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## Flaccidity of geometric index for nonsingular vector fields

DANIEL ASIMOV

### 1. Introduction

We consider nonsingular vector fields  $V$  on compact connected  $C^\infty$  orientable manifolds  $M^n$  without boundary. It is of interest to find criteria for detecting the occurrence of closed orbits of  $V$ . If we were to look instead at diffeomorphisms  $f: M \rightarrow M$  the analogous question would be to find the periodic points of  $f$ . We may at least count the periodic points algebraically via the Lefschetz fixed point formula, if  $\{x \mid f^k(x) = x\} = \text{Fix}(f^k)$  is finite:

$$\sum_{f^k(x)=x} i_{f^k}(x) \doteq \sum_{j=0}^n (-1)^j \text{tr}(f_{*j}^k: H_j(M; \mathbb{R}) \xrightarrow{\sim}). \quad (1)$$

Here  $i_{f^k}(x)$  is the local index of the fixed point  $x$  of  $f^k$ , and the right hand side is, by definition, the Lefschetz number  $\Lambda(f^k)$  of  $f^k$ .

We define here an index for nonsingular Morse–Smale (NMS) vector fields [2] as follows. Let  $V$  be an NMS field on  $M$ . Let  $C_i$   $1 \leq i \leq r$  be all the closed orbits of  $V$ . Let  $\bar{C}_i \in H_1(M; \mathbb{Z})$  be the homology class of  $C_i$  (oriented by  $V$ ) and let  $\varepsilon_i = \pm 1$  be the fixed point index of the Poincaré map induced by the flow of  $V$  around  $C_i$ . Then we define

$$J(V) = \sum_{i=1}^r \varepsilon_i \bar{C}_i \quad \text{in} \quad H_1(M; \mathbb{Z}) \quad (2)$$

to be the *geometric index of the NMS field*  $V$ .

Let  $\mathcal{V}$  denote all nonsingular vector fields on  $M$ , with the  $C^0$  topology. In Section 2 we describe some situations where  $J$  remains constant over a large open set or on distinct homotopic NMS vector fields. In particular we show that if  $V$  is the suspension of a Morse–Smale diffeomorphism, then  $J(V)$  is constant through large perturbations of  $V$ .

In Section 3, however, we show by example that if the dimension of  $M$  is greater than 3, then for each  $\alpha \in H_1(M; \mathbb{Z})$  and for each homotopy class  $\mathcal{D}$  of

nonsingular vector fields on  $M$ , there exists a  $V_\alpha \in \mathcal{D}$  such that  $V_\alpha$  is NMS and  $J(V_\alpha) = \alpha$ . Fuller's Theorem 1 [8] then imposes conditions on any homotopy between  $V_\alpha$  and  $V_\beta$ ,  $\alpha \neq \beta$ . We note that NMS vector fields are structurally stable [9].

## 2. Examples

A. Let  $p: E^n \rightarrow S^1$  be a smooth oriented fibre bundle with fibre the compact manifold  $F^{n-1}$ . Let  $\{V_s\}_{0 \leq s \leq 1}$  denote a homotopy of nonsingular vector fields on  $E$  which are never tangent to a fibre. Assume that  $V_0$  and  $V_1$  are NMS, with orientable stable and unstable manifolds. Then we have

PROPOSITION A.

$$J(V_0) = J(V_1). \quad (3)$$

*Proof.* Without loss of generality we may assume that for all  $(x, s) \in E \times I$  we have

$$Dp(V_s(x)) = 2\pi \cdot d/d\theta, \quad (4)$$

where  $Dp: TE \rightarrow TS^1$  is the tangent mapping of  $p$ . Let  $p_*: H_1 E \rightarrow H_1 S^1$  be the map on integer homology. Identify  $H_1 S^1$  with  $\mathbb{Z}$  by sending the counterclockwise generator to 1. Now set  $J_k(V_0) = \sum_{\bar{C}_i \in p_*^{-1}(k)} \varepsilon_i \bar{C}_i$ . Then

$$J(V_0) = \sum_{k=1}^{\infty} J_k(V_0) \quad (5)$$

and similarly

$$J(V_1) = \sum_{k=1}^{\infty} J_k(V_1). \quad (6)$$

Hence the proposition follows from showing  $J_k(V_0) = J_k(V_1)$  for all  $k = 1, 2, \dots$

We notice that by (4) the time-one map  $\varphi_s^1$  of the flow  $\{\varphi_s^t\}_{t \in \mathbb{R}}$  of  $V_s$  is a map  $\varphi_s^1: E \rightarrow E$  which preserves fibres. Similarly for  $\varphi_s^k$ ,  $k$  any integer. Let  $p^*(E)$  denote the space  $\{(x, y) \in E \times E \mid p(x) = p(y)\}$ , the total space of the pullback by  $p$

of the bundle  $E \xrightarrow{p} S^1$ . Then  $p^*(E)$  fibres over  $S^1$  with fibre  $F \times F$ , via  $(x, y) \mapsto p(x) = p(y) \in S^1$ . We define the subset  $\Delta \subset p^*(E)$  by  $\Delta = \{(x, x) \in p^*(E)\}$ . Also let  $\bar{p}: p^*(E) \rightarrow E$  be the projection given by  $\bar{p}(x, y) = x$ .

Now we may form the map  $\Gamma_s^1: E \rightarrow p^*(E)$  defined by  $\Gamma_s^1(x) = (x, \varphi_s^1(x))$ . This map  $\Gamma_s^1$  is transverse to  $\Delta \subset p^*(E)$  precisely when the map  $\varphi_s^1$  restricted to any fibre  $F$  has no eigenvalues equal to 1. In particular, this is the case for  $s = 0, 1$ . It is clear from the definitions that as a point set,  $\bar{p}(\Gamma_i^1(E) \cap \Delta)$  is the union of all the closed orbits of  $V_i$  which go around  $E$  once, i.e., those closed orbits comprising  $J_1(V_i)$ , for  $i = 0, 1$ .

Now considering homology intersection  $\cdot$ , we let  $[\Delta] \in H_n(p^*(E))$  denote the fundamental class of the diagonal, i.e., the image of  $[E] \in H_n(E)$  under  $x \mapsto (x, x)$ . Let  $\Gamma_{s*}^1: H_n(E) \rightarrow H_n(p^*(E))$  and  $\bar{p}_*: H_1(p^*(E)) \rightarrow H_1(E)$  denote the induced maps on homology. Then

$$\Lambda_s^1 = \bar{p}_*(\Gamma_{s*}^1([E]) \cdot [\Delta]) \quad \text{in } H_1(E) \quad (7)$$

is an integer homology class independent of  $s$ , and so

$$\Lambda_0^1 = \Lambda_1^1. \quad (8)$$

Let  $f_i: F \rightarrow F$  denote  $\varphi_i^1|F$  for the fibre  $F = p^{-1}(x)$  of  $E$ . Let  $y \in F$  be the fixed point of  $f_i$  corresponding to the closed orbit  $C_y$  of  $V_i$ . Then  $C_y$  contributes  $\varepsilon_y \bar{C}_y \in H_1(E)$  to the class  $\Lambda_i^1$ , where

$$\varepsilon_y = \det \begin{pmatrix} I_{n-1} & I_{n-1} \\ \hline Df(y) & I_{n-1} \end{pmatrix}$$

i.e.,

$$\begin{aligned} \varepsilon_y &= \det(I_{n-1} - Df(y)) \\ &= i_{f_i}(y), \end{aligned} \quad (9)$$

the local fixed point index at  $y$  (since 1 is not an eigenvalue of  $Df(y)$ ). Thus we have shown that

$$\Lambda_i^1 = J_1(V_i) \quad i = 0, 1 \quad (10)$$

and so by (8) this shows  $J_1(V_0) = J_1(V_1)$ .

Now using the fact that we assumed all stable and unstable manifolds to be *orientable* for  $V_i$ ,  $i = 0, 1$ , we may easily check that for  $i = 0, 1$ , we have

$$\pi_k * J_1(\tilde{V}_i) = k \sum_{d|k} J_d(V_i) \in H_1(E). \quad (11)$$

Here  $\pi_k : \tilde{E}_k \rightarrow E$  denotes the canonical  $k$ -fold covering space over  $E$  (since  $E$  is a fibre bundle over  $S^1$ ) and  $\tilde{V}_i$  denotes the unique lift of  $V_i$  to a nonsingular vector field on  $\tilde{E}_k$ .

Hence  $J_k(V_i)$  is expressible in terms of  $\pi_k * J_1(\tilde{V}_i)$  and  $J_d(V_i)$  for  $1 \leq d < k$ . Now  $J_1(\tilde{V}_i)$  is independent of  $i = 0, 1$  by the considerations we applied above to  $J_1(V_i)$  (since the homotopy  $V_s$  gives rise to a homotopy  $\tilde{V}_s$ ,  $0 \leq s \leq 1$ ). Hence we also have

$$\pi_k * J_1(\tilde{V}_0) = \pi_k * J_1(\tilde{V}_1) \quad (12)$$

and by induction, therefore,

$$J_k(V_0) = J_k(V_1), \quad k \geq 1. \quad (13)$$

Hence summing over  $k$  we obtain

$$J(V_0) = J(V_1), \quad \text{as desired.} \quad (14)$$

*Remarks.* 1. Proposition A could have also been obtained using Fuller's Theorem 1 of [8].

2. The assumption of orientable stable manifolds insures that  $L_{f^k}(y)$  is independent of  $k$ . If we omitted this assumption the theorem would be false. For example, the  $180^\circ$  rotation of the 2-sphere  $S^2$  may be perturbed to a Morse-Smale (M.-S.) diffeomorphism  $f$  having 2 fixed orientation-reversing saddles (of index 1) at the poles, and four alternating sources and sinks of period 2 each (and index 1) along the equator (see Figure 1). Let  $f_s$  be a homotopy of  $f = f_0$  to the gradient M.-S. diffeomorphism  $f_1$  given by  $z \mapsto z^2$  for  $z \in S^2$ . Then if  $V_s$  are the corresponding "suspension" vector fields on  $S^1 \times S^2$ , we have

$$J(V_0) = 6 \quad \text{but} \quad J(V_1) = 2,$$

where we have identified  $H_1(S^1 \times S^2)$  canonically with  $\mathbb{Z}$ .

*Example B.* For a second example, consider the 2-torus  $T^2$ . It follows from [8] that if  $V_0, V_1$  are two NMS vector fields on  $T^2$ , then they are homotopic through nonsingular vector fields if and only if  $J(V_0) = J(V_1)$ .

*Example C.* We consider the case of a vector field  $V$  tangent to the fibres of a principal circle bundle  $E^n$  over a compact manifold  $M^{n-1}$ . Let  $G$  denote a small gradient Morse-Smale vector field on  $M$  (not necessarily nonsingular). A choice of connection on  $E$  enables us to lift  $G$  to a unique  $S^1$ -invariant vector field  $\tilde{G}$  on  $E$ .

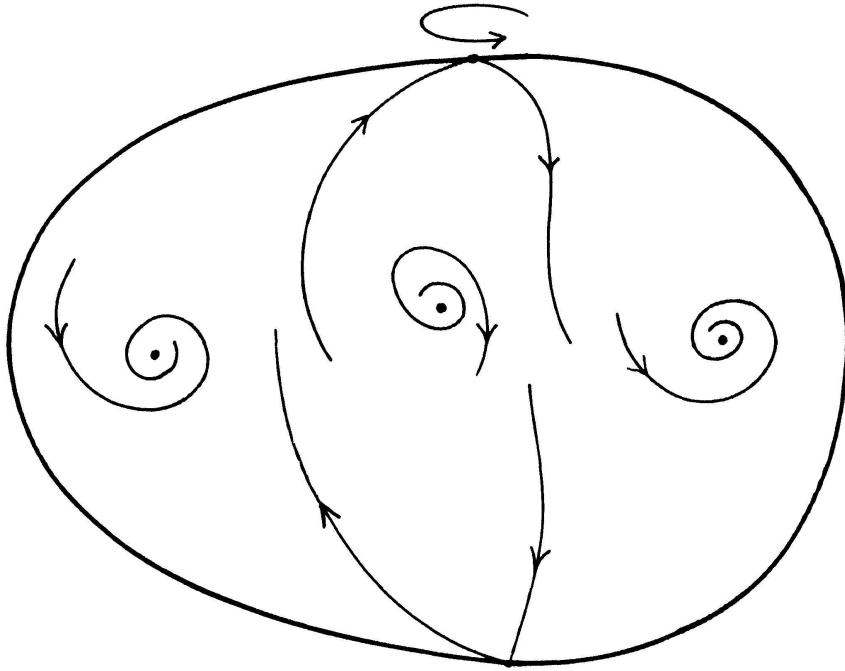


Figure 1.

It is easy to verify that  $V' = V + \tilde{G}$  is a NMS vector field on  $E$ . Furthermore, considering the cases  $n$  odd and  $n$  even separately, we may check that in fact

$$J(V') = \chi(M) \cdot \bar{F} \quad \text{in} \quad H_1(E), \quad (15)$$

where  $\bar{F}$  is the homology class of the oriented fibre  $F \approx S^1$ .

By picking topologically distinct gradient fields  $G$  one may create countably many topologically distinct NMS fields on  $E$  all homotopic to one another and all having the same geometric index.

*Example D.* Let  $M^n$  have the round handle decomposition

$$M \approx Q_1 + Q_2 + \cdots + Q_r, \quad (16)$$

where the  $Q_i$ ,  $1 \leq i \leq r$ , are round handles of various indices attached successively (see [2], [3]), and we assume  $n \geq 4$ . We may assume  $M$  carries a NMS vector field  $V$  compatible with the decomposition.

Then there is an operation which creates a new round handle decomposition and a corresponding compatible vector field  $V'$ . We introduce a cancelling pair of round handles in between  $Q_i$  and  $Q_{i+1}$ :

$$M \approx Q_1 + \cdots + Q_i + R^j + R^{j+1} + Q_{i+1} + \cdots + Q_r. \quad (17)$$

Here  $R^j \approx S^1 \times D^j \times D^{n-j-1}$  and  $R^{j+1} \approx S^1 \times D^{j+1} \times D^{n-j-2}$ . The union  $R^j \cup R^{j+1} \approx S^1 \times (h^j \cup h^{j+1})$  where  $h^j + h^{j+1}$  represents a cancelling pair of ordinary handles of dimension  $n-1$ .  $R^j + R^{j+1}$  is attached in such a way that

$$Q_1 + \cdots + Q_i + R^j + R^{j+1} \approx Q_1 + \cdots + Q_i. \quad (18)$$

On each of  $R^j$ ,  $R^{j+1}$  we define a nonsingular vector field that is essentially the neighborhood of a single hyperbolic closed orbit whose Poincaré map has  $j$  or  $j+1$  contracting dimensions, respectively. As in [2], this defines a new NMS vector field, say  $V'$ , which is homotopic to  $V$  and such that also

$$J(V') = J(V), \quad (19)$$

since we have added two closed orbits which are homologous but whose Poincaré maps have fixed point indices of opposite sign, namely  $(-1)^{n-1-j}$  and  $(-1)^{n-2-j}$ , respectively.

### 3.

**THEOREM 1.** *We assume  $n = \dim M \geq 4$ . Let  $\alpha \in H_1(M; \mathbb{Z})$  be arbitrary, and let  $\mathcal{D}$  be any homotopy class of nonsingular vector fields on  $M$ . Then there is a NMS vector field  $V \in \mathcal{D}$  such that  $J(V) = \alpha$ . It follows from structural stability that there is in fact an entire  $C^1$  neighborhood  $N(V)$  such that*

$$V' \in N(V) \Rightarrow J(V') = \alpha.$$

*Proof.* We begin by constructing an example of a certain NMS vector field  $X_1$  on the solid torus  $B = S^1 \times D^2$ . We construct  $X_1$  by a round handle decomposition  $B = R^0 + R^1$ . As in Section 5 of [2],  $R^i$ ,  $i = 0, 1$  is supplied with a nonsingular vector field having exactly one closed orbit, whose Poincaré map is hyperbolic with  $i$  contracting dimensions. In coordinates the vector field on  $R^0 = S^1 \times D^2$  is given by  $d/d\theta + x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2$ , and the vector field on  $R^1 = S^1 \times D^1 \times D^1$  is given by  $d/d\varphi - y_1 \partial/\partial y_1 + y_2 \partial/\partial y_2$ . The attaching region  $\partial_-(R^1)$  is defined by  $\{(\varphi, y_1, y_2) \in R^1 \mid |y_1| = 1\}$  and is the disjoint union  $S^1 \times S^0 \times D^1$  of two annuli  $A_{-1} = S^1 \times \{-1\} \times D^1$  and  $A_1 = S^1 \times \{1\} \times D^1$ .

We must specify, up to isotopy, the attaching map

$$h : S^1 \times S^0 \times D^1 \rightarrow \partial(R^0)$$

(where  $\partial(R^0) = \partial(S^1 \times D^2)$  is a 2-torus). Any  $h$  satisfying the following conditions will suffice for our purposes:

- a) The composition  $A_{-1} \xrightarrow{h|_{A_{-1}}} \partial(R^0) \subseteq R^0$  induces an isomorphism  $H_1(A_{-1}) \rightarrow H_1(R^0)$ .
- b)  $h(A_1)$  deforms to a point in  $\partial(R^0)$ .
- c)  $A_{-1}$  and  $A_1$  are embedded in  $\partial(R^0)$  with opposite orientations.

LEMMA 1. *If  $h: \partial_+(R^1) \rightarrow \partial(R^0)$  satisfies a), b), c) above, then the quotient space  $R^0 + R^1 = R^0 \cup R^1 / x \sim h(x)$  is diffeomorphic to a solid torus.*

*Proof.* By the isotopy extension lemma [6] an isotopy of an embedding of the core circle of  $A_i$ ,  $i = \pm 1$  extends to an isotopy of all of  $\partial_-(R^1)$ . Using this it is straightforward to show that if  $h$  satisfies a), b), c) then up to isotopy  $h(A_{-1})$  and  $h(A_1)$  are as depicted in Figure 2 (after embedding  $R^0$  appropriately in  $\mathbb{R}^3$ ).

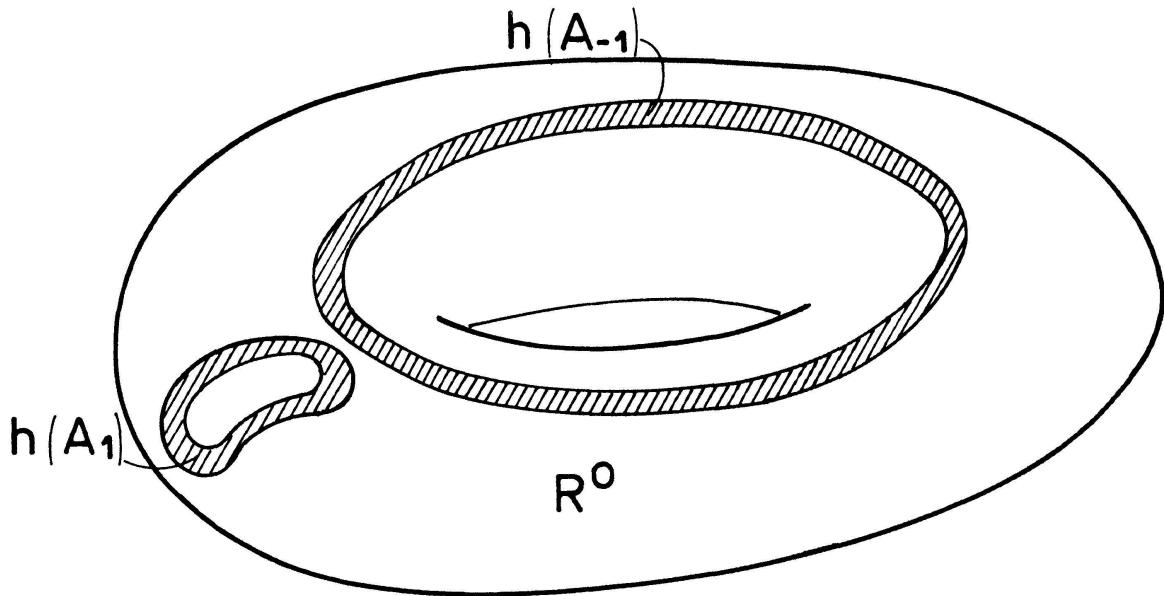


Figure 2.

Hence  $R^0 + R^1$  will resemble Figure 3.

Then performing the isotopy indicated in Figure 4a) to d) shows the lemma.

Now  $X_1$  is defined as the NMS vector field on  $B = S^1 \times D^2$  induced from the round handle decomposition of  $B$  described above. Let us assume  $X_1$  is normal to  $\partial B$ .

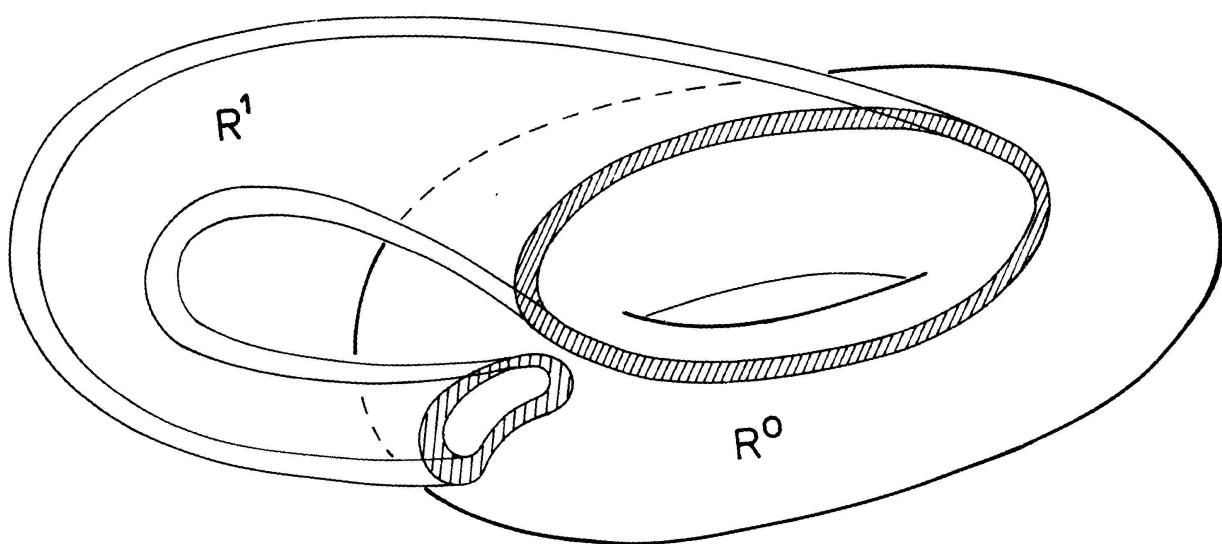
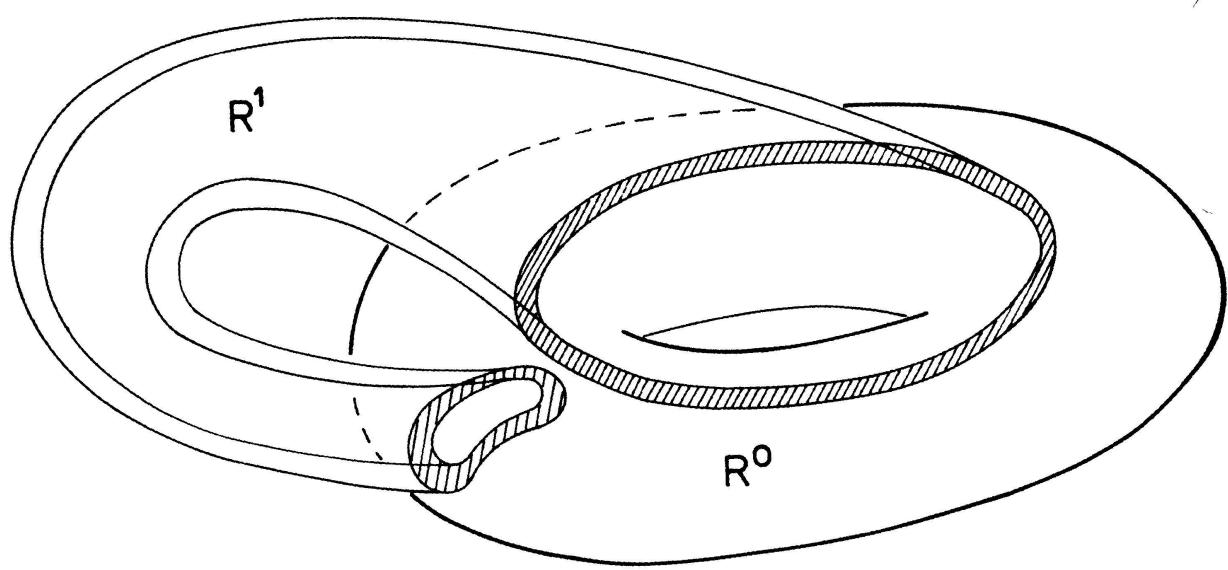
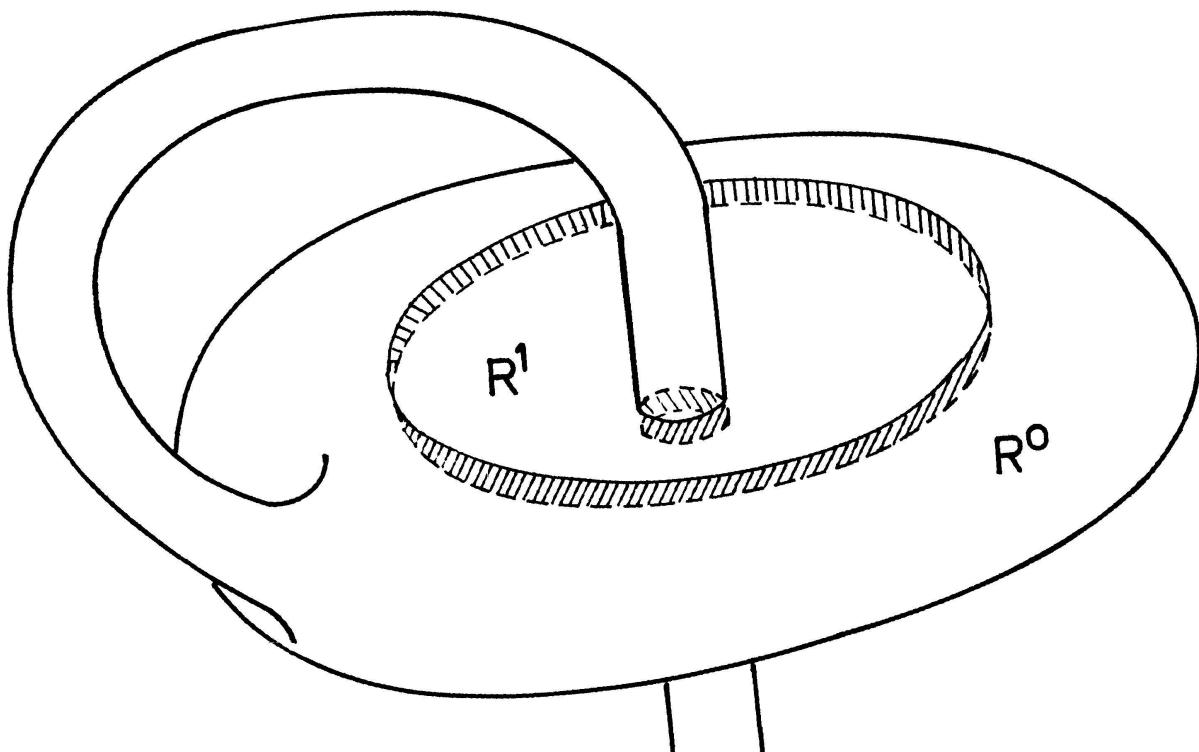


Figure 3.

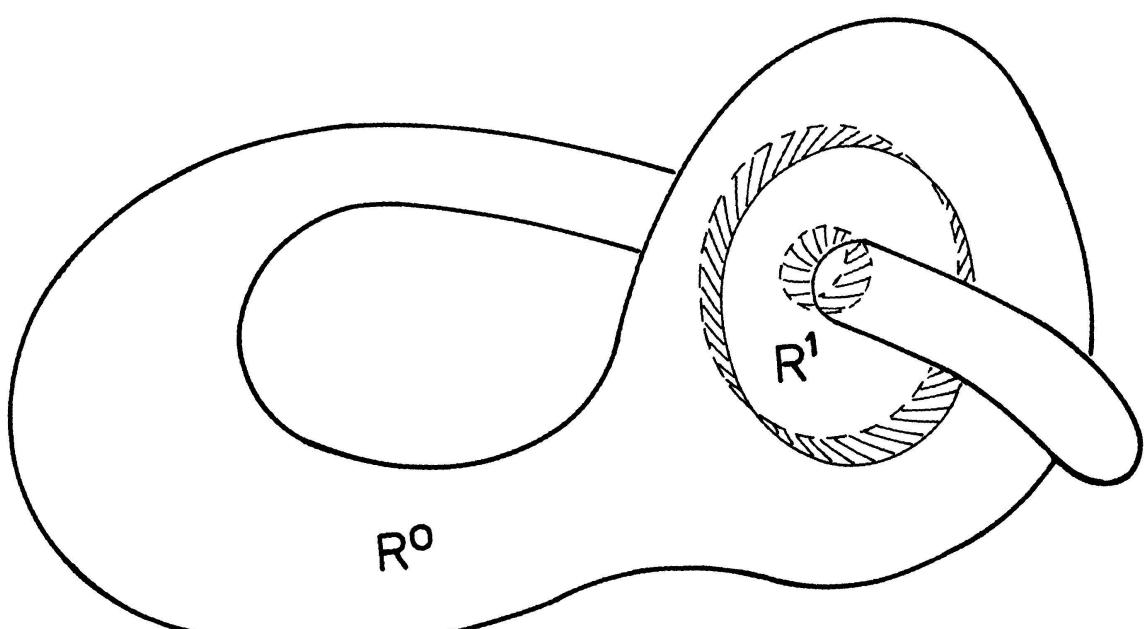


a)

Figure 4.



b)



c)

Figure 4 (cont'd).

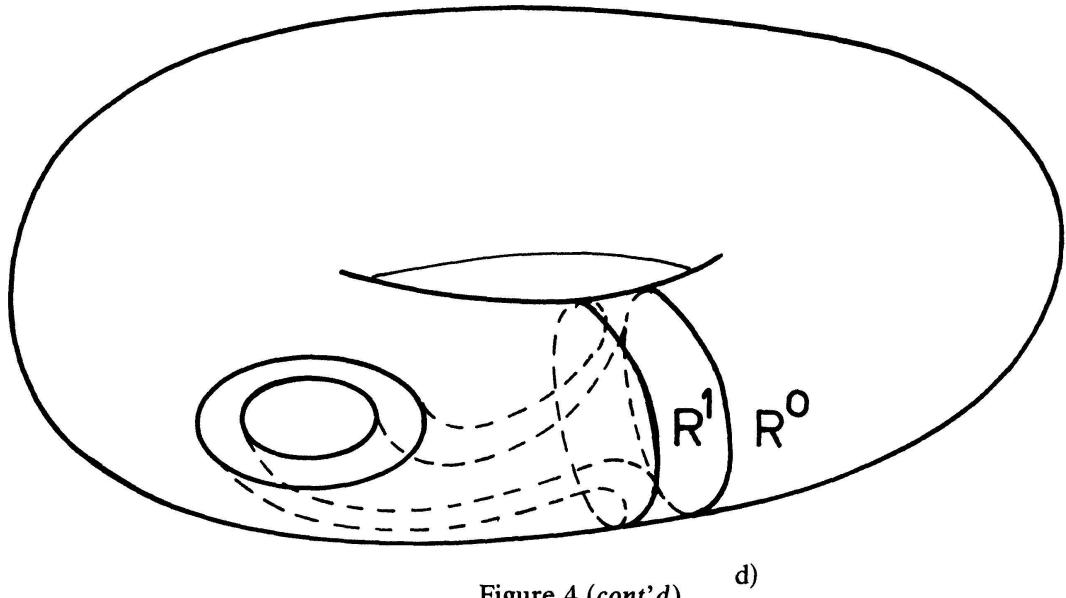


Figure 4 (cont'd). d)

LEMMA 2. *The first obstruction between  $X_1$  and  $V_0$  standard source vector field  $Y$  on  $B$  vanishes in*

$$H^2(B, \partial B; \pi_2(S^2)) \approx H_1(B; \mathbb{Z}) \approx \mathbb{Z}.$$

*Proof.* Let  $\gamma = d_1(X_1, Y) \in H^2(B, \partial B; \pi_2(S^2))$ . Then  $\gamma$  is detected by its Kronecker product  $\langle \gamma, w \rangle$  with the generator  $w$  of  $H_2(B, \partial B; \mathbb{Z}) \approx \mathbb{Z}$ . Now  $w$  is represented by  $\{x\} \times D^2 \subset S^1 \times D^2 = B$ . Then  $\langle \beta, w \rangle = \langle d_1(X_1, Y), w \rangle$  can be identified with an element  $c \in \pi_2(S^2) \approx \mathbb{Z}$  obtained as follows. Let  $S^2$  be the union of two copies of  $\{x\} \times D^2$  identified on their boundaries. Then choosing any framing for  $T(B)$  restricted to  $\{x\} \times D^2$ , each of the nonsingular vector fields  $X_1$  and  $Y$ , once they have been normalized to unit length, defines a map  $\{x\} \times D^2 \rightarrow S^2$ . These two maps agree on  $\{x\} \times \partial D^2$  and hence induce a map  $S^2 \rightarrow S^2$  which is well defined up to homotopy, and its homotopy class is  $c$ .

Now we notice that a circle which surrounds  $A_1$  (cf. Figure 2) bounds a 2-disc  $D$  unique up to homotopy rel  $\partial D$ , embedded in  $R^0$  so that  $D \cap \partial R^0 = \partial D \cap \partial R^0$  (see Figure 5). This disc  $D$  will correspond under the isotopy shown in Figure 4 to a representative of  $c$ . On  $D$  it is easy to see by inspection that up to homotopy both  $X_1$  and  $Y$  can be represented in continuous coordinates  $x_1, x_2$  along  $D$ , and  $x_3$  normal to  $D$ , via

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \sqrt{1 - x_1^2 - x_2^2} \frac{\partial}{\partial x_3}.$$

Thus  $c = 0$  in  $\pi_2(S^2)$  and the lemma is proved.

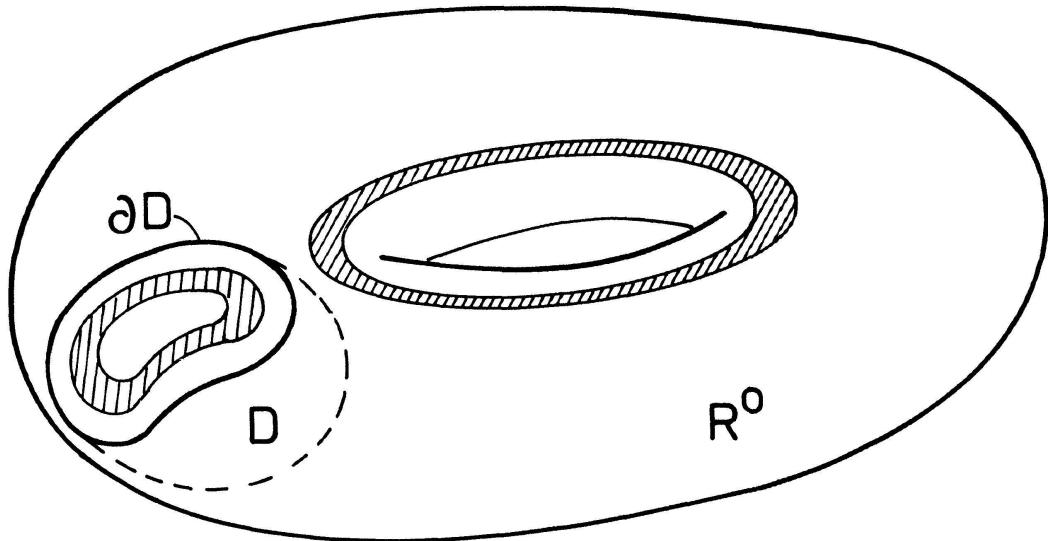


Figure 5.

LEMMA 3. *There is a NMS vector field  $X$  on  $B$ , exiting on  $\partial B$ , such that*

- a)  $X$  is homotopic to the source field  $Y$  rel  $(\partial B)$ , and
- b)  $J(X) = 0$ ; in fact each closed orbit of  $X$  bounds a disc in  $B$ .

*Remark.* Since NMS vector fields are structurally stable [9],  $X$  shows that Fuller's example [7] is not just a "pathological" phenomenon.

*Proof of Lemma 3.* We know from the work of Hopf and Boltyanskii [5] that if the first obstruction between two vector fields vanishes, then they are homotopic if the second obstruction vanishes (as computed using homotopic fields which agree on the codimension-one skeleton). Since  $X_1$  and  $Y$  agree on  $\{x\} \times D^2 \subset S^1 \times D^2 = B$ , an arbitrarily small perturbation will cause them to actually agree on  $[x, y] \times D^2$ . Hence they agree on the 2-skeleton of a certain cell decomposition of  $B$ , namely on  $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$ . (We continue to denote the perturbed fields by  $X_1$  and  $Y$ .)

Now we observe that the second obstruction  $d_2(X_1, Y) = m\eta$ , some positive multiple of a generator  $\eta$  of  $H^3(B, \partial B; \pi_3(S^2)) \approx \mathbb{Z}$ .

Let  $C_i$ ,  $1 \leq i \leq m$  be disjoint nullhomotopic circles embedded in  $\partial B$ . In a small tubular neighborhood of each  $C_i$  we attach a cancelling pair of round handles  $\bar{R}_i^0 + \bar{R}_i^1$  (see Figure 6).

We supply  $\bar{R}_i^0$  and  $\bar{R}_i^1$  each with a single hyperbolic closed orbit whose Poincaré map has, respectively, 0 or 1 contracting dimension. We also require that the two closed orbits thus created have opposite sense (i.e., represent both generators of  $H_1(\bar{R}_i^0 + \bar{R}_i^1) \approx H_1(S^1 \times D^2) \approx \mathbb{Z}$ ). This resulting vector field is  $X$  (NMS after a small  $C^1$  perturbation by [1]).

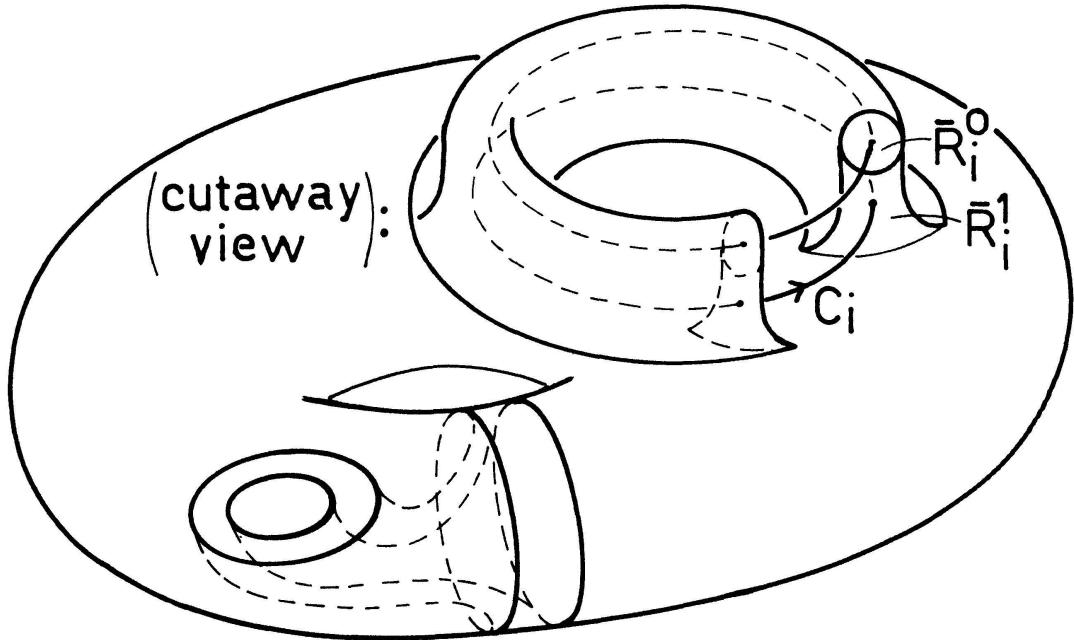


Figure 6.

Let us consider the effect of adding only *one* such cancelling pair  $\bar{R}^0 + \bar{R}^1$  to  $B = R^0 + R^1$ . We extend the vector field on  $B + \bar{R}^0 + \bar{R}^1$  to a collar on  $\partial B$  just as in [3], and we call this field by the name  $X'$ . We also extend  $X_1$  to the collar  $(\partial B) \times I$ , letting it be the vertical vector field  $\partial/\partial t$  there. We may now identify  $B \cup (\partial B) \times I$  with  $B$  by the usual isotopic deformation down the collar.

This gives us two vector fields on  $B$ , which we still call  $X_1$  and  $X'$ . The construction above, done carefully, will result in their agreeing on the 2-skeleton  $\{x\} \times D^2 \cup \{y\} \times D^2 \cup \partial B$  of  $B$ . To compute  $d_2(X_1, X') \in H^3(B, \partial B; \pi_3(S^2))$  we use the same argument as in Lemmas 10 and 11 of [3], only noticing that the framed cobordism class of  $Q_0^{-1}(\mathcal{G})$  is indeed the generator of  $\Phi_1$ , the framed cobordism group of 1-dimensional framed submanifolds in  $D^3$ . This follows immediately from [4]. Hence  $d_2(X_1, X')$  generates  $H^3(B, \partial B; \pi_3(S^2))$ , so is  $\pm\eta$ . By changing the sense of both closed orbits in  $\bar{R}^0 + \bar{R}^1$  if necessary, we may assume

$$d_2(X_1, X') = \eta. \quad (20)$$

Now we add all  $m$  cancelling pairs  $\bar{R}_i^0 + \bar{R}_i^1$  to  $B = R^0 + R^1$ , (with the proper sense to the closed orbits of the corresponding vector fields). Just as above we identify the resulting space  $B + R_1^0 + R_1^1 + \cdots + R_m^0 + R_m^1$  with  $B$  itself. Then applying induction to the argument of the last paragraph we obtain

$$d_2(X_1, X) = m\eta. \quad (21)$$

Thus by the addition formula for difference cocycles, we have

$$\begin{aligned}
 d_2(X, Y) &= d_2(X, X_1) + d_2(X_1, Y) \\
 &= -d_2(X_1, X) + d_2(X_1, Y) \\
 &= -m\eta + m\eta = 0.
 \end{aligned} \tag{22}$$

Hence by [5],  $X$  is homotopic to  $Y$  rel  $(\partial B)$ . By construction, each of the  $2+2m=2(m+1)$  closed orbits of  $X$  bounds a disc in  $B$ . Hence  $J(X)=0$ .

LEMMA 4. *Let  $W$  be an NMS vector field on the compact manifold  $P$ . In case  $\partial P \neq \emptyset$  we assume  $W$  to exit on  $\partial P$ . Let  $Z^k$  denote a simple hyperbolic source vector field on  $D^k$ , such as the positive vector field*

$$Z^k(x_1, \dots, x_n) = \sum_{i=1}^k x_i \frac{\partial}{\partial x_i} \tag{23}$$

*Then the direct sum vector field  $W \oplus Z^k$  is exiting on  $\partial(P \times D^k)$  (after appropriate smoothing of the corner  $\partial P \times \partial D^k$ ) and by a  $C^1$ -small perturbation  $W \oplus Z^k$  can be made NMS.*

*Proof.* It is straightforward to verify that the nonwandering set of  $W \oplus Z^k$  is a finite union of hyperbolic closed orbits. Transversal intersection of stable and unstable manifolds is then obtainable by a  $C^1$ -small perturbation that preserves the truth of the previous sentence [1]. To see that  $W \oplus Z^k$  exits on  $\partial(P \times D^k)$  it is necessary to choose a “convex” straightening of the angle [6] of the product of two half-spaces. We use charts where  $W$  and  $Z^k$  are unit normal fields to the boundaries and the result follows.

LEMMA 5. *Let  $n \geq 3$ . There is a NMS vector field  $X^n$  on the solid torus  $S^1 \times D^{n-1}$  such that*

- a)  $X^n$  exits on  $S^1 \times \partial D^{n-1} = \partial(S^1 \times D^{n-1})$ .
- b)  $X^n$  is homotopic to a hyperbolic source NMS vector field on  $S^1 \times D^{n-1}$ , through nonsingular vector fields which remain everywhere transverse to  $\partial(S^1 \times D^{n-1})$ .
- c)  $J(X^n) = 0$ .

*Proof.* Let  $\{X_t\}_{0 \leq t \leq 1}$  denote a homotopy through nonsingular fields between  $X_0 = X$  and  $X_1 = Y$  on  $S^1 \times D^2$ , as in Lemma 3. We define

$$\bar{X}^n = X \oplus Z^{n-3} \tag{24}$$

on  $S^1 \times D^{n-1}$  (obtained as in Lemma 4 on  $S^1 \times D^3 \times D^{n-3}$  by rounding the corners). Then

$$\{\bar{X}_t^n \oplus Z^{n-3}\}_{0 \leq t \leq 1} \quad (25)$$

provides a homotopy of nonsingular vector fields on  $S^1 \times D^{n-1}$  which are each transverse to  $\partial(S^1 \times D^{n-1})$  by Lemma 4. It is easy to see that  $\bar{X}_1^n = Y \oplus Z^{n-3}$  is a standard hyperbolic source vector field on  $S^1 \times D^{n-1}$ . Finally by Lemma 4 again,  $\bar{X}^n = X \oplus Z^{n-3}$  can be  $C^1$ -small perturbed to a field  $X^n$  which is NMS. The closed orbits of  $X^n$  will correspond to those of  $\bar{X}^n$  and may be assumed to have the same Poincaré maps. Then by definition of  $Z^{n-3}$  we have

$$J(X^n) = (-1)^{n-3} i_* J(X)$$

where  $i_*: H_1(S^1 \times D^2) \xrightarrow{\sim} H_1(S^1 \times D^{n-1})$ . Hence by Lemma 3, b), we are done.

*Conclusion of proof of Theorem 1.* In [3] we showed the existence of a NMS vector field  $V_0$  in any desired homotopy class  $\mathcal{D}$ . As in [3], Lemma 5 (cf. Example D, Section 1 of this paper) we may homotope  $V_0$  to another NMS field  $V'_0$  obtained from  $V_0$  by adding a cancelling pair of new round handles of index 0 and 1. By (19),  $J(V_0) = J(V'_0) = \alpha_0 \in H_1(M)$ , say. We may arrange that the  $S^1$  direction of each of these new round handles represents the element  $(-1)^n(\alpha - \alpha_0)$  in  $H_1(M)$ .

We finally define  $V$  as follows:

$$V(x) = \begin{cases} V'_0(x) & \text{if } x \text{ is outside the new round 0-handle} \\ X^n(x) & \text{if } x \text{ is inside the new round 0-handle.} \end{cases}$$

Here we are identifying  $S^1 \times D^{n-1}$  of Lemma 5 with the new round 0-handle. We assume a smooth interpolation near a collar of  $\partial(S^1 \times D^{n-1})$  if necessary, to fit  $V'_0$  and  $X^n$  together smoothly. We also assume a  $C^1$ -small perturbation if necessary to make  $V$  NMS (as in [3], Lemma 4).

Hence  $V$  is a NMS vector field in  $\mathcal{D}$ , and by choice of embedding of the new round handles and by Lemma 5, c), we have

$$\begin{aligned} J(V) &= J(V'_0) + (-1)^{n-2}((-1)^n(\alpha - \alpha_0)) \\ &= \alpha_0 + \alpha - \alpha_0 = \alpha \quad \text{as desired.} \end{aligned}$$

*Remarks.* The above technique shows that in dimension 3, the conclusion of Theorem 2 holds when there exists a round handle decomposition.

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