy $h(A_{-1})$ and $h(A_1)$ are as depicted in Figure 2 (after embedding R^0 appropriately in \mathbb{R}^3).

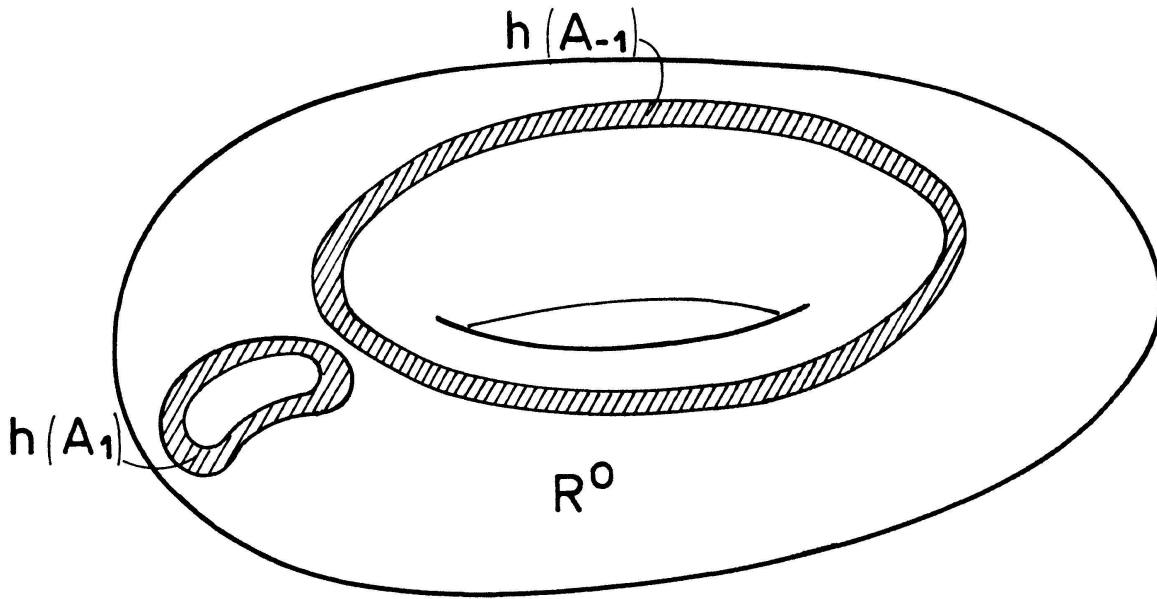


Figure 2.

Hence $R^0 + R^1$ will resemble Figure 3.

Then performing the isotopy indicated in Figure 4a) to d) shows the lemma.

Now X_1 is defined as the NMS vector field on $B = S^1 \times D^2$ induced from the round handle decomposition of B described above. Let us assume X_1 is normal to ∂B .

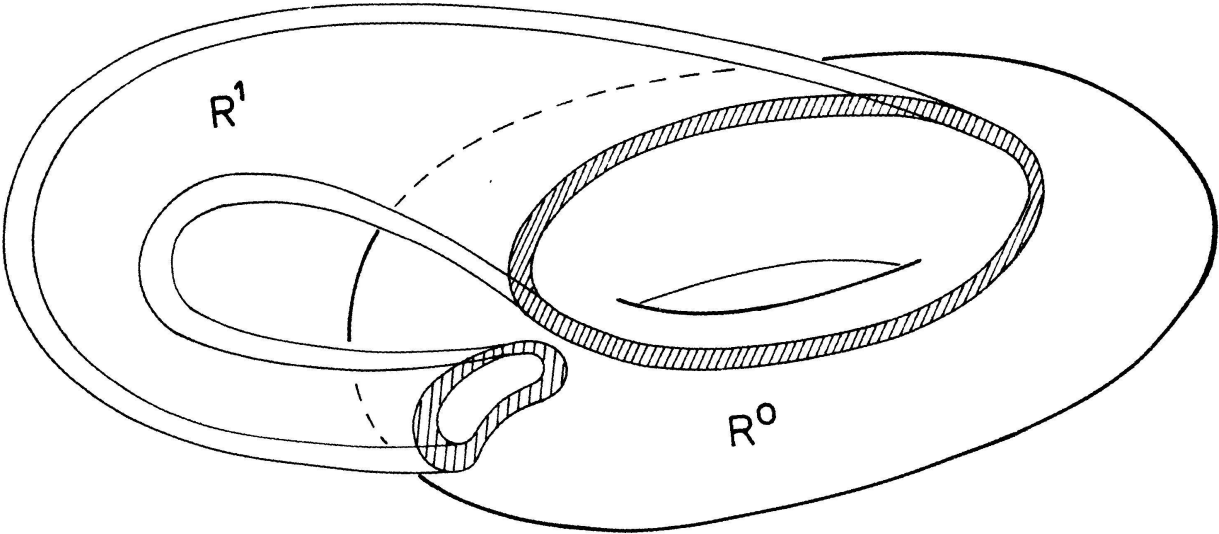


Figure 3.

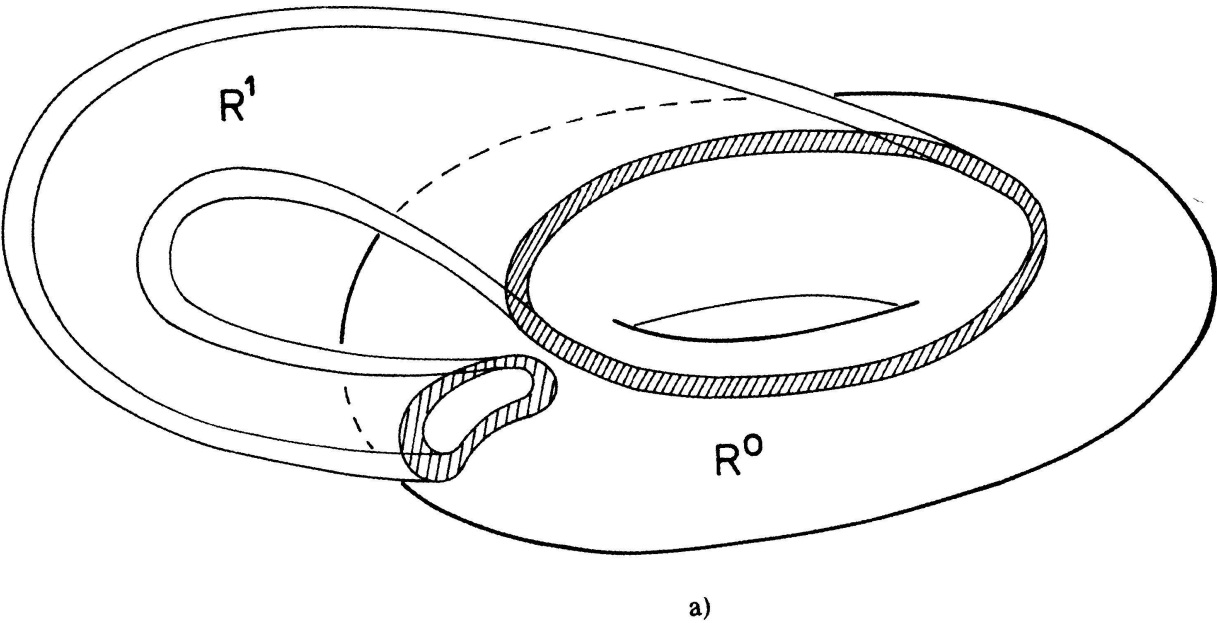


Figure 4.

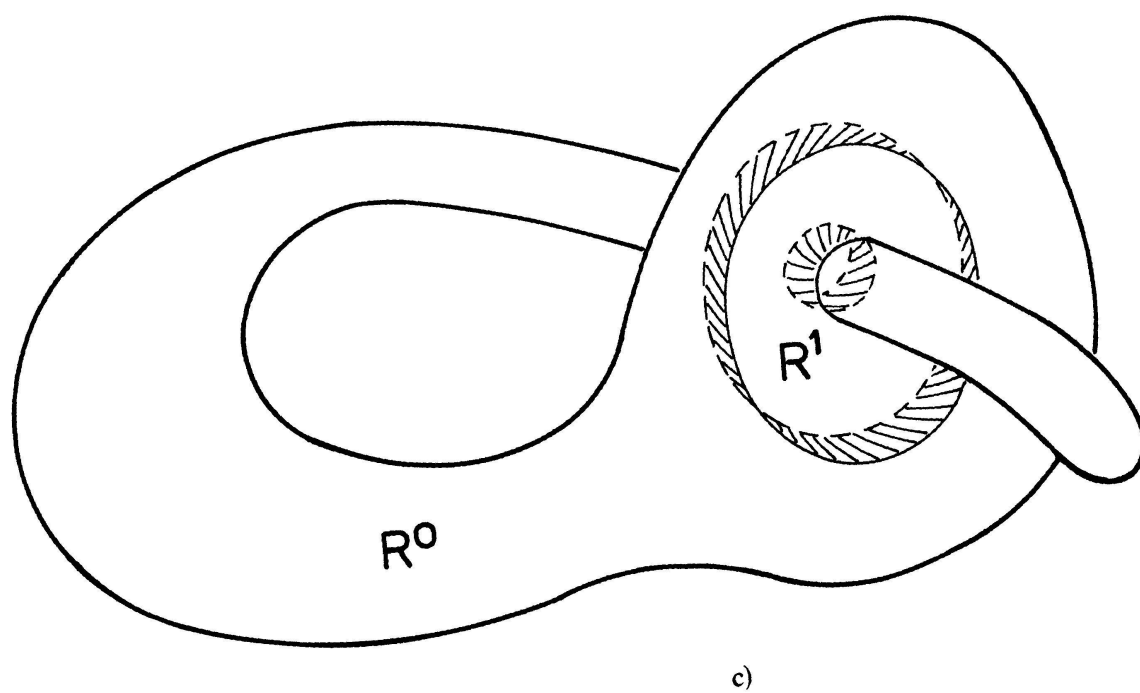
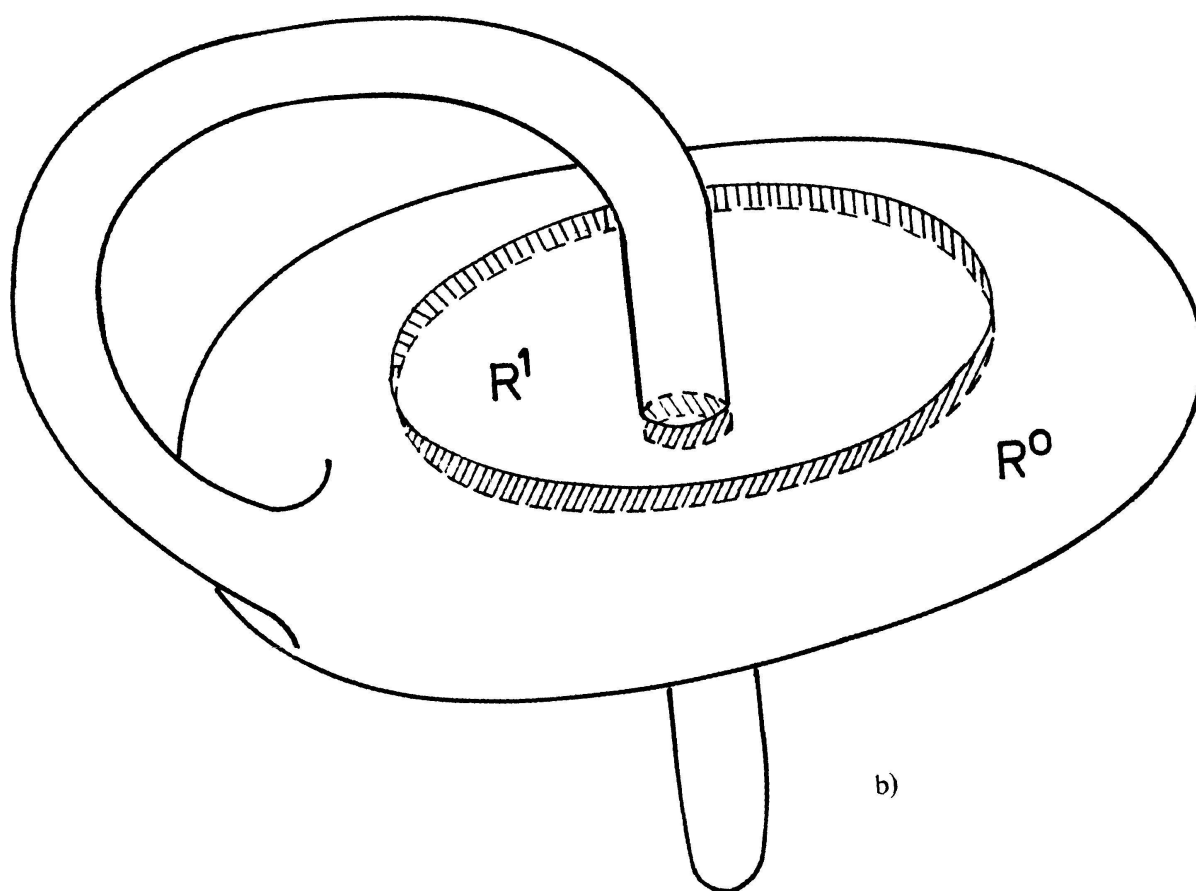


Figure 4 (cont'd).

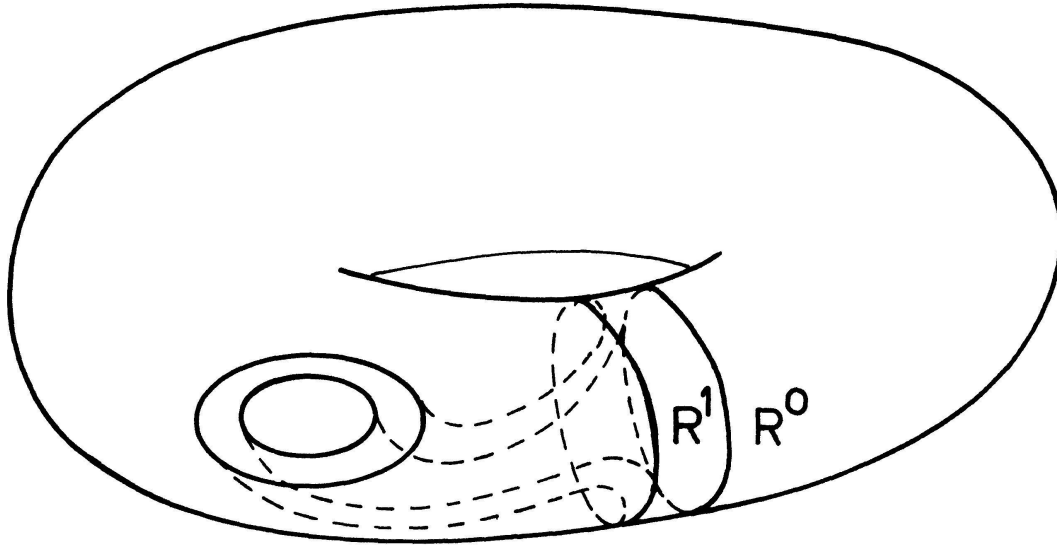


Figure 4 (cont'd).

d)

LEMMA 2. *The first obstruction between X_1 and V_0 standard source vector field Y on B vanishes in*

$$H^2(B, \partial B; \pi_2(S^2)) \approx H_1(B; \mathbb{Z}) \approx \mathbb{Z}.$$

Proof. Let $\gamma = d_1(X_1, Y) \in H^2(B, \partial B; \pi_2(S^2))$. Then γ is detected by its Kronecker product $\langle \gamma, w \rangle$ with the generator w of $H_2(B, \partial B; \mathbb{Z}) \approx \mathbb{Z}$. Now w is represented by $\{x\} \times D^2 \subset S^1 \times D^2 = B$. Then $\langle \beta, w \rangle = \langle d_1(X_1, Y), w \rangle$ can be identified with an element $c \in \pi_2(S^2) \approx \mathbb{Z}$ obtained as follows. Let S^2 be the union of two copies of $\{x\} \times D^2$ identified on their boundaries. Then choosing any framing for $T(B)$ restricted to $\{x\} \times D^2$, each of the nonsingular vector fields X_1 and Y , once they have been normalized to unit length, defines a map $\{x\} \times D^2 \rightarrow S^2$. These two maps agree on $\{x\} \times \partial D^2$ and hence induce a map $S^2 \rightarrow S^2$ which is well defined up to homotopy, and its homotopy class is c .

Now we notice that a circle which surrounds A_1 (cf. Figure 2) bounds a 2-disc D unique up to homotopy rel ∂D , embedded in R^0 so that $D \cap \partial R^0 = \partial D \cap \partial R^0$ (see Figure 5). This disc D will correspond under the isotopy shown in Figure 4 to a representative of c . On D it is easy to see by inspection that up to homotopy both X_1 and Y can be represented in continuous coordinates x_1, x_2 along D , and x_3 normal to D , via

$$x_1 \partial / \partial x_1 + x_2 \partial / \partial x_2 + \sqrt{1 - x_1^2 - x_2^2} \partial / \partial x_3.$$

Thus $c = 0$ in $\pi_2(S^2)$ and the lemma is proved.

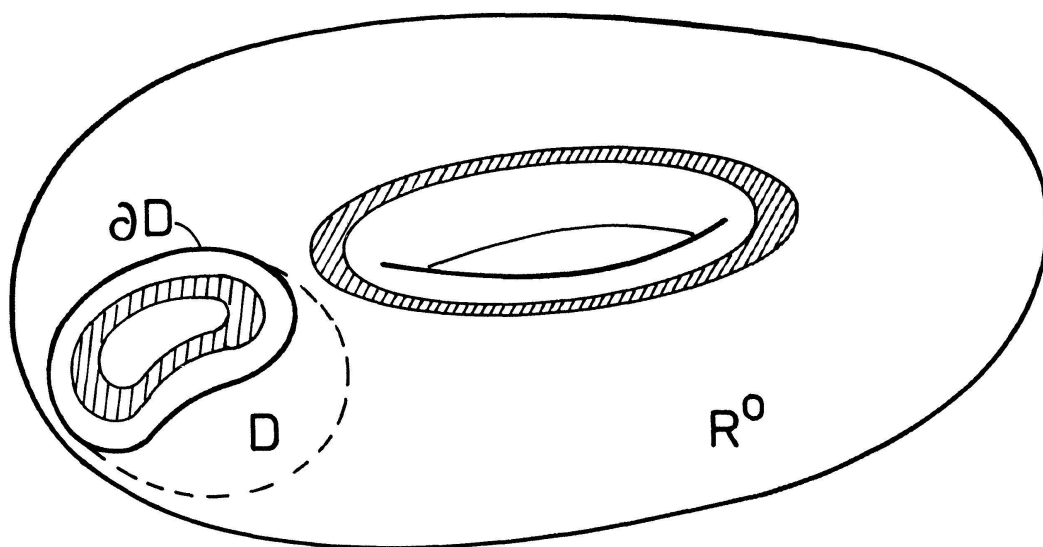


Figure 5.

LEMMA 3. *There is a NMS vector field X on B , exiting on ∂B , such that*

- a) *X is homotopic to the source field Y rel (∂B) , and*
- b) *$J(X) = 0$; in fact each closed orbit of X bounds a disc in B .*

Remark. Since NMS vector fields are structurally stable [9], X shows that Fuller's example [7] is not just a "pathological" phenomenon.

Proof of Lemma 3. We know from the work of Hopf and Boltyanskii [5] that if the first obstruction between two vector fields vanishes, then they are homotopic if the second obstruction vanishes (as computed using homotopic fields which agree on the codimension-one skeleton). Since X_1 and Y agree on $\{x\} \times D^2 \subset S^1 \times D^2 = B$, an arbitrarily small perturbation will cause them to actually agree on $[x, y] \times D^2$. Hence they agree on the 2-skeleton of a certain cell decomposition of B , namely on $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$. (We continue to denote the perturbed fields by X_1 and Y .)

Now we observe that the second obstruction $d_2(X_1, Y) = m\eta$, some positive multiple of a generator η of $H^3(B, \partial B; \pi_3(S^2)) \approx \mathbb{Z}$.

Let C_i , $1 \leq i \leq m$ be disjoint nullhomotopic circles embedded in ∂B . In a small tubular neighborhood of each C_i we attach a cancelling pair of round handles $\bar{R}_i^0 + \bar{R}_i^1$ (see Figure 6).

We supply \bar{R}_i^0 and \bar{R}_i^1 each with a single hyperbolic closed orbit whose Poincaré map has, respectively, 0 or 1 contracting dimension. We also require that the two closed orbits thus created have opposite sense (i.e., represent both generators of $H_1(\bar{R}_i^0 + \bar{R}_i^1) \approx H_1(S^1 \times D^2) \approx \mathbb{Z}$). This resulting vector field is X (NMS after a small C^1 perturbation by [1]).

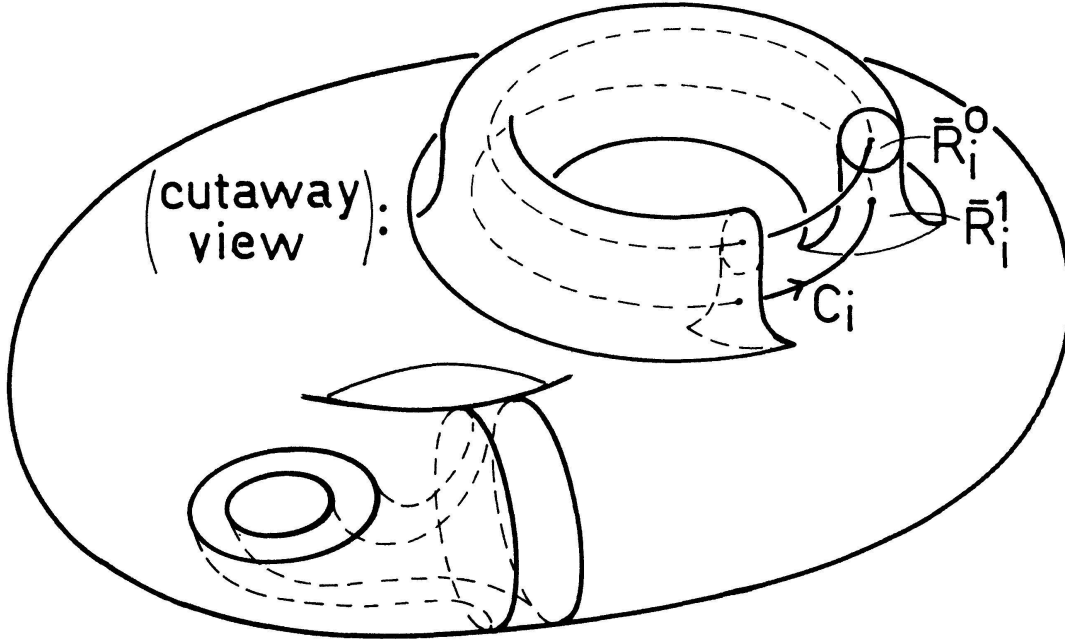


Figure 6.

Let us consider the effect of adding only *one* such cancelling pair $\bar{R}^0 + \bar{R}^1$ to $B = R^0 + R^1$. We extend the vector field on $B + \bar{R}^0 + \bar{R}^1$ to a collar on ∂B just as in [3], and we call this field by the name X' . We also extend X_1 to the collar $(\partial B) \times I$, letting it be the vertical vector field $\partial/\partial t$ there. We may now identify $B \cup (\partial B) \times I$ with B by the usual isotopic deformation down the collar.

This gives us two vector fields on B , which we still call X_1 and X' . The construction above, done carefully, will result in their agreeing on the 2-skeleton $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$ of B . To compute $d_2(X_1, X') \in H^3(B, \partial B; \pi_3(S^2))$ we use the same argument as in Lemmas 10 and 11 of [3], only noticing that the framed cobordism class of $Q_0^{-1}(\mathcal{G})$ is indeed the generator of Φ_1 , the framed cobordism group of 1-dimensional framed submanifolds in D^3 . This follows immediately from [4]. Hence $d_2(X_1, X')$ generates $H^3(B, \partial B; \pi_3(S^2))$, so is $\pm\eta$. By changing the sense of both closed orbits in $\bar{R}^0 + \bar{R}^1$ if necessary, we may assume

$$d_2(X_1, X') = \eta. \quad (20)$$

Now we add all m cancelling pairs $\bar{R}_i^0 + \bar{R}_i^1$ to $B = R^0 + R^1$, (with the proper sense to the closed orbits of the corresponding vector fields). Just as above we identify the resulting space $B + R_1^0 + R_1^1 + \cdots + R_m^0 + R_m^1$ with B itself. Then applying induction to the argument of the last paragraph we obtain

$$d_2(X_1, X) = m\eta. \quad (21)$$

Thus by the addition formula for difference cocycles, we have

$$\begin{aligned} d_2(X, Y) &= d_2(X, X_1) + d_2(X_1, Y) \\ &= -d_2(X_1, X) + d_2(X_1, Y) \\ &= -m\eta + m\eta = 0. \end{aligned} \tag{22}$$

Hence by [5], X is homotopic to $Y \text{ rel } (\partial B)$. By construction, each of the $2 + 2m = 2(m + 1)$ closed orbits of X bounds a disc in B . Hence $J(X) = 0$.

LEMMA 4. *Let W be an NMS vector field on the compact manifold P . In case $\partial P \neq \emptyset$ we assume W to exit on ∂P . Let Z^k denote a simple hyperbolic source vector field on D^k , such as the positive vector field*

$$Z^k(x_1, \dots, x_n) = \sum_{i=1}^k x_i \partial/\partial x_i \tag{23}$$

Then the direct sum vector field $W \oplus Z^k$ is exiting on $\partial(P \times D^k)$ (after appropriate smoothing of the corner $\partial P \times \partial D^k$) and by a C^1 -small perturbation $W \oplus Z^k$ can be made NMS.

Proof. It is straightforward to verify that the nonwandering set of $W \oplus Z^k$ is a finite union of hyperbolic closed orbits. Transversal intersection of stable and unstable manifolds is then obtainable by a C^1 -small perturbation that preserves the truth of the previous sentence [1]. To see that $W \oplus Z^k$ exits on $\partial(P \times D^k)$ it is necessary to choose a “convex” straightening of the angle [6] of the product of two half-spaces. We use charts where W and Z^k are unit normal fields to the boundaries and the result follows.

LEMMA 5. *Let $n \geq 3$. There is a NMS vector field X^n on the solid torus $S^1 \times D^{n-1}$ such that*

- a) X^n exits on $S^1 \times \partial D^{n-1} = \partial(S^1 \times D^{n-1})$.
- b) X^n is homotopic to a hyperbolic source NMS vector field on $S^1 \times D^{n-1}$, through nonsingular vector fields which remain everywhere transverse to $\partial(S^1 \times D^{n-1})$.
- c) $J(X^n) = 0$.

Proof. Let $\{X_t\}_{0 \leq t \leq 1}$ denote a homotopy through nonsingular fields between $X_0 = X$ and $X_1 = Y$ on $S^1 \times D^2$, as in Lemma 3. We define

$$\bar{X}^n = X \oplus Z^{n-3} \tag{24}$$

on $S^1 \times D^{n-1}$ (obtained as in Lemma 4 on $S^1 \times D^3 \times D^{n-3}$ by rounding the corners). Then

$$\{\bar{X}_t^n \oplus Z^{n-3}\}_{0 \leq t \leq 1} \quad (25)$$

provides a homotopy of nonsingular vector fields on $S^1 \times D^{n-1}$ which are each transverse to $\partial(S^1 \times D^{n-1})$ by Lemma 4. It is easy to see that $\bar{X}_1^n = Y \oplus Z^{n-3}$ is a standard hyperbolic source vector field on $S^1 \times D^{n-1}$. Finally by Lemma 4 again, $\bar{X}^n = X \oplus Z^{n-3}$ can be C^1 -small perturbed to a field X^n which is NMS. The closed orbits of X^n will correspond to those of \bar{X}^n and may be assumed to have the same Poincaré maps. Then by definition of Z^{n-3} we have

$$J(X^n) = (-1)^{n-3} i_* J(X)$$

where $i_*: H_1(S^1 \times D^2) \xrightarrow{\cong} H_1(S^1 \times D^{n-1})$. Hence by Lemma 3, b), we are done.

Conclusion of proof of Theorem 1. In [3] we showed the existence of a NMS vector field V_0 in any desired homotopy class \mathcal{D} . As in [3], Lemma 5 (cf. Example D, Section 1 of this paper) we may homotope V_0 to another NMS field V'_0 obtained from V_0 by adding a cancelling pair of new round handles of index 0 and 1. By (19), $J(V_0) = J(V'_0) = \alpha_0 \in H_1(M)$, say. We may arrange that the S^1 direction of each of these new round handles represents the element $(-1)^n(\alpha - \alpha_0)$ in $H_1(M)$.

We finally define V as follows:

$$V(x) = \begin{cases} V'_0(x) & \text{if } x \text{ is outside the new round 0-handle} \\ X^n(x) & \text{if } x \text{ is inside the new round 0-handle.} \end{cases}$$

Here we are identifying $S^1 \times D^{n-1}$ of Lemma 5 with the new round 0-handle. We assume a smooth interpolation near a collar of $\partial(S^1 \times D^{n-1})$ if necessary, to fit V'_0 and X^n together smoothly. We also assume a C^1 -small perturbation if necessary to make V NMS (as in [3], Lemma 4).

Hence V is a NMS vector field in \mathcal{D} , and by choice of embedding of the new round handles and by Lemma 5, c), we have

$$\begin{aligned} J(V) &= J(V_0) + (-1)^{n-2}((-1)^n(\alpha - \alpha_0)) \\ &= \alpha_0 + \alpha - \alpha_0 = \alpha \quad \text{as desired.} \end{aligned}$$

Remarks. The above technique shows that in dimension 3, the conclusion of Theorem 2 holds when there exists a round handle decomposition.

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*School of Mathematics,
Institute for Advanced Study,
Princeton, N.J. 08540,
U.S.A.*

and

*School of Mathematics,
University of Minnesota,
Minneapolis, Minn. 55455
U.S.A.*

*Current Address: Department of Mathematics,
University of North Carolina,
Chapel Hill, N.C. 27514
U.S.A.*

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