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Autor: Chandrasekharan, K.

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Exponential sums associated with the Dedekind zeta-function

by K. CHANDRASEKHARAN AND RAGHAVAN NARASIMHAN

(To Carl Ludwig Siegel for his eightieth birthday)

§1. Introduction

Let K be an algebraic number field of degree n, and let $\zeta_K(s)$ be the Dedekind zeta-function associated with it. For Re $s>1$, $\zeta_K(s)=\sum_{k=1}^{\infty} a_k k^{-s}$, where a_k denotes the number of integral ideals in K with norm k. The function $\zeta_K(s)$ is meromorphic in the complex s-plane with a simple pole at $s=1$. If r_1 denotes the number of real conjugates of K, and $2r_2$ the number of imaginary conjugates, and D the discriminant, $\zeta_K(s)$ satisfies the functional equation $\xi(s)=\xi(1-s)$, where

$$\xi(s)=\Gamma^{r_1}(\tfrac{1}{2}s)\Gamma^{r_2}(s)B^{-s}\zeta_K(s),$$

with $B=2^{r_2}\pi^{n/2}(|D|)^{-1/2}$, $r_1+2r_2=n$. It is known that $a_k=O(k^{\varepsilon})$, for every $\varepsilon>0$ [2, p. 55], while

$$\sum_{k\leq x} a_k=\lambda x+O(x^{(n-1)/(n+1)}),$$

where λ is a positive constant determined by the field K [5, §2.6].

Our object is to prove the following

THEOREM. *If* $n\geq 3$, η *real,* $\eta\neq 0$, *then*

$$\sum_{k\leq x} a_k \exp(2\pi i k^{1/(n-1)}\eta)=O(x^{1-(1/2(n-1))}\log x),$$

for $x\geq 2$, *the 'O' depending on* η.

The case $n=2$ was considered by us in two previous papers [3, 4]. In [3] we showed that there was a connexion between the order of magnitude of the corresponding sums and the existence of an infinity of zeros of the associated zeta-function on the critical line. In [4] we proved an approximate reciprocity formula for the sum $\sum_{k\leqslant x} a_k \exp(2\pi ik\eta)$. For $n \geqslant 3$ we begin in the same way as before, and relate the sum $\sum_{k\leqslant x} a_k \exp(2\pi ik\eta)$ to the sum

$$\sum_{k\leqslant c_0 x^{n-1}} k^{(2-n)/2(n-1)} a_k \exp(2\pi ik^{1/(n-1)} \cdot \eta^{-1/(n-1)} \cdot H),$$

where H is a constant. The difference between the two sums can be expressed as a sum of terms for each of which we determine the asymptotic behaviour, which is different in different ranges of k. The principal terms in the asymptotic expansions give the required result with x^{θ}, for any positive θ, instead of $\log x$. Replacing the asymptotic expansion, in some places, by a direct estimate, which is slightly more sophisticated, we get the stated theorem.

§2. Preliminaries

It is known that the functional equation for $\zeta_K(s)$ implies for $\rho \geqslant 0$, ρ integral, the identity [1, (4.23)]

$$\frac{1}{\Gamma(\rho+1)} \sum_{\lambda_k \leqslant x}{}' a_k (x-\lambda_k)^{\rho} = Q_\rho(x) + \sum_{k=1}^{\infty} a_k \lambda_k^{-1-\rho} I_\rho(\lambda_k x), \tag{2.1}$$

for $x>0$, provided that $\rho > \frac{1}{2}(n-1)$. The dash in the summation on the left-hand side of (2.1) indicates that if $\rho=0$, and $x=\lambda_k$, the last term should be halved. Here $\lambda_k = B \cdot k$, B being defined as in §1, and

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{B^{-s} \zeta_K(s) \Gamma(s)}{\Gamma(s+\rho+1)} x^{s+\rho}\, ds,$$

where $\mathscr{C}$ is a curve which encloses all the singularities of the integrand. Clearly

$$Q_\rho(x) = cx^{1+\rho} + \sum_{j=0}^{\rho} c_j x^{\rho-j},$$

where c equals the residue of $\zeta_K(s)$ at $s=1$ multiplied by $\{B\Gamma(\rho+2)\}^{-1}$. The function I_ρ in (2.1) is defined, for $x>0$, by

$$I_\rho(x) = \frac{1}{2\pi i} \int_{\mathscr{C}'_\rho} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(\rho+2-s)\Delta(1-s)} x^{1+\rho-s}\, ds, \qquad \Delta(s) = \Gamma^{r_1}(\tfrac{1}{2}s)\Gamma^{r_2}(s), \tag{2.2}$$

where $\mathscr{C}'_\rho$ denotes the path of integration extending from $c_\rho - i\infty$ to $c_\rho - iR$, thence to $c_\rho + r - iR$, $c_\rho + r + iR$, $c_\rho + iR$, and $c_\rho + i\infty$, with r and R chosen suitably large, and with $c_\rho = \frac{1}{2} + (\rho/n) - \varepsilon$, $0 < \varepsilon < 1/2n$.

The following asymptotic formula plays a key rôle here:

$$I_\rho(x) = \sum_{\nu=0}^{m} e'_\nu x^{\omega_\rho - (\nu/n)} \cos(hx^{1/n} + \pi_\nu) + O(x^{\omega_\rho - (m+1/n)}), \tag{2.3}$$

where $\omega_\rho = \frac{1}{2} - (1/2n) + \rho(1-(1/n))$, $h = n2^{(n-r_2)/n}$, $\pi_\nu = \pi_\nu(\rho) = -\pi((n/2) + (\rho/2) + \frac{1}{4}(r_1+3) - (\nu/2))$. It was proved in [2, Lemma 1] for $\rho \geqslant 0$. But the formula and the proof are, in fact, valid for all real ρ.

Since $I_0(x)$ is continuous at $x = 0$, with the value $I_0(0) = \Delta(1)/\Delta(0)$,

$$\int_0^\varepsilon I'_0(x)\,dx \to 0 \quad \text{as} \quad \varepsilon \to 0+, \tag{2.4}$$

where I'_0 stands for the derivative of I_0. Clearly $I'_0(x) = I_{-1}(x)$, for $x > 0$.

§3. Some basic lemmas

We shall obtain an expression for the sum $\sum_{k \leqslant x} a_k \exp(2\pi i \lambda_k \eta)$, with an arbitrary $\eta > 0$, and $\lambda_k = B \cdot k$, as a sum of three sums of integrals, and then estimate those integrals separately.

Let $a_0 = 0$, $\lambda_0 = 0$, and

$$A(x) = \sum_{\lambda_k \leqslant x}' a_k, \qquad A(x) = 0, \quad \text{for} \quad 0 \leqslant x < \lambda_1. \tag{3.1}$$

Let $f(t) = \exp(2\pi i \eta t)$, $\eta > 0$, $t \geqslant 0$. Let β be a smoothing function defined as follows: $\beta(t) \in C^\infty(-\infty, \infty)$, $\beta(t) = 1$ in a neighbourhood of $t \leqslant \lambda_N$ (where N is a fixed positive integer), $\beta(t) = 0$ in a neighbourhood of $t \geqslant \lambda_{N+1}$, and $0 \leqslant \beta(t) \leqslant 1$ everywhere. Since $\lambda_{N+1} - \lambda_N$ is bounded below by a positive constant, β can be so chosen that all its derivatives are bounded in $(-\infty, \infty)$. We then have

$$\sum_{k=0}^{\infty} a_k f(\lambda_k)\beta(\lambda_k) = \int_0^\infty f(t)\beta(t)\,dA(t). \tag{3.2}$$

Integrating this by parts r times, where r is an integer so large that the infinite series in (2.1) converges absolutely, and uniformly, for $x > 0$, and $\rho \geqslant r > 0$, we obtain

$$\int_0^\infty f(t)\beta(t)\,dA(t)=(-1)^r\int_0^\infty A^r(t)\cdot(f(t)\beta(t))^{(r+1)}\,dt, \tag{3.3}$$

where

$$A^r(t)=\frac{1}{\Gamma(r+1)}\sum_{\lambda_k\le t}a_k(t-\lambda_k)^r.$$

Writing $F(t)=f(t)\beta(t)$, we have, from (2.1),

$$(-1)^r\int_0^\infty A^r(t)\cdot F^{(r+1)}(t)\,dt=(-1)^r\int_0^{\lambda_{N+1}}F^{(r+1)}(t)\cdot Q_r(t)\,dt$$
$$+(-1)^r\int_0^{\lambda_{N+1}}\left(\sum_{k=1}^\infty a_k\lambda_k^{-1-r}I_r(\lambda_k t)\right)\cdot F^{(r+1)}(t)\,dt. \tag{3.4}$$

Consider that part of the last integral given by

$$(-1)^r\sum_{\lambda_k>y}a_k\lambda_k^{-1-r}\int_0^{\lambda_{N+1}}F^{(r+1)}(t)\cdot I_r(\lambda_k t)\,dt.$$

This is

$$\ll\sum_{\lambda_k>y}a_k\lambda_k^{-1-r}\int_0^{\lambda_{N+1}}\lambda_k^{\omega_r}(1+t)^{\omega_r}\,dt \qquad \text{(see (2.3))}$$
$$\ll\sum_{\lambda_k>y}a_k\lambda_k^{-((1/2)+(1/2n)+(r/n))}\int_0^{\lambda_{N+1}}(1+t)^{(1/2)+r(1-(1/n))-(1/2n)}\,dt$$
$$\ll\sum_{\lambda_k>y}a_k\lambda_k^{-((1/2)+(1/2n)+(r/n))}\cdot\lambda_{N+1}^{(3/2)+r(1-(1/n))-(1/2n)}$$
$$\ll y^{(1/2)-(1/2n)-(r/n)}\lambda_{N+1}^{(3/2)+r(1-(1/n))-(1/2n)}. \tag{3.5}$$

Choose N and y such that

$$\lambda_N\le x<\lambda_{N+1},\qquad y=c_0x^{n-1+\varepsilon_0},\qquad \varepsilon_0>0. \tag{3.6}$$

Then (3.5) is

$$\ll x^{(3/2)+r(1-(1/n))-(1/2n)-(r/n)(n-1+\varepsilon_0)+n((1/2)-(1/2n))},\quad \text{if}\quad \varepsilon_0<1,$$
$$\ll x^{1+(n/2)-(r\varepsilon_0/n)}$$
$$\ll x^{-q},$$

for a given $q>0$, if r is chosen large enough. Hence, for any $q>0$,

$$(-1)^r \sum_{\lambda_k>y} a_k\lambda_k^{-1-r} \int_0^{\lambda_{N+1}} F^{(r+1)}(t)\cdot I_r(\lambda_k t)\,dt = O(x^{-q}) \tag{3.7}$$

for $r=r(q)$. Using (3.7) in (3.3) and (3.4), we get

$$\int_0^\infty f(t)\beta(t)\,dA(t) = (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t)\cdot Q_r(t)\,dt + O(x^{-q})$$
$$+(-1)^r \sum_{\lambda_k\le y} a_k\lambda_k^{-1-r} \int_0^{\lambda_{N+1}} I_r(\lambda_k t)\cdot F^{(r+1)}(t)\,dt. \tag{3.8}$$

The last sum of integrals equals

$$\sum_{\lambda_k\le y} a_k\lambda_k^{-1} \int_0^{\lambda_{N+1}} F'(t)\cdot I_0(\lambda_k t)\,dt$$
$$= -\sum_{\lambda_k\le y} a_k \int_0^{\lambda_{N+1}} I_{-1}(\lambda_k t)\cdot F(t)\,dt - \sum_{\lambda_k\le y} \frac{a_k}{\lambda_k}\cdot I_0(0)$$
$$= -\sum_{\lambda_k\le y} a_k \int_0^{\lambda_N} I_{-1}(\lambda_k t) f(t)\,dt - \sum_{\lambda_k\le y} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t)\cdot f(t)\cdot\beta(t)\,dt$$
$$-\sum_{\lambda_k\le y} \frac{a_k}{\lambda_k}\cdot I_0(0). \tag{3.9}$$

From (3.9), (3.8), (3.6), and (3.2), we have

$$\sum_{\lambda_k\le x} a_k f(\lambda_k) = \sum_{k=0}^{N} a_k f(\lambda_k)$$
$$= \sum_{k=1}^{N} a_k \exp(2\pi i\eta\lambda_k)$$
$$= (-)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t)\cdot Q_r(t)\,dt + O(x^{-q})$$
$$- \sum_{\lambda_k\le y=c_0x^{n-1+\varepsilon_0}} a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\,dt$$
$$- \sum_{\lambda_k\le y=c_0x^{n-1+\varepsilon_0}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t)\cdot f(t)\cdot\beta(t)\,dt$$
$$- \sum_{\lambda_k\le y} \frac{a_k}{\lambda_k}\cdot I_0(0). \tag{3.10}$$

Now let

$$c_0 = \left(\frac{2\pi n\eta}{h}\right)^n, \tag{3.11}$$

where h is defined as in (2.3). Then $\lambda_k \leq c_0 x^{n-1+\varepsilon_0}$ implies that

$$\mu \leq n x^{1-(1/n)+\varepsilon'}, \quad \text{where} \quad \mu = \frac{h}{2\pi\eta} \cdot \lambda_k^{1/n}, \qquad \varepsilon' = \frac{\varepsilon_0}{n}. \tag{3.12}$$

Then the first sum on the right-hand side of (3.10) equals

$$\begin{aligned}
&\sum_{\lambda_k \leq c_0 x^{n-1+\varepsilon_0}} a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \\
&= \sum_{\lambda_k \leq c_0 x^{n-1}} + \sum_{c_0 x^{n-1} < \lambda_k \leq c_0 x^{n-1+\varepsilon_0}} \\
&= \left[\sum_{\mu \leq n x^{1-(1/n)}} + \sum_{n x^{1-(1/n)} < \mu \leq n x^{1-(1/n)+\varepsilon'}}\right] a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \\
&= \sum_{\mu \leq n x^{1-(1/n)}} a_k \int_0^{\infty} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \\
&\qquad - \sum_{\mu \leq n x^{1-(1/n)}} a_k \int_{\lambda_N}^{\infty} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \\
&\quad + \sum_{n x^{1-(1/n)} < \mu \leq n x^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt,
\end{aligned} \tag{3.13}$$

provided that the integral from 0 to ∞ converges, which we shall prove (in Lemma 2). Combining (3.13) with (3.10), we obtain the following

LEMMA 1. *If ε' is arbitrary with $0 < \varepsilon' < (1/n)$, and $\lambda_N \leq x < \lambda_{N+1}$, then*

$$\begin{aligned}
\sum_{k=1}^{N} a_k \exp(2\pi i\eta\lambda_k) &= (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t)\, dt + O(x^{-q}) \\
&\quad - \sum_{\mu \leq n x^{1-(1/n)+\varepsilon'}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t) \exp(2\pi i\eta t) \cdot \beta(t) \cdot dt \\
&\quad - \sum_{\mu \leq n x^{1-(1/n)+\varepsilon'}} \frac{a_k}{\lambda_k} I_0(0) \\
&\quad - \sum_{\mu \leq n x^{1-(1/n)}} a_k \int_0^{\infty} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt
\end{aligned}$$

$$+\sum_{\mu\le nx^{1-(1/n)}} a_k \int_{\lambda_N}^{\infty} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$-\sum_{nx^{1-(1/n)}<\mu\le nx^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt,$$

where η *is real,* $\eta>0$, $\beta(t)\in C^\infty(-\infty,\infty)$, β *defined as in* (3.2), $F(t)=\exp(2\pi i\eta t)\,\beta(t)$, *and* $\mu=(h/2\pi\eta)\cdot\lambda_k^{1/n}$ (μ *defined as in* (3.12) *and h as in* (2.3)).

LEMMA 2. *The integral*

$$\int_0^\infty \exp(i\eta t)\, I_{-1}(t)\, dt$$

converges for $\eta>0$.

Proof. We first observe that

$$\int_1^\infty t^a \exp(i(\eta t\pm ht^{1/n}))\, dt$$

converges if $a<0$. For, if $u(t)=(\eta t\pm ht^{1/n})$, then

$$\left|\frac{du}{dt}\right|\ge\eta-\frac{h}{n}t^{(1/n)-1}\ge\tfrac{1}{2}\eta, \quad \text{if} \quad t\ge\left(\frac{2h}{n\eta}\right)^{n/(n-1)},$$

so that, if R and R' are large enough, we have, by the second mean-value theorem,

$$\left|\int_R^{R'} t^a\cos\{u(t)\}\, dt\right|\le\frac{2R^a}{\eta}, \qquad \left|\int_R^{R'} t^a\sin\{u(t)\}\, dt\right|\le\frac{2R^a}{\eta},$$

and, in particular,

$$\int_R^{R'} t^a\exp(iu(t))\, dt=o(1), \quad \text{as} \quad R, R'\to\infty.$$

Now from (2.3) we have

$$I_{-1}(t)=\sum_{\nu=0}^{m} e'_\nu t^{\omega_{-1}-(\nu/n)}\cos(ht^{1/n}+\pi_\nu)+O(t^{\omega_{-1}-(m+1)/n}),$$

and

$$\int_1^\infty t^{\omega_{-1}-(1/n)(m+1)}\,dt<\infty,$$

if m is large, while $\omega_{-1}-\nu/n<0$ for $\nu\geq 0$. Hence

$$\int_1^\infty \exp(i\eta t)\,I_{-1}(t)\,dt$$

converges. On the other hand, the limit

$$\lim_{\varepsilon\to 0+}\int_\varepsilon^1 \exp(i\eta t)\,I_{-1}(t)\,dt$$

$$=\lim_{\varepsilon\to 0+}\{(I_0(1)\exp(i\eta)-I_0(\varepsilon)\exp(i\eta\varepsilon))-i\eta\int_\varepsilon^1 I_0(t)\exp(i\eta t)\,dt\}$$

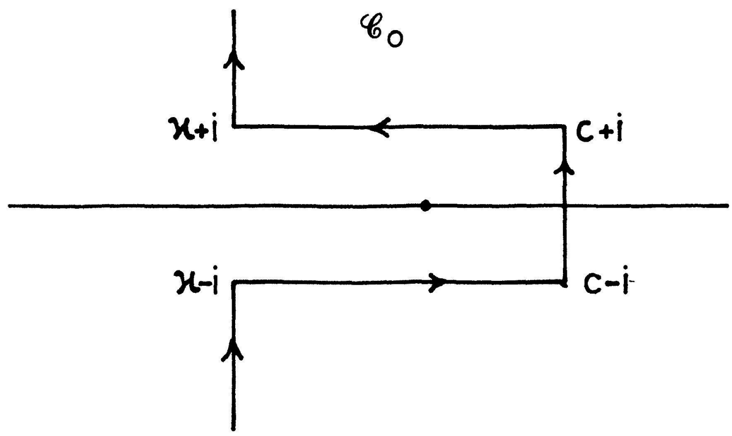
exists, and is finite, since $I_0(t)$ is continuous at $t=0$ (as remarked in (2.4)).

We shall now express the integral in Lemma 2 as a contour integral in the complex s-plane.

LEMMA 3. *For $\xi>0$, we have*

$$\int_0^\infty \exp(i\xi t)\,I_{-1}(t)\,dt=\frac{1}{2\pi i}\int_{\mathscr{C}_0}\frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\cdot\left(\frac{\xi}{i}\right)^{s-1}ds,$$

where $\Delta(s)$ is defined as in (2.2), and $\mathscr{C}_0$ denotes the path of integration indicated in the diagram, with $0<c<1$, and κ a large, negative number.



Proof. Because of Lemma 2,

$$\int_0^\infty \exp(i\xi t)\, I_{-1}(t)\, dt = \lim_{\varepsilon \to 0+} \int_0^\infty \exp(i\xi t)\, I_{-1}(t) \exp(-\varepsilon t)\, dt$$

$$= \lim_{z \to -i\xi} \int_0^\infty \exp(-zt)\, I_1(t)\, dt, \quad (3.14)$$

where $z = \varepsilon - i\xi$, $\varepsilon > 0$. Now

$$I_{-1}(t) = \frac{1}{2\pi i} \int_{\mathscr{C}_0} \frac{\Delta(s)}{\Delta(1-s)}\, t^{-s}\, ds,$$

since $\Delta(s)/\Delta(1-s)$ has no singularities in Re $s > 0$. If κ is large, and negative, the integral from $\kappa + i$ to $\kappa + i\infty$, and the integral from $\kappa - i\infty$ to $\kappa - i$ converge absolutely, since

$$\left|\frac{\Delta(s)}{\Delta(1-s)}\right| \ll (1 + |\tau|)^{n\sigma - (n/2)},$$

where $s = \sigma + i\tau$. Moreover Re $s \leq c < 1$, everywhere on $\mathscr{C}_0$. Hence

$$\int_0^\infty \exp(-zt)\, I_{-1}(t)\, dt = \frac{1}{2\pi i} \int_{\mathscr{C}_0} \frac{\Delta(s)}{\Delta(1-s)}\, ds \int_0^\infty \exp(-zt)\, t^{-s}\, dt, \qquad \operatorname{Re} z > 0$$

$$= \frac{1}{2\pi i} \int_{\mathscr{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot z^{s-1}\, ds. \quad (3.15)$$

Let $z = |z| \cdot \exp(i\theta)$. Then $|z^{s-1}| = |z|^{\sigma-1} \cdot \exp(-\theta\tau)$. Since Re $z > 0$, we have $|\theta| < \frac{1}{2}\pi$, so that $|z^{s-1}| \leq |z|^{\sigma-1} \exp(\frac{1}{2}\pi\, |\tau|)$. On the other hand,

$$\left|\frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)}\right| \sim \sqrt{2\pi} \cdot \exp(-\tfrac{1}{2}\pi\, |\tau|) \cdot |\tau|^{n\sigma - (n/2) + (1/2) - \sigma}, \quad \text{as} \quad |\tau| \to \infty.$$

Hence the integrand in (3.15) is $O(|\tau|^{\sigma(n-1) - ((n-1)/2)}) = O(|\tau|^{-2})$, if κ is large and negative, and $|\tau|$ large. Thus the integral in (3.15) converges uniformly for Re $z > 0$.

Now let $z \to -i\xi$, $\xi > 0$. Then

$$\lim_{z \to -i\xi} \frac{1}{2\pi i} \int_{\mathscr{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\, z^{s-1}\, ds = \frac{1}{2\pi i} \int_{\mathscr{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot \left(\frac{\xi}{i}\right)^{s-1} ds.$$

Comparing this with (3.14) and (3.15), we get the lemma.

§4. Some asymptotic expansions

LEMMA 4. *We have*

$$\int_0^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$= \frac{1}{2\eta} \sum_{\nu=0}^{m} m_k^{((2-n)/2(n-1))-(\nu/(n-1))}$$

$$\times \{c_\nu \cos(q m_k^{1/(n-1)} + \pi\omega' + \tfrac{1}{2}\pi\nu) - i d_\nu \sin(q m_k^{1/(n-1)} + \pi\omega' + \tfrac{1}{2}\pi\nu)\}$$

$$-\frac{1}{\lambda_k} \sum_{1\le l<((n+2m)/2(n-1))} \frac{\Delta(l)\cdot(-1)^{l-1}}{\Delta(1-l)\cdot(l-1)!} \cdot \left(\frac{-2\pi i\eta}{\lambda_k}\right)^{l-1} + O(m_k^{-((n+2m)/2(n-1))}),$$

where $\eta>0$, $m_k=\lambda_k/2\pi\eta$, $q=(n-1)2^{r_1/(n-1)}$, $c_0=d_0=\pi^{-1}(n-1)^{-1/2}2^{r_1/2(n-1)}$, $\omega'=-\frac{1}{2}+r_1/4$, *$l$ integral.*

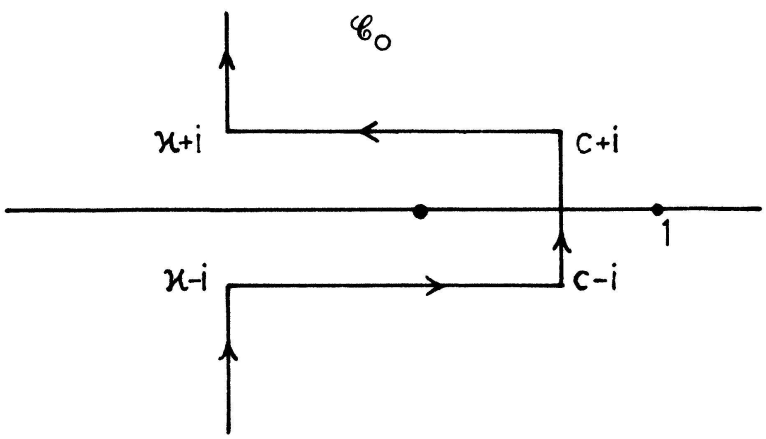
Proof. Let

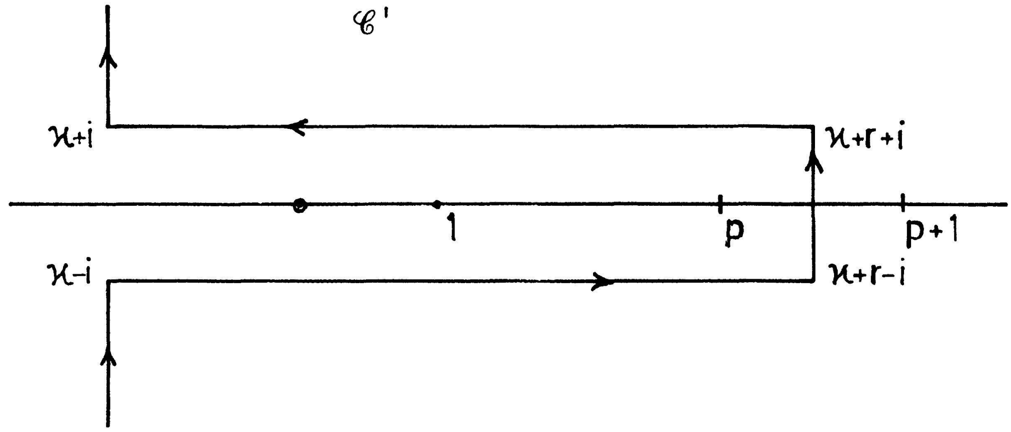
$$J=\int_0^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt, \qquad m_k=\frac{\lambda_k}{2\pi\eta}, \qquad \eta>0.$$

Then

$$J=\frac{1}{\lambda_k}\int_0^\infty \exp(it/m_k) I_{-1}(t)\, dt = \frac{1}{\lambda_k}\cdot\frac{1}{2\pi i}\int_{\mathscr{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\cdot\left(\frac{1}{im_k}\right)^{s-1} ds,$$

after Lemma 3. Deform the path of integration $\mathscr{C}_0$ into $\mathscr{C}'$, by choosing p to be a sufficiently large integer, and $p<\kappa+r<p+1$, as indicated in the diagram.





We then have

$$J=\frac{1}{\lambda_k}\left[\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\cdot\left(\frac{1}{im_k}\right)^{s-1}ds-\sum_{l=1}^{p}\frac{\Delta(l)\cdot(-1)^{l-1}}{\Delta(1-l)\cdot(l-1)!}\left(\frac{-2\pi i\eta}{\lambda_k}\right)^{l-1}\right]$$

$$=\left(\frac{i}{2\pi\eta}\right)\cdot\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\cdot m_k^{-s}\left(\cos\frac{\pi s}{2}-i\sin\frac{\pi s}{2}\right)ds$$

$$-\frac{1}{\lambda_k}\left[\sum_{l=1}^{p}\frac{\Delta(l)(-1)^{l-1}}{\Delta(1-l)\cdot(l-1)!}\cdot\left(\frac{-2\pi i\eta}{\lambda_k}\right)^{l-1}\right]. \tag{4.2}$$

We seek an expansion for

$$J_1=\left(\frac{i}{2\pi\eta}\right)\cdot\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)}\cdot m_k^{-s}\left(\cos\frac{\pi s}{2}-i\sin\frac{\pi s}{2}\right)ds. \tag{4.3}$$

Since

$$\cos\frac{\pi s}{2}=\frac{\pi}{\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)},\qquad \sin\frac{\pi s}{2}=\frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)},$$

we have

$$J_1=\left(\frac{i}{2\pi\eta}\right)\cdot\frac{1}{2\pi i}\int_{\mathscr{C}'}\{V_0(s)-iV_1(s)\}\cdot\pi\cdot m_k^{-s}\,ds, \tag{4.4}$$

where

$$V_0(s)=\frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)},\qquad V_1(s)=\frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}.$$

Choose

$$U_0(s) = -\frac{1}{\pi}\cdot b^{-s}\Gamma(S)\sin\{\pi(\tfrac{1}{2}S+\omega')\},$$

$$U_1(s) = \frac{1}{\pi}\cdot b^{-s}\Gamma(S)\cos\{\pi(\tfrac{1}{2}S+\omega')\},$$

where

$$S=(n-1)s-\tfrac{1}{2}n+1, \qquad \omega' = -\frac{1}{2}+\frac{r_1}{4}, \qquad b=(n-1)^{n-1}\cdot 2^{r_1}.$$

By comparing the Stirling expansions of V_0 and U_0 on the one hand, and of V_1, U_1 on the other, we get (as in [2, Lemma 1])

$$V_0(s) = b^{1/2}U_0(s)\left\{1+\sum_{\nu=1}^{m}\frac{e_\nu}{S^\nu}+O(|s|^{-m-1})\right\}$$

$$= -\frac{b^{(1/2)-s}}{\pi}\cdot\Gamma(S)\sin\left(\frac{\pi S}{2}+\pi\omega'\right)\cdot\left\{1+\sum_{\nu=1}^{m}\frac{e_\nu}{S^\nu}+O(|s|^{-m-1})\right\},$$

and

$$V_1(s) = b^{1/2}U_1(s)\left\{1+\sum_{\nu=1}^{m}\frac{e'_\nu}{S^\nu}+O(|s|^{-m-1})\right\}$$

$$= \frac{b^{(1/2)-s}}{\pi}\cdot\Gamma(S)\cos\left(\frac{\pi S}{2}+\pi\omega'\right)\left\{1+\sum_{\nu=1}^{m}\frac{e'_\nu}{S^\nu}+O(|s|^{-m-1})\right\}.$$

Now, following the same procedure as in [2, Lemma 1], we can get

$$\frac{1}{2\pi i}\int_{\mathscr{C}'} V_0(s)x^{-s}\,ds = -\sum_{\nu=0}^{m} d_\nu x^{((2-n)/2(n-1))-\nu/(n-1)}\sin\left(qx^{1/(n-1)}+\pi\omega'+\frac{\pi\nu}{2}\right)$$
$$+O(x^{-((n+2m)/2(n-1))}), \quad (4.5)$$

and

$$\frac{1}{2\pi i}\int_{\mathscr{C}'} V_1(s)x^{-s}\,ds = \sum_{\nu=0}^{m} c_\nu x^{((2-n)/2(n-1))-(\nu/(n-1))}\cos\left(qx^{1/(n-1)}+\pi\omega'+\frac{\pi\nu}{2}\right)$$
$$+O(x^{-((n+2m)/2(n-1))}), \quad (4.6)$$

where

$$q = b^{1/(n-1)} = (n-1)2^{r_1/(n-1)}, \qquad c_0 = \frac{q^{-((n/2)-1)}}{n-1} \cdot \frac{b^{1/2}}{\pi} = d_0,$$

provided that

$$p \geq \frac{n+2m}{2(n-1)}.$$

From (4.4) we have

$$J_1 = \frac{1}{2\eta} \cdot \frac{1}{2\pi i} \int_{\mathscr{C}'} \{V_1(s) + iV_0(s)\} m_k^{-s}\, ds.$$

From (4.5) and (4.6) we therefore obtain

$$2\eta \cdot J_1 = \sum_{\nu=0}^{m} m_k^{((2-n)/2(n-1))-\nu/(n-1)} \times \left\{ c_\nu \cos\left(q m_k^{1/(n-1)} + \pi\omega' + \frac{\pi\nu}{2}\right) - i d_\nu \sin\left(q m_k^{1/(n-1)} + \pi\omega' + \frac{\pi\nu}{2}\right)\right\} + O(m_k^{-((n+2m)/2(n-1))}), \quad (4.7)$$

where $c_0 = d_0$. However, from (4.2) we have

$$J = J_1 - \frac{1}{\lambda_k} \sum_{l=1}^{p} \frac{\Delta(l) \cdot (-1)^{l-1}}{\Delta(1-l) \cdot (l-1)!} \cdot \left(\frac{-2\pi i\eta}{\lambda_k}\right)^{l-1}, \quad (4.8)$$

for a suitably chosen integer p. Now (4.8) and (4.7) give us the required result.

LEMMA 5. *Let a be a real number, $a \neq 0$, $\mu = (h/2\pi\eta)\lambda_k^{1/n}$, $\eta > 0$, and $h = n2^{(n-r_2)/n}$. Let $\varphi(t) = \varphi(t, \mu) = t - \mu t^{1/n}$, $F_0(t) = F_0(t, \mu) = t^a/\varphi'(t)$, $F_{l+1}(t) = F_{l+1}(t, \mu) = 1/\varphi'(t) \cdot (d/dt)F_l(t)$, for $l = 0, 1, 2, \ldots$. Then*

$$F_l(t) = \frac{t^{a-l}}{(\varphi'(t))^{l+1}} \sum_{k=0}^{l} c_{l,k} \frac{\left(\frac{1}{n}\mu t^{(1/n)-1}\right)^k}{\left(1 - \frac{1}{n}\mu t^{(1/n)-1}\right)^k},$$

with $c_{0,0}=1$, $c_{l,-1}=0$, $c_{l,l+1}=0$, *and*

$$c_{l+1,k}=c_{l,k}\left\{(a-l)-k\left(1-\frac{1}{n}\right)\right\}-c_{l,k-1}\left\{(l+1)\left(1-\frac{1}{n}\right)+(k-1)\left(1-\frac{1}{n}\right)\right\}$$

for $k=0,1,\ldots,l+1$.

Analogously, if

$$\psi(t)=\psi(t,\mu)=t+\mu t^{1/n},\qquad G_0(t)=G_0(t,\mu)=\frac{t^a}{\psi'(t)},$$

$$G_{l+1}(t)=G_{l+1}(t,\mu)=\frac{1}{\psi'(t)}\cdot\frac{d}{dt}G_l(t),$$

for $l=0,1,2,\ldots$, *then*

$$G_l(t)=\frac{t^{a-l}}{(\psi'(t))^{l+1}}\sum_{k=0}^{l}d_{l,k}\frac{\left(\frac{1}{n}\mu t^{(1/n)-1}\right)^k}{\left(1+\frac{1}{n}\mu t^{(1/n)-1}\right)^k},$$

with $d_{0,0}=1$, $d_{l,-1}=0$, $d_{l,l+1}=0$, *and*

$$d_{l+1,k}=d_{l,k}\left\{(a-l)+k\left(1-\frac{1}{n}\right)\right\}+d_{l,k-1}\left\{(l+1)\left(1-\frac{1}{n}\right)+(k-1)\left(1-\frac{1}{n}\right)\right\}.$$

The proof follows by induction on l.

LEMMA 6. *Let* $a=a(\nu)=\omega_{-1}-\nu/n$, *for* $\nu=0,1,2,\ldots$. *If* $\delta>0$, δ *sufficiently small, and* $0<\eta_0<\eta$, *and* $\mu\leq(n-\delta)x^{1-(1/n)}$, *we have*

$$\int_x^\infty t^a\exp(2\pi i\eta\varphi(t))\,dt=\exp(2\pi i\eta\varphi(x))\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+O(x^{a-L-1}),$$

where the 'O' depends on L *and* δ, *but is uniform in* μ, *and* $F_{l,\nu}$ *is the function* F_l *of Lemma* 5, *with* $a=a(\nu)$.

Analogously we have

$$\int_x^\infty t^a\exp(2\pi i\eta\psi(t))\,dt=\exp(2\pi i\eta\psi(x))\sum_{l=0}^{L}\frac{(-1)^{l+1}G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+O(x^{a-L-1}),$$

where $G_{l,\nu}$ *is the function* G_l *of Lemma* 5, *with* $a=a(\nu)$.

Proof. We note that $\omega_{-1}=(1/2n)-\frac{1}{2}$, and that x is bounded below. Since $t\geq x$, we have

$$1-\frac{\mu}{n}t^{(1/n)-1}\geq 1-\frac{\mu}{n}x^{(1/n)-1}\geq\frac{\delta}{n},$$

and

$$\frac{\mu}{n}t^{(1/n)-1}\leq\frac{\mu}{n}x^{(1/n)-1}<1.$$

Hence, from Lemma 5, we have

$$F_{l,\nu}(t)=O(t^{a-l}),\qquad t\geq x,\qquad \mu\leq(n-\delta)x^{1-(1/n)},$$

where the 'O' depends on a, l, and δ, but *not* on t, x, or μ. By partial integration, we have

$$\begin{aligned}\int_x^\infty t^a\exp(2\pi i\eta\varphi(t))\,dt&=\int_x^\infty\frac{t^a}{\varphi'(t)}\cdot\exp(2\pi i\eta\varphi(t))\cdot\varphi'(t)\,dt\\&=\exp(2\pi i\eta\varphi(x))\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+\frac{(-1)^{L+1}}{(2\pi i\eta)^{L+1}}\\&\quad\times\int_x^\infty F_{L+1,\nu}(t)\cdot\varphi'(t)\cdot\exp(2\pi i\eta\varphi(t))\,dt\\&=\exp(2\pi i\eta\varphi(x))\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+O(x^{a-L}).\end{aligned}$$

Replacing L by $L+1$, we get the lemma.

LEMMA 7. *Under the same conditions as in Lemma 6, we have*

$$\begin{aligned}\int_x^\infty\exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt&=\sum_{\nu=0}^{n(m+1)}\lambda_k^{\omega_{-1}-(\nu/n)}\sum_{l=0}^{m+1}\\&\quad\times(\exp(2\pi i\eta\varphi(x))\cdot b_{l,\nu}F_{l,\nu}(x)+\exp(2\pi i\eta\psi(x))\,b'_{l,\nu}G_{l,\nu}(x))+O(x^{\omega_{-1}-m}),\end{aligned}$$

where $\omega_{-1}=(1/2n)-\frac{1}{2}$, *and* $F_{l,\nu}$, $G_{l,\nu}$ *stand for the functions* F_l, G_l *defined in Lemma* 5, *with* $a=a(\nu)=\omega_{-1}-(\nu/n)$, $\nu=0,1,2,\ldots.$

Proof. On using the asymptotic expansion for $I_{-1}(t)$, we have

$$\int_x^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt = \sum_{\nu=0}^{m} e'_\nu \int_x^\infty (\lambda_k t)^{\omega_{-1}-(\nu/n)} \times \exp(2\pi i\eta t)\cos(h\lambda_k^{1/n} t^{1/n} + \pi_\nu)\, dt + O_m((\lambda_k x)^{\omega_{-1}-((m+1)/n)+1}),$$

where $h = n2^{(n-r_2)/n}$, and $\pi_\nu = \pi_\nu(-1) = -\pi(\frac{1}{2}n + (r_1/4 + \frac{1}{4}) - (\nu/2))$. Thus

$$\int_x^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$
$$= \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Big(b_\nu \int_x^\infty t^{\omega_{-1}-(\nu/n)} \exp(2\pi i\eta t - h\lambda_k^{1/n} t^{1/n})\, dt$$
$$+ b'_\nu \int_x^\infty t^{\omega_{-1}-(\nu/n)} \exp(2\pi i\eta t + h\lambda_k^{1/n} t^{1/n})\, dt\Big) + O_m((\lambda_k x)^{\omega_{-1}-((m+1)/n)+1}).$$

If we now use Lemma 6, we get

$$\int_x^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt = \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Big\{b_\nu \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}$$
$$+ b'_\nu \exp(2\pi i\eta\psi(x)) \sum_{l=0}^{L} \frac{(-1)^{l+1}G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + O(x^{\omega_{-1}-(\nu/n)-L-1})\Big\}$$
$$+ O_m(x^{\omega_{-1}-((m+1)/n)+1}).$$

If $L = [(m+1)/n]$, then $L+1 > (m+1)/n$, hence

$$\int_x^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt = \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Big(\exp(2\pi i\eta\varphi(x)) \sum_{0\le l\le (m+1)/n} b_{l,\nu}F_{l,\nu}(x)$$
$$+ \exp(2\pi i\eta\psi(x)) \sum_{0\le l\le (m+1)/n} b'_{l,\nu}G_{l,\nu}(x)\Big) + O_m(x^{\omega_{-1}-((m+1)/n)+1}).$$

Replacing m by $n(m+1)$, we get the lemma.

Our treatment of the estimates for the range $\mu \ge (n+\delta)x^{1-(1/n)}$ is based on the next two lemmas.

LEMMA 8. *For a fixed ξ, such that $0 < \xi_0 \le \xi \le \xi_1$, we have, as $\mu \to \infty$, the following asymptotic expansion (in decreasing powers of μ):*

$$F_\nu(\xi, \mu) = \frac{1}{\mu^{\nu+1}}\Big(\sum_{l=0}^{L} \frac{d_{l,\nu}}{\mu^l} + o(\mu^{-L})\Big),$$

where the $d_{l,\nu}$ are continuous functions of ξ for $0 < \xi_0 \le \xi \le \xi_1$.

Proof. We have only to use the formula

$$F_\nu(\xi, \mu) = \frac{\xi^{a-\nu}}{(\varphi'(\xi))^{\nu+1}} \sum_{m=0}^{\nu} c_{\nu,m} \frac{\left(\frac{1}{n}\mu\xi^{(1/n)-1}\right)^m}{\left(1-\frac{1}{n}\mu\xi^{(1/n)-1}\right)^m},$$

(see Lemma 5) and the Bionomial Theorem.

LEMMA 9. *Suppose for a fixed integer $K \geqslant 0$, we have*

$$f(\mu) = f(x, \mu) = \sum_{k=0}^{K} \mu^{n_k}(a_k(\xi) + b_k(\xi) \exp(i\mu\xi) + b_k'(\xi) \exp(-i\mu\xi)) + o(\mu^{n_K}),$$

as $\mu \to \infty$, where $n_0 > n_1 > n_2 > \cdots \to -\infty$, for fixed x in a compact set in $\mathbf{R}_1$, and this holds uniformly in ξ for $a \leqslant \xi \leqslant b$, where $a_k(\xi)$ and $b_k(\xi)$ are continuous in ξ, and the function $f(x, \mu)$ is independent of ξ, then $a_k(\xi)$ is a constant, say a_k, and $b_k(\xi) = b_k'(\xi) = 0$ for all $k \leqslant K$.

Proof. By hypothesis,

$$\mu^{-n_0} f(\mu) = a_0(\xi) + b_0(\xi) \exp(i\mu\xi) + b_0'(\xi) \exp(-i\mu\xi) + o(1),$$

as $\mu \to \infty$, uniformly in ξ. Integrating this with respect to ξ, we get

$$(\beta - a)\mu^{-n_0} f(\mu) = \int_a^\beta a_0(\xi)\, d\xi + o(1),$$

as $\mu \to \infty$, for $a \leqslant \xi \leqslant \beta < b$, because of the Riemann–Lebesgue Lemma. Hence the limit $\lim_{\mu\to\infty} \mu^{-n_0} f(\mu)$ exists, and equals, say, c which is independent of ξ. It follows that $a_0(\xi) = c$. Hence

$$\lim_{\mu\to\infty} \{b_0(\xi) \exp(i\mu\xi) + b_0'(\xi) \exp(-i\mu\xi)\} = 0,$$

or $\lim_{\mu\to\infty} b_0(\xi) \exp(2i\mu\xi) + b_0'(\xi) = 0$. Again, by integration, we see that $b_0'(\xi) \equiv 0$, hence also $b_0(\xi) \equiv 0$. A repetition of the argument leads to the lemma.

COROLLARY. *If*

$$f(\mu) = \sum_{k=0}^{K} \mu^{n_k}\{a_k(\xi) + b_k(\xi) \cos(\mu^\alpha \xi) + b_k'(\xi) \sin(\mu^\alpha \xi)\} + o(\mu^{n_K}),$$

for $\alpha > 0$, *uniformly in* ξ *for* $a \leq \xi \leq b$, *as* $\mu \to \infty$, *then* $a_k(\xi)$ *is independent of* ξ, *and* $b_k(\xi) \equiv b_k'(\xi) \equiv 0$.

LEMMA 10. *If* $\delta > 0$, *and sufficiently small, and* $0 < \eta_0 \leq \eta$, *and* $\mu \geq (n+\delta)x^{1-(1/n)}$, *we have*

$$\int_0^x \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$
$$= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b_\nu \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$
$$+ b_\nu' \exp(2\pi i\eta\psi(x)) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\Bigg) + O(\mu^{-L-1}) + O(\lambda_k^{\omega_0-(m/n)-1}),$$

where $\omega_0 = \frac{1}{2} - (1/2n)$ *(as in* (2.3)), *and* $\omega_{-1} = (1/2n) - \frac{1}{2}$. *If* $L = m$, *the term* $O(\mu^{-L-1})$ *can be dropped.*

Proof. Let ξ be a number such that $\frac{1}{4}\lambda_1 \leq \xi \leq \frac{1}{2}\lambda_1$, say. Then

$$\int_0^x \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt = \int_0^\xi + \int_\xi^x .$$

If $a = a(\nu) = \omega_{-1} - (\nu/n) = (1/2n) - \frac{1}{2} - (\nu/n)$, for $\nu = 0, 1, 2, \ldots$, we obtain by partial integration,

$$\int_\xi^x t^a \exp(2\pi i\eta\varphi(t))\, dt$$
$$= \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} - \exp(2\pi i\eta\varphi(\xi)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}$$
$$+ (-1)^{L+1} \int_\xi^x \frac{F_{L+1,\nu}(t,\mu)\varphi'(t)}{(2\pi i\eta)^{L+1}} \exp(2\pi i\eta\varphi(t))\, dt.$$

To estimate the last integral, we use, after Lemma 5,

$$\varphi'(t)F_k(t) = \frac{t^{a-k}}{\{\varphi'(t)\}^k} \sum_{l=0}^{k} c_{k,l} \left(\frac{\frac{1}{n}\mu t^{(1/n)-1}}{1 - \frac{1}{n}\mu t^{(1/n)-1}}\right)^l .$$

Here $t \leq x$, and $\mu \geq (n+\delta)x^{1-(1/n)}$, so that

$$\frac{1}{n}\mu t^{(1/n)-1} \geq \frac{1}{n}\mu x^{(1/n)-1} \geq \frac{n+\delta}{n} = 1 + \frac{\delta}{n},$$

while

$$\frac{t^{-1}}{\varphi'(t)}=\frac{1}{t\left(1-\frac{1}{n}\mu t^{(1/n)-1}\right)}=\frac{1}{t-\frac{1}{n}\mu t^{1/n}}=\frac{-nt^{-1/n}}{\mu-nt^{1-(1/n)}}.$$

For $1\leqslant\xi\leqslant t\leqslant x$, the numerator is bounded, while

$$\mu-nt^{1-(1/n)}\geqslant\mu-nx^{1-(1/n)}\geqslant\mu-\frac{n\mu}{n+\delta}=\frac{\delta\mu}{n+\delta},$$

since $nx^{1-(1/n)}\leqslant n\mu/(n+\delta)$. Hence

$$\left|\frac{t^{-1}}{\varphi'(t)}\right|\leqslant\frac{n(n+\delta)}{\delta}\cdot\frac{\xi^{-1/n}}{\mu},$$

so that

$$\left(\frac{t^{-1}}{\varphi'(t)}\right)^k=O(\mu^{-k}),$$

provided that $k\geqslant 1$, $\frac{1}{4}\lambda_1\leqslant\xi\leqslant\frac{1}{2}\lambda_1$. On the other hand,

$$\frac{\frac{1}{n}\mu t^{(1/n)-1}}{1-\frac{1}{n}\mu t^{(1/n)-1}}=\frac{u}{1-u}=-\left(1+\frac{1}{u-1}\right),\quad\text{with}\quad u\geqslant 1+\frac{\delta}{n},$$

hence

$$\left|\frac{\frac{1}{n}\mu t^{(1/n)-1}}{1-\frac{1}{n}\mu t^{(1/n)-1}}\right|<C,\text{ say.}$$

Thus $\varphi'(t)F_k(t)=O(t^a\mu^{-k})$, for $k\geqslant 1$. It follows that

$$\begin{aligned}(-1)^{L+1}\int_\xi^x\frac{F_{L+1,\nu}(t,\mu)}{(2\pi i\eta)^{L+1}}\cdot\varphi'(t)\cdot\exp(2\pi i\eta\varphi(t))\,dt&=O\left(\int_\xi^x t^a\mu^{-(L+1)}\,dt\right)\\&=O(\mu^{-(L+1)})\cdot(x^{a+1}-\xi^{a+1})\\&=O(x^{\omega_{-1}+1}\mu^{-(L+1)})\\&=O(\mu^{n(\omega_{-1}+1)/(n-1)-(L+1)}),\qquad\omega_{-1}=\frac{1}{2n}-\frac{1}{2}.\end{aligned}$$

Thus we obtain (after replacing L by a larger L' if necessary),

$$\int_\xi^x t^a \exp(2\pi i\eta\varphi(t))\,dt = \exp(2\pi i\eta\varphi(x))\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$

$$-\exp(2\pi i\eta\varphi(\xi))\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}} + O(\mu^{-L-1}), \quad (4.9)$$

for $\mu \geqslant (n+\delta)x^{1-(1/n)}$, where the '$O$' does *not* depend on ξ.

If we replace φ by ψ, we obtain an analogue with $G_{l,\nu}(x,\mu)$ instead of $F_{l,\nu}(x,\mu)$. It follows that

$$\int_\xi^x \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Big(b_\nu'' \exp(2\pi i\eta\varphi(x,\mu))\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$

$$- b_\nu'' \exp(2\pi i\eta\varphi(\xi,\mu))\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Big) + O(\mu^{-L-1})$$

$$+ \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Big(b_\nu''' \exp(2\pi i\eta\psi(x,\mu))\sum_{l=0}^{L}\frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$

$$- b_\nu''' \exp(2\pi i\eta\psi(\xi,\mu))\sum_{l=0}^{L}\frac{(-1)^l G_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Big) + O(\mu^{-L-1})$$

$$+ O(\lambda_k^{\omega_{-1}-((m+1)/n)}).$$

On the other hand

$$\int_0^\xi \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt$$

$$= \frac{I_0(\lambda_k\xi)\exp(2\pi i\eta\xi)}{\lambda_k} - \frac{I_0(0)}{\lambda_k} - \frac{2\pi i\eta}{\lambda_k}\int_0^\xi \exp(2\pi i\eta t)\,I_0(\lambda_k t)\,dt,$$

where

$$\int_0^\xi \exp(2\pi i\eta t)\,I_0(\lambda_k t)\,dt = \left[\frac{I_1(\lambda_k t)\cdot\exp(2\pi i\eta t)}{\lambda_k}\right]_0^\xi$$

$$-\int_0^\xi \frac{2\pi i\eta}{\lambda_k}\cdot I_1(\lambda_k t)\cdot\exp(2\pi i\eta t)\,dt$$

$$= \sum_{\nu=1}^{m}\frac{c_\nu(\xi)I_\nu(\lambda_k\xi)}{\lambda_k^\nu} + \frac{(-1)^m}{\lambda_k^m}\int_0^\xi (2\pi i\eta)^m\cdot I_m(\lambda_k t)$$

$$\times\exp(2\pi i\eta t)\,dt.$$

Since

$$I_m(x) = O((1+x)^{\omega_0+m(1-(1/n))}),$$

and ξ is bounded, we have

$$-\frac{2\pi i\eta}{\lambda_k}\int_0^\xi \exp(2\pi i\eta t)\, I_0(\lambda_k t)\, dt = \sum_{\nu=1}^{m} \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O\left(\frac{1}{\lambda_k^{m+1}}\right)\cdot \int_0^\xi |I_m(\lambda_k t)|\, dt$$

$$= \sum_{\nu=1}^{m} \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O(\lambda_k^{\omega_0-(m/n)-1}).$$

Thus altogether we have

$$\int_0^\xi \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt = \sum_{\nu=0}^{m} \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} - \frac{I_0(0)}{\lambda_k} + O(\lambda_k^{\omega_0-(m/n)-1}), \qquad (4.10)$$

the 'O' being independent of ξ.

Combining (4.10) and (4.9), we get

$$\int_0^x \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^{m} \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O(\lambda_k^{\omega_0-(m/n)+1})$$

$$+ \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b''_\nu \exp(2\pi i\eta\varphi(x,\mu)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} - b''_\nu \exp(2\pi i\eta\varphi(\xi,\mu)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Bigg)$$

$$+ \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b'''_\nu \exp(2\pi i\eta\psi(x,\mu)) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} - b'''_\nu \exp(2\pi i\eta\psi(\xi,\mu)) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Bigg)$$

$$+ O(\mu^{-L-1}) + O(\lambda_k^{\omega_{-1}-(m/n)-(1/n)}),$$

if $\mu \geq (n+\delta)x^{1-(1/n)}$, $\delta > 0$, δ sufficiently small, and $\mu = (h/2\pi\eta)\cdot \lambda_k^{1/n}$.

If we now use the asymptotic expansion (2.3) for I_ν, $0 \leq \nu \leq m$, and apply Lemmas 8 and 9, we get the required result.

LEMMA 11. *If* $(n-\delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)}$, $\delta > 0, 0 < 2\varepsilon/n < 1$, *then*

$$\int_x^\infty t^a \exp(2\pi i\eta\varphi(t))\,dt$$
$$= \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + O_L(x^{a-L+((L+1)2\varepsilon/n)}),$$

and, for all $\mu > 0$,

$$\int_x^\infty t^a \exp(2\pi i\eta\psi(t))\,dt = \exp(2\pi i\eta\psi(x)) \sum_{l=0}^{L} \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + O_L(x^{a-L}),$$

where, as before, $a = a(\nu) = \omega_{-1} - (\nu/n)$; $\nu = 0, 1, 2, \ldots$.

If $\varepsilon > 0$ (*and not necessarily* $2\varepsilon < n$), *we have*

$$\int_x^\infty t^a \exp(2\pi i\eta\varphi(t))\,dt \ll |F_{0,0}(x)|.$$

Proof. By partial integration we have

$$\int_x^\infty t^a \exp(2\pi i\eta\varphi(t))\,dt = \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}$$
$$+ \frac{(-1)^{L+1}}{(2\pi i\eta)^{L+1}} \int_x^\infty F_{L+1,\nu}(t) \cdot \varphi'(t) \cdot \exp(2\pi i\eta\varphi(t))\,dt.$$

In order to estimate the last integral, in the range under consideration, we have to estimate $F_{k,\nu}(t)$ anew. We have

$$\left(\frac{\frac{1}{n}\mu t^{(1/n)-1}}{1-\frac{1}{n}\mu t^{(1/n)-1}}\right) = \left(\frac{\frac{1}{n}\mu}{t^{1-(1/n)} - \frac{1}{n}\mu}\right) \leq \frac{x^{1-(1/n)}}{\frac{1}{n}x^{1-(1/n)-\varepsilon/n}} \leq nx^{\varepsilon/n},$$

since $\mu \leq nx^{1-(1/n)}$, and $t^{1-(1/n)} - (\mu/n) \geq t^{1-(1/n)} - x^{1-(1/n)} + (1/n)x^{1-(1/n)-(\varepsilon/n)} \geq (1/n)x^{1-(1/n)-(\varepsilon/n)}$, for $t \geq x$. On the other hand,

$$\frac{t^{a-k}}{(\varphi'(t))^{k+1}} = \frac{t^a}{\left(t - \frac{\mu}{n}t^{1/n}\right)^k} \cdot \frac{1}{\left(1 - \frac{\mu}{n}t^{(1/n)-1}\right)}$$
$$\leq \frac{t^a}{\left(t^{1/n} \cdot \frac{1}{n}x^{1-(1/n)-(\varepsilon/n)}\right)^k \cdot \left|1 - \frac{\mu}{n}t^{(1/n)-1}\right|},$$

since

$$t-\frac{\mu}{n}t^{1/n}=t^{1/n}\left(t^{1-(1/n)}-\frac{\mu}{n}\right)\geqslant t^{1/n}\cdot\frac{1}{n}x^{1-(1/n)-(\varepsilon/n)}.$$

Therefore we have

$$|F_{k,\nu}(t)|\ll\frac{x^{k\varepsilon/n}\cdot t^{a}}{t^{k/n}(x^{1-(1/n)-(\varepsilon/n)})^{k}\cdot|\varphi'(t)|},$$

on using the formula for $F_k(t)$ given in Lemma 5. It follows that

$$\frac{(-1)^{L+1}}{(2\pi i\eta)^{L+1}}\int_x^\infty F_{L+1,\nu}(t)\cdot\varphi'(t)\cdot\exp(2\pi i\eta\varphi(t))\,dt$$
$$\ll\frac{x^{(L+1)(\varepsilon/n)}}{x^{(L+1)(1-(1/n)-(\varepsilon/n))}}\int_x^\infty t^{a-(L+1)/n}\,dt$$
$$\ll x^{a-L+((L+1)2\varepsilon/n)},$$

hence the first part of the lemma.

The proof of the second part is similar. We use the fact that

$$\frac{\frac{1}{n}\mu t^{(1/n)-1}}{1+\frac{1}{n}\mu t^{(1/n)-1}}\leqslant 1,\quad\text{and}\quad\frac{t^{a-k}}{\{\psi'(t)\}^{k+1}}\leqslant t^{a-k}.$$

To prove the final remark, we note that

$$\int_x^\infty t^a\exp(2\pi i\eta\varphi(t))\,dt=\int_x^{x(1+\gamma)}\frac{t^a}{\varphi'(t)}\cdot\exp(2\pi i\eta\varphi(t))\cdot\varphi'(t)\,dt$$
$$+\int_{x(1+\gamma)}^\infty t^a\exp(2\pi i\eta\varphi(t))\,dt.$$

Since $t^a/\varphi'(t)$ is decreasing, we may apply the second mean-value theorem to the first term (that is, separately to the real and imaginary part). To the second term we may apply Lemma 6.

LEMMA 12. *If* $(n-\delta)x^{1-(1/n)}<\mu\leqslant nx^{1-(1/n)}-x^{1-(1/n)-(\varepsilon/n)}$, $0<2\varepsilon/n<1$, $\delta>0$, *δ sufficiently small, then we have*

$$\int_x^\infty\exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt=\sum_{\nu=0}^{m}\lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b_\nu\exp(2\pi i\eta\varphi(x,\mu))\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$
$$+b'_\nu\exp(2\pi i\eta\psi(x,\mu))\sum_{l=0}^{L}\frac{(-1)^{l+1}G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\Bigg)+O_m(x^{\omega_{-1}-((m+1)/n)+1}),$$

provided that $L=[(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$ *(and not necessarily* $2\varepsilon < n$*), we have*

$$\int_x^\infty \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. We use the asymptotic expansion for $I_{-1}(\lambda_k t)$, and then Lemma 11 gives the required result.

LEMMA 13. *If* $nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leq \mu < (n+\delta)x^{1-(1/n)}$, *and* $0<\gamma<1$, $0<2\varepsilon/n<1$, then

$$\begin{aligned}\int_{x(1-\gamma)}^{x} &\exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \\ &= \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b_\nu \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} \\ &\quad + b'_\nu \exp(2\pi i\eta\psi(x)) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\Bigg) \\ &\quad - \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)}\Bigg(b_\nu \exp(2\pi i\eta\varphi\{x(1-\gamma)\}) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x(1-\gamma),\mu)}{(2\pi i\eta)^{l+1}} \\ &\quad + b'_\nu \exp(2\pi i\eta\psi\{x(1-\gamma)\}) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x(1-\gamma),\mu)}{(2\pi i\eta)^{l+1}}\Bigg) \\ &\quad + O(x^{\omega_{-1}-((m+1)/n)+1}),\end{aligned}$$

provided that $L = [(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$, *then we have*

$$\int_{x(1-\gamma)}^{x} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. We shall first estimate

$$\int_{x(1-\gamma)}^{x} t^a \exp(2\pi i\eta\varphi(t))\, dt, \quad \text{with} \quad a = a(\nu) = \omega_{-1} - \frac{\nu}{n}.$$

Here $t \leq x$, $(\mu/n) - t^{1-(1/n)} \geq (\mu/n) - x^{1-(1/n)} \geq x^{1-(1/n)-(\varepsilon/n)}/n$, so that

$$\left|\frac{\frac{\mu}{n} t^{(1/n)-1}}{1-\frac{\mu}{n} t^{(1/n)-1}}\right| = \left|\frac{\frac{\mu}{n}}{t^{1-(1/n)} - \frac{\mu}{n}}\right| = \left|\frac{-\frac{\mu}{n}}{\frac{\mu}{n} - t^{1-(1/n)}}\right| \leq \frac{x^{1-(1/n)}}{\frac{1}{n} x^{1-(1/n)-(\varepsilon/n)}} \leq nx^{\varepsilon/n}.$$

Hence

$$|F_{L+1,\nu}(t)\cdot\varphi'(t)|\ll\frac{x^{(L+1)(\varepsilon/n)}\cdot t^{a}}{\left|t^{1-(1/n)}-\dfrac{\mu}{n}\right|^{L+1}\cdot t^{(L+1)/n}}$$
$$\ll x^{(L+1)(\varepsilon/n)}\cdot t^{a-((L+1)/n)}\cdot x^{-(L+1)(1-(1/n)-(\varepsilon/n))}$$

It follows that

$$\int_{x(1-\gamma)}^{x}F_{L+1,\nu}(t)\cdot\varphi'(t)\cdot\exp(2\pi i\eta\varphi(t))\,dt\ll x^{(L+1)(2\varepsilon/n)-(L+1)(1-(1/n))}$$
$$\times\int_{x(1-\gamma)}^{x}t^{a-(L+1)/n}\,dt$$
$$\ll x^{a-L+(L+1)(2\varepsilon/n)}.$$

Hence

$$\int_{x(1-\gamma)}^{x}t^{a}\exp(2\pi i\eta\varphi(t))\,dt=\exp(2\pi i\eta\varphi(x))\sum_{l=0}^{L}\frac{(-1)^{l}F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$
$$-\exp(2\pi i\eta\phi(x(1-\gamma)))\sum_{l=0}^{L}\frac{(-1)^{l}F_{l,\nu}(x(1-\gamma),\mu)}{(2\pi i\eta)^{l+1}}$$
$$+O(x^{a-L+(L+1)(2\varepsilon/n)}).$$

An analogous result holds with ψ in place of φ, and $G_{l,\nu}$ in place of $F_{l,\nu}$. If we use the asymptotic formula for $I_{-1}(\lambda_k t)$, and choose $L=[(m+1)/(n-2\varepsilon)]$, $0<2\varepsilon/n<1$, we get the first part of the lemma.

If $2\varepsilon/n\geqslant 1$, we simply use the asymptotic expansion for $I_{-1}(\lambda_k t)$, and observe that while

$$\left|\int_{x(1-\gamma)}^{x}t^{a}\exp(2\pi i\eta\psi(t))\,dt\right|\ll|G_{0,0}(x)|\ll|F_{0,0}(x)|,$$

the second mean-value theorem may be applied to the integral

$$\int_{x(1-\gamma)}^{x}t^{a}\exp(2\pi i\eta\varphi(t))\,dt.$$

LEMMA 14. *If* $nx^{1-(1/n)}+x^{1-(1/n)-(\varepsilon/n)}\leqslant\mu<(n+\delta)x^{1-(1/n)}$, $0<\gamma<1$, $\delta>0$,

$\varepsilon > 0$, *then*

$$\int_0^{x(1-\gamma)} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)} \Big(b_\nu \exp(2\pi i\eta\phi(x(1-\gamma))) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x(1-\gamma))}{(2\pi i\eta)^{l+1}}$$

$$+ b'_\nu \exp(2\pi i\eta\psi(x(1-\gamma))) \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x(1-\gamma))}{(2\pi i\eta)^{l+1}} \Big) + O(\lambda_k^{\omega_0-(m/n)-1}).$$

Proof. Since

$$\frac{\mu}{\{x(1-\gamma)\}^{1-(1/n)}} \geqslant \frac{n}{(1-\gamma)^{1-(1/n)}} > (n+\delta_1), \quad \text{say},$$

an application of Lemma 10 gives the result. Note that $\omega_0 - (m/n) - 1 = \omega_{-1} - (m+1)/n$.

LEMMA 15. *If* $nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leqslant \mu < (n+\delta)x^{1-(1/n)}$, $0 < \gamma < 1$, $\delta > 0$, $0 < 2\varepsilon/n < 1$, *then*

$$\int_0^{x} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^{m} \lambda_k^{\omega_{-1}-(\nu/n)} \Big(b_\nu \exp(2\pi i\eta\varphi(x)) \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + b'_\nu \exp(2\pi i\eta\psi(x))$$

$$\times \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} \Big) + O(x^{\omega_{-1}-((m+1)/n)+1}) + O(\lambda_k^{\omega_{-1}-((m+1)/n)}),$$

where $L = [(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$, *then*

$$\frac{I(0)}{\lambda_k} + \int_0^{x} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. This results from combining Lemmas 14 and 13.

LEMMA 16. *If* $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leqslant nx^{1-(1/n)}$, *and* $0 < \gamma < 1$, $\varepsilon > 0$, *then*

$$\int_{x(1+\gamma)}^{\infty} t^a \exp(2\pi i\eta\varphi(t))\, dt$$

$$= \exp(2\pi i\eta\varphi(x(1+\gamma))) \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x(1+\gamma))}{(2\pi i\eta)^{l+1}} + O(x^{a-L-1}),$$

where $a = a(\nu) = \omega_{-1} - (\nu/n)$; $\nu = 0, 1, 2, \ldots$.

Proof. Here we have

$$\frac{\mu}{\{x(1+\gamma)\}^{1-(1/n)}} \leqslant \frac{n}{(1+\gamma)^{1-(1/n)}} \leqslant n-\delta_1, \quad \text{say,}$$

so that the required result follows from Lemma 6.

LEMMA 17. *If* $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leqslant nx^{1-(1/n)}$, $0<\gamma<1$, γ *sufficiently small, and* $\varepsilon > 0$, *then*

$$\int_x^{x(1+\gamma)} t^a \exp(2\pi i\eta\varphi(t))\,dt$$

$$= \tfrac{1}{2}nx^{a+1}\left\{\sum_{p=0}^{N-1} \frac{\{A_p(1+\gamma)\exp(2\pi i\eta\phi(x(1+\gamma))) - B_p(1)\exp(2\pi i\eta\varphi(x))\}}{(2\pi x\eta)^{p+1}} + \sum_{m=0}^{N-1} \frac{\alpha'_m \exp(2\pi i x\eta P(v_0))}{(2\pi x\eta)^{m+(1/2)}}\right.$$

$$\left. - \sum_{m=0}^{N-1} \frac{\beta'_m \exp(2\pi i x\eta P(v_0))}{(2\pi x\eta)^m} \int_0^\tau s^{-1/2}\exp(2\pi i x\eta s)\,ds\right\} + O(x^{a-N}),$$

where $P(v) = v^n - \alpha v$, $\alpha = \mu x^{(1/n)-1}$, $v_0 = (\alpha/n)^{1/(n-1)}$, $\tau = P(1) - P(v_0)$, *and*

$$\int_0^\tau s^{-1/2}\exp(2\pi i x\eta s)\,ds = O(x^{-1/2}),$$

uniformly in τ. *The terms* A_p, B_p, α'_m, β'_m *depend on* α, *and, for fixed p or m, are continuous functions of* α *in a neighbourhood of the point* $\alpha = n$.

Proof. We have

$$\int_x^{x(1+\gamma)} t^a \exp(2\pi i\eta\varphi(t))\,dt$$

$$= x^{a+1}\int_1^{1+\gamma} u^a \exp(2\pi i x\eta(u - \mu x^{(1/n)-1}\cdot u^{1/n}))\,du, \qquad (t = ux)$$

$$= x^{a+1}\int_1^{1+\gamma} u^a \exp(2\pi i x\eta(u - \alpha u^{1/n}))\,du$$

$$= nx^{a+1}\int_1^{(1+\gamma)^{1/n}} \exp(2\pi i x\eta P(v))\, v^{an+n-1}\,dv. \qquad (u = v^n)$$

Now $\partial P/\partial v = 0$, for $v = (\alpha/n)^{1/(n-1)} = v_0$. In order to make the substitution $P(v) - P(v_0) = s$, consider the expansion:

$$P(v) - P(v_0) = (f(v - v_0))^2,$$

where $f(z)$ is a power series of the form $(z/\sqrt{2})\sqrt{P''(v_0)} + \cdots$. (Note that $P'(v_0) = 0$, while $P'(v) < 0$ for $v < v_0$, and $P'(v) > 0$ for $v > v_0$. Further $P''(v_0)$ is close to $n(n-1)$ if x is large, in the range of μ under consideration.) We then have $f(v - v_0) = s^{1/2}$. Since there exists an F, such that $F[f(z)] = z$ with

$$F(z) = \frac{\sqrt{2}}{\sqrt{P''(v_0)}} z + \cdots,$$

we have

$$v - v_0 = F(s^{1/2}),$$

where $F(s)$ is holomorphic in a neighbourhood of the origin, say in $|s| < 2R$, which is independent of α. The above integral therefore equals

$$\tfrac{1}{2} n x^{a+1} \cdot \exp(2\pi i x \eta P(v_0))$$

$$\times \int_{s = P(1) - P(v_0)}^{s = P((1+\gamma)^{1/n}) - P(v_0)} (v_0 + F(s^{1/2}))^{an+n-1} s^{-1/2} F'(s^{1/2}) \exp(2\pi i x \eta s)\, ds,$$

provided that γ is so small that $P((1+\gamma)^{1/n}) - P(v_0) < R$, say. Now set

$$g(s) = (v_0 + F(s))^{an+n-1} F'(s),$$

$$\tau = P(1) - P(v_0), \qquad C = P((1+\gamma)^{1/n}) - P(v_0).$$

Note that $\tau = O(x^{-2\varepsilon/n})$, while C is bounded below by a positive constant. We then have

$$\int_x^{x(1+\gamma)} t^a \exp(2\pi i \eta \varphi(t))\, dt$$

$$= \tfrac{1}{2} n x^{a+1} \exp(2\pi i x \eta P(v_0)) \int_\tau^C g(s^{1/2}) s^{-1/2} \exp(2\pi i x \eta s)\, ds.$$

Now $g(s) = \sum_{m=0}^{\infty} g_m s^m$, uniformly for $|s| \leq C < R$. For $N \geq 1$, let

$$g(s) = \sum_{m \leq 2N} g_m s^m + \sum_{m=2N+1}^{\infty} g_m s^m = \sum_{m \leq 2N} g_m s^m + \tilde{g}_N(s), \quad \text{say.}$$

Then the function $H_N(s) = \tilde{g}_N(s) s^{-1/2}$ is N times continuously differentiable in

$s \geqslant 0$, and the first $N-1$ derivatives vanish at $s=0$. Hence

$$\int_0^C H_N(s) \exp(2\pi i x\eta s)\, ds = \sum_{p=0}^{N-1} \frac{(-1)^p \exp(2\pi i x\eta C)\, H_p(C)}{(2\pi i x\eta)^{p+1}} + O(x^{-N-1}).$$

Now

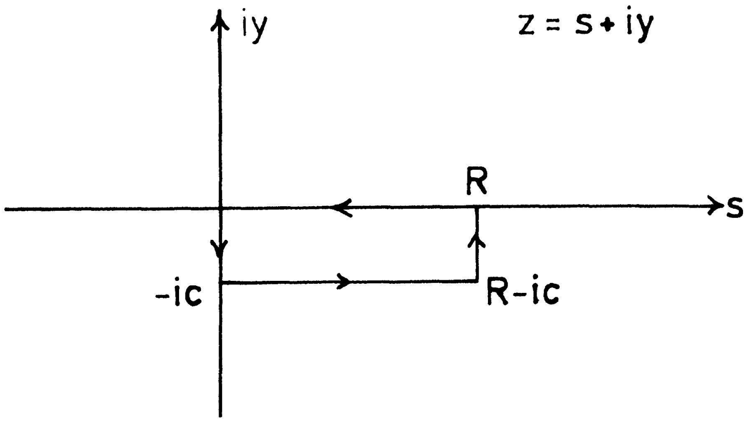
$$\int_0^C s^{(m-1)/2} \exp(2\pi i x\eta s)\, ds$$

$$= \begin{cases} \displaystyle\sum_{p=0}^{((m-1)/2)-1} \frac{a_p(C) \exp(2\pi i x\eta C)}{(2\pi i x\eta)^{p+1}} + \frac{(-1)^{(m-1)/2}\left(\frac{m-1}{2}\right)!\, (\exp(2\pi i x\eta C)-1)}{(2\pi i x\eta)^{((m-1)/2)-1}}, \\ \qquad \text{if } m \text{ is an odd integer} \geqslant 1; \\ \displaystyle\sum_{p=0}^{(m-1)/2} \frac{a'_p(C) \exp(2\pi i x\eta C)}{(2\pi i x\eta)^{p+1}} + \frac{(-1)^{m/2}\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\cdots\frac{1}{2}}{(2\pi i x\eta)^{m/2}} \\ \qquad \times \displaystyle\int_0^C s^{-1/2} \exp(2\pi i x\eta s)\, ds, \\ \qquad \text{if } m \text{ is an even integer} \geqslant 0. \end{cases}$$

On the other hand, as $x \to \infty$,

$$\int_0^C s^{-1/2} \exp(2\pi i x\eta s)\, ds = \frac{b_0}{(2\pi x\eta)^{1/2}} + \sum_{r=0}^{N-1} \frac{b_r(C) \exp(2\pi i x\eta C)}{(2\pi x\eta)^{r+1}} + O(x^{-N-1}).$$

To prove this, one has only to consider the contour integral $\int z^{-1/2} \exp(-xz)\, dz$, for $x > 0$, in the complex z-plane, taken along the rectangle with corners at the points $z=0$, $-iC$, $R-iC$, and R, with $R>0$, and then let $R \to \infty$. We then get



$$\int_0^C y^{-1/2} \exp(ixy)\, dy$$

$$= i^{-3/2}\left[\int_0^\infty s^{-1/2} \exp(-xs)\, ds - \exp(ixC) \int_0^\infty (s-iC)^{-1/2} \exp(-xs)\, ds\right].$$

If we note that the first integral equals $\sqrt{\pi x^{-1/2}}$, and integrate the second integral by parts a sufficient number of times, we get the result.

Putting the above results together, we obtain

$$\int_0^C g(s^{1/2})s^{-1/2}\exp(2\pi ix\eta s)\,ds=\sum_{p=0}^{N-1}\frac{\alpha_p(C)\exp(2\pi ix\eta C)}{(2\pi ix\eta)^{p+1}}$$
$$+\sum_{m=0}^{N}\frac{\alpha'_m}{(2\pi x\eta)^{m+(1/2)}}+\sum_{q=0}^{N-1}\frac{\alpha''_q}{(2\pi x\eta)^{q+1}}+O(x^{-N-1}).$$

In the same way we see that

$$\int_0^\tau g(s^{1/2})s^{-1/2}\exp(2\pi ix\eta s)\,ds=\sum_{p=0}^{N-1}\frac{\beta_p(\tau)\exp(2\pi ix\eta\tau)}{(2\pi ix\eta)^{p+1}}+\sum_{q=0}^{N-1}\frac{\alpha''_q}{(2\pi x\eta)^{q+1}}$$
$$+\sum_{m=0}^{N}\frac{\beta'_m}{(2\pi x\eta)^m}\int_0^\tau s^{-1/2}\exp(2\pi ix\eta s)\,ds+O(x^{-N-1}).$$

Thus

$$\int_\tau^C g(s^{1/2})s^{-1/2}\exp(2\pi ix\eta s)\,ds=\sum_{p=0}^{N-1}\frac{\alpha_p(C)\exp(2\pi ix\eta C)-\beta_p(\tau)\exp(2\pi ix\eta\tau)}{(2\pi ix\eta)^{p+1}}$$
$$+\sum_{m=0}^{N}\frac{\alpha'_m}{(2\pi x\eta)^{m+(1/2)}}-\sum_{m=0}^{N}\frac{\beta'_m}{(2\pi x\eta)^m}\int_0^\tau s^{-1/2}\exp(2\pi ix\eta s)\,ds+O(x^{-N-1}).$$

Hence

$$\int_x^{x(1+\gamma)} t^a\exp(2\pi i\eta\varphi(t))\,dt$$
$$=\tfrac{1}{2}nx^{a+1}\Bigg\{\sum_{p=0}^{N-1}\frac{(A_p(1+\gamma)\exp(2\pi i\eta\phi(x(1+\gamma)))-B_p(1)\exp(2\pi i\eta\varphi(x)))}{(2\pi x\eta)^{p+1}}$$
$$+\sum_{m=0}^{N}\frac{\alpha'_m\exp(2\pi ix\eta P(v_0))}{(2\pi x\eta)^{m+(1/2)}}-\sum_{m=0}^{N}\beta'_m\exp(2\pi ix\eta P(v_0))$$
$$\times\int_0^\tau s^{-1/2}\exp(2\pi ix\eta s)\,ds\Bigg\}+O(x^{a-N}),$$

since $xP((1+\gamma)^{1/n})=\varphi(x(1+\gamma))$, and $xP(1)=\varphi(x)$. (The constants in the above formula are independent of N.) We have $xP(v_0)=\mu^{n/(n-1)}\cdot n^{-1/(n-1)}$, $\mu=(h/2\pi\eta)\cdot\lambda_k^{1/n}$. Note also that if

$$K(\tau,x)=\int_0^\tau s^{-1/2}\exp(2\pi ix\eta s)\,ds,$$

then

$$K(\tau, x)=x^{1/2}\int_0^{\tau x} y^{-1/2}\exp(2\pi i\eta y)\cdot x^{-1}\,dy=x^{-1/2}\int_0^{\tau x} y^{-1/2}\exp(2\pi i\eta y)\,dy.$$

Since $K(b)=\int_0^b s^{-1/2}\exp(2\pi i s)\,ds$ tends to zero as $b\to 0+$, and $K(b)$ tends to a finite limit as $b\to+\infty$, $K(b)$ is bounded everywhere, hence

$$K(\tau, x)=O(x^{-1/2})$$

uniformly in τ.

LEMMA 18. *If* $nx^{1-(1/n)}-x^{1-(1/n)-(\varepsilon/n)}<\mu\leq nx^{1-(1/n)}$, $\varepsilon>0$, *then*

$$\int_x^\infty t^a\exp(2\pi i\eta\psi(t))\,dt=\exp(2\pi i\eta\psi(x))\sum_{l=0}^{L}\frac{(-1)^{l+1}G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+O(x^{a-L-1}),$$

where $a=a(\nu)=\omega_{-1}-(\nu/n)$.

The proof here is simpler than in the case of $\varphi(t)$, since $\psi(t)=t+\mu t^{1/n}$, so that $\psi'(t)$ is bounded both above and below for $t>0$. (See Lemma 6.)

LEMMA 19. *If* $nx^{1-(1/n)}-x^{1-(1/n)-(\varepsilon/n)}<\mu\leq nx^{1-(1/n)}$, $\varepsilon>0$, *then*

$$\begin{aligned}\int_x^\infty \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt=&\sum_{\nu=0}^{nN}\sum_{m=0}^{N-1}u_{\nu,m}x^{a-(\nu/n)+1}\cdot\lambda_k^{a-(\nu/n)}\frac{\exp(2\pi i\eta\varphi(x))}{(2\pi x\eta)^{m+1}}\\&+\sum_{\nu=0}^{nN}\sum_{m=0}^{N-1}u'_{\nu,m}x^{a-(\nu/n)+1}\cdot\lambda_k^{a-(\nu/n)}\frac{\exp(2\pi i\eta xP(v_0))}{(2\pi x\eta)^{m+(1/2)}}\\&+\sum_{\nu=0}^{nN}\sum_{m=0}^{N-1}u''_{\nu,m}x^{a-(\nu/n)+1}\cdot\lambda_k^{a-(\nu/n)}\cdot\frac{\exp(2\pi i\eta xP(v_0))}{(2\pi x\eta)^{m}}\\&\times\int_0^\tau s^{-1/2}\exp(2\pi i x\eta s)\,ds\\&+\sum_{\nu=0}^{nN}\sum_{m=0}^{N-1}\frac{(-1)^l G_{m,\nu}(x)}{(2\pi i\eta)^{m+1}}\cdot\lambda_k^{a-(\nu/n)}+O(x^{a-N}).\end{aligned}$$

The coefficients $u_{\nu,m}$, $u'_{\nu,m}$, $u''_{\nu,m}$ *are continuous functions of* α *for fixed* ν *and* m.

The proof follows upon putting together Lemmas 16, 17, and 18, after using the asymptotic expansion for $I_{-1}(\lambda_k t)$, and then noting that the integral on the left-hand side is independent of γ, hence in the expansion for it (obtained with the help of those lemmas) in decreasing powers of x, all the terms involving γ should vanish (by the kind of argument used in the proof of Lemma 10).

LEMMA 20. *If* $nx^{1-(1/n)}<\mu\leq nx^{1-(1/n)}+x^{1-(1/n)-\varepsilon'}$, $0<\varepsilon'<1$, *then*

$$\int_0^x \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\, dt$$

has the same expansion as in Lemma 19, *but for the additional summand* $-I_0(0)/\lambda_k$.

The proof follows upon considering the integrals $\int_0^{x(1-\gamma)}$ and $\int_{x(1-\gamma)}^x$, with $0<\gamma<1$. The first integral can be treated as in Lemma 10, and the second as in Lemma 17.

LEMMA 21. *If* $\lambda_N\leq x<\lambda_{N+1}$, *then*

$$\begin{aligned}\int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\beta(t)\, dt &= \exp(2\pi i\eta\varphi(x)) \sum c_{j,p,q,\nu}\lambda_k^{a-j}x^{a-j}(\varphi^{(p)}(x))^q\\ &\quad\times\int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l\cdot\beta(t)\cdot\exp(2\pi i\eta(t-x)\varphi'(x))\, dt\\ &\quad+\exp(2\pi i\eta\psi(x))\sum c'_{j,p,q,\nu}\lambda_k^{a-j}x^{a-j}(\psi^{(p)}(x))^q\\ &\quad\times\int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l\cdot\beta(t)\cdot\exp(2\pi i\eta(t-x)\psi'(x))\, dt\\ &\quad+O(\lambda_k^{1/n}x^{(1/n)-M-1}),\end{aligned}$$

where $a=\omega_{-1}-(\nu/n)$, *and the summation on the right-hand side extends over:* $0\leq\nu<n(M+1)$, $0\leq j\leq M$, $0\leq q\leq M$, $2\leq p\leq M$, $j+pq=l$, *all integers, and* β *is the function defined as in* §3.

Proof. From the asymptotic expansion of $I_{-1}(t)$, we have

$$\begin{aligned}\int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i\eta t)\, I_{-1}(\lambda_k t)\cdot\beta(t)\, dt &= \sum_{\nu=0}^{m} e'_\nu\lambda_k^{\omega_{-1}-(\nu/n)}\int_{\lambda_N}^{\lambda_{N+1}} t^{\omega_{-1}-(\nu/n)}\cdot\exp(2\pi i\eta\varphi(t))\\ &\quad\times\beta(t)\, dt+\sum_{\nu=0}^{m} e'_\nu\lambda_k^{\omega_{-1}-(\nu/n)}\int_{\lambda_N}^{\lambda_{N+1}} t^{\omega_{-1}-(\nu/n)}\cdot\exp(2\pi i\eta\psi(t))\\ &\quad\times\beta(t)\, dt+O(x^{\omega_{-1}-(m+1/n)}).\end{aligned}$$

Now $t^a = x^a + \sum_{j\geq 1} c_j \cdot x^{a-j}(t-x)^j$, for $\lambda_N \leq t \leq \lambda_{N+1}$, and $(t-x)^j = O(1)$, for $j \geq 1$. Further

$$\varphi(t) = \varphi(x) + \sum_{1\leq p\leq M} \frac{(t-x)^p}{p!} \cdot \varphi^{(p)}(x) + O(\lambda_k^{1/n} \cdot x^{(1/n)-M-1}).$$

Hence

$$\begin{aligned}
&\int_{\lambda_N}^{\lambda_{N+1}} t^a \exp(2\pi i\eta\varphi(t)) \cdot \beta(t) \cdot dt \\
&= \sum_{j=0}^{M} c_j x^{a-j} \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^j \exp(2\pi i\eta\varphi(t)) \cdot \beta(t)\, dt + O(x^{-M-1}) \\
&= \exp(2\pi i\eta\varphi(x)) \sum_{j=0}^{M} c_j x^{a-j} \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^j \exp(2\pi i\eta(t-x)\varphi'(x)) \cdot \beta(t) \\
&\quad \times \exp\left\{(2\pi i\eta)\left(\sum_{2\leq p\leq M} \frac{(t-x)^p}{p!}\varphi^{(p)}(x)\right)\right\} dt + O(x^{-M-1}) + O(\lambda_k^{1/n} x^{(1/n)-M-1}) \\
&= \exp(2\pi i\eta\varphi(x)) \sum_{j,p,q} c'_{j,p,q} x^{a-j} (\varphi^{(p)}(x))^q \\
&\quad \times \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l \cdot \beta(t) \cdot \exp(2\pi i\eta(t-x)\varphi'(x))\, dt + O(\lambda_k^{1/n} x^{(1/n)-M-1}),
\end{aligned}$$

the last sum extending over: $0 \leq j \leq M$, $2 \leq p \leq M$, $q \geq 0$, $j + pq = l$, all integers. Note that the last integral equals

$$\int_{\lambda_N - x}^{\lambda_{N+1}-x} t^{j+pq} \cdot \beta(x+t) \cdot \exp(2\pi i\eta t\varphi'(x))\, dt = O(1),$$

and, if we integrate by parts, we see that this is also $\ll 1/|\varphi'(x)|$.

A similar result holds with ψ in place of φ. If we choose $m+1 \geq n(M+1)$, we get the lemma.

LEMMA 22. *For the first integral in Lemma* 1, *we have the estimate:*

$$(-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) \cdot dt = O(1).$$

Proof. By partial integration, we see that

$$(-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t)\cdot Q_r(t)\,dt$$

$$= \int_0^{\lambda_{N+1}} F'(t)\cdot Q_0(t)\,dt = -Q_0(0) - \int_0^{\lambda_{N+1}} F(t)\cdot Q_{-1}(t)\cdot dt$$

$$= -Q_0(0) - R_0\left\{\int_0^{\lambda_N} \exp(2\pi i\eta t)\,dt + \int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i\eta t)\,\beta(t)\,dt\right\}$$

$$= O(1),$$

since $\lambda_{N+1}-\lambda_N$, and β are bounded; and

$$Q_{-1}(x) = \frac{1}{2\pi i}\int_{\mathscr{C}} B^{-s}\zeta_K(s)x^{s-1}\,ds, \qquad x>0,$$

so that $Q_{-1}(x)$ equals B^{-1} times the residue of $\zeta_K(s)$ at $s=1$, and $Q_0(0)$ (which is defined by continuity) equals $\zeta_K(0)$.

§5. Proof of theorem

If $0<\varepsilon'<1/n$, $\lambda_N\leqslant x<\lambda_{N+1}$, and $\eta>0$, we have, by Lemma 1,

$$\sum_{k=1}^{N} a_k \exp(2\pi i\eta\lambda_k) = W_1+W_2+W_3+W_4+W_5+W_6, \tag{5.1}$$

where

$$W_1 = (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t)Q_r(t)\,dt,$$

$$W_2 = O(x^{-q'}), \quad \text{for any} \quad q'>0,$$

$$W_3 = -\sum_{\mu\leqslant nx^{1-(1/n)+\varepsilon'}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t)\cdot\exp(2\pi i\eta t)\cdot\beta(t)\,dt,$$

$$W_4 = -\sum_{\mu\leqslant nx^{1-(1/n)}} a_k \int_0^{\infty} \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt - \sum_{\mu\leqslant nx^{1-(1/n)}} \frac{a_k I_0(0)}{\lambda_k},$$

$$W_5 = \sum_{\mu\leqslant nx^{1-(1/n)}} a_k \int_{\lambda_N}^{\infty} \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt,$$

$$W_6 = -\sum_{nx^{1-(1/n)}<\mu\leqslant nx^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i\eta t)\,I_{-1}(\lambda_k t)\,dt - \sum_{nx^{1-(1/n)}<\mu\leqslant nx^{1-(1/n)+\varepsilon'}} \frac{a_k I_0(0)}{\lambda_k}.$$

We suppose, to begin with, that $\lambda_N = x$. By Lemma 22, we have

$$W_1 = O(1). \tag{5.2}$$

To estimate W_5 we write it as a sum of three separate sums, namely

$$W_5 = W_{5,1} + W_{5,2} + W_{5,3}$$
$$= \sum_{\mu \leq (n-\delta)x^{1-(1/n)}} + \sum_{(n-\delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)}} + \sum_{nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}}.$$

Now, by Lemma 7, we have

$$W_{5,1} = \sum_{\mu \leq (n-\delta)x^{1-(1/n)}} a_k \int_x^\infty \exp(2\pi i \eta t)\, I_{-1}(\lambda_k t)\, dt$$
$$\ll x^a \sum_{\lambda_k \leq (c_0-\delta_1)x^{n-1}} a_k \lambda_k^a \ll x^{(n/2)-(1/2)}, \tag{5.3}$$

if we note that, in this range, $F_{0,0}(x) = O(x^a)$, $G_{0,0}(x) = O(x^a)$, (where $a = (1/2n) - \frac{1}{2}$), and use the fact that for $-1 < \alpha \leq 0$, we have

$$\sum_{\lambda_k \leq y} a_k \lambda_k^\alpha \sim c y^{1+\alpha}, \quad \text{as} \quad y \to \infty, \tag{5.4}$$

since $\sum_{\lambda_k \leq y} a_k \sim \lambda' y$.

By the last part of Lemma 12, we have

$$W_{5,2} \ll \sum_{(c_0-\delta_1)x^{n-1} < \lambda_k < c_0 x^{n-1} - c_1 x^{n-1-\varepsilon/n}} \left(\frac{a_k \lambda_k^a x^a}{1 - \frac{1}{n} \mu x^{(1/n)-1}} \right)$$
$$\ll x^{a + a(n-1)} \sum_{(1-\delta_2)y < \lambda_k < y-z} \frac{a_k}{1 - \lambda_k^{1/n} y^{-1/n}},$$

with $y = c_0 x^{n-1}$, $z = c_1 x^{n-1-\varepsilon/n}$. (Here, as elsewhere, we take $\delta_1 > 0$, $\delta_2 > 0$, and sufficiently small. We adopt the usual convention that an empty sum is zero and an empty product is 1.) Since $A(t) = \sum_{\lambda_k \leq t} a_k = \lambda' t + R(t)$, where $R(t) = O(t^{1-2/(n+1)})$, we have

$$\sum_{(1-\delta_2)y < \lambda_k < y-z} \frac{a_k}{1 - \lambda_k^{1/n} y^{-1/n}} \ll \int_{(1-\delta_2)y}^{y-z} \frac{dA(t)}{1 - (t/y)^{1/n}}$$
$$= \lambda' \int_{(1-\delta_2)y}^{y-z} \frac{dt}{1 - (t/y)^{1/n}} + \int_{(1-\delta_2)y}^{y-z} \frac{dR(t)}{1 - (t/y)^{1/n}}.$$

Now

$$c\int_{(1-\delta_2)y}^{y-z} \frac{dt}{1-(t/y)^{1/n}} = \lambda' y \int_{(1-\delta_2)}^{1-z/y} \frac{(1-s)}{(1-s^{1/n})} \frac{1}{(1-s)}\, ds$$

$$= O\left(y \log \frac{y}{z}\right) = O(x^{n-1} \log(1+x)).$$

Similarly

$$\int_{(1-\delta_2)y}^{y-z} \frac{dR(t)}{1-(t/y)^{1/n}} = O\left(y^{1-n} \cdot \frac{y}{z}\right) = O(x^{n-1}), \quad \text{if} \quad 0 < \varepsilon \leqslant n-1.$$

Thus we have altogether

$$W_{5,2} = O(x^{(n-1)/2} \log(1+x)), \quad \text{if} \quad 0 < \varepsilon \leqslant n-1. \tag{5.5}$$

To handle $W_{5,3}$ we use Lemma 17, and obtain

$$W_{5,3} = \sum_{nx^{1-(1/n)}-x^{1-(1/n)-(\varepsilon/n)} < \mu \leqslant nx^{1-(1/n)}} a_k \int_x^\infty \exp(2\pi i \eta t)\, I_{-1}(\lambda_k t)\, dt$$

$$\ll \sum_{c_0 x^{n-1} - c_2 x^{n-1-\varepsilon/n} < \lambda_k \leqslant c_0 x^{n-1}} a_k \cdot \lambda_k^a \cdot x^{a+(1/2)} \qquad \left(a = \frac{1}{2n} - \frac{1}{2}\right)$$

$$\ll x^{1/2n} \sum a_k \lambda_k^a \ll x^{(1/2n)+a(n-1)} \sum_{c_0 x^{n-1} - c_2 x^{n-1-\varepsilon/n} < \lambda_k \leqslant c_0 x^{n-1}} a_k, \quad \text{since} \quad \lambda_k \ll x^{n-1}$$

$$\ll x^{(1/2n)+a(n-1)+(n-1-(\varepsilon/n))} \ll x^{(n/2)-(\varepsilon/n)}, \tag{5.6}$$

since $\sum_{y-z<\lambda_k \leqslant y} a_k = c_5 z + O(y^{1-(1/n)})$, say, for $0 < z < y$. Now (5.6), (5.5), and (5.3) give

$$W_5 = O(x^{(n-1)/2}) + O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\varepsilon/n)}), \tag{5.7}$$

if $\varepsilon \leqslant n-1$.

The estimation of W_6 is similar. We first split it up into three parts:

$$W_6 = W_{6,1} + W_{6,2} + W_{6,3}$$

$$= -\sum_{(n+\delta)x^{1-(1/n)} \leqslant \mu \leqslant nx^{1-(1/n)+\varepsilon'}} - \sum_{nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leqslant \mu < (n+\delta)x^{1-(1/n)}}$$

$$- \sum_{nx^{1-(1/n)} < \mu \leqslant nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)}},$$

in each of which there are two summands. We use Lemma 10 in $W_{6,1}$ and obtain

$$W_{6,1} = -\sum_{(n+\delta)x^{1-(1/n)} \leqslant \mu \leqslant nx^{1-(1/n)+\varepsilon'}} a_k \int_0^x \exp(2\pi i \eta t)\, I_{-1}(\lambda_k t)\, dt$$
$$= O\Big(\sum a_k \lambda_k^a \cdot |F_{0,0}(x)|\Big) = O\Big[\sum a_k \cdot \lambda_k^a \cdot x^a \Big(\frac{1}{n}\mu x^{(1/n)-1} - 1\Big)^{-1}\Big]$$
$$= O(x^{(n-1)/2} \log(1+x)), \quad \text{if} \quad 0 < \varepsilon \leqslant n-1, \tag{5.8}$$

as in the case of $W_{5,2}$ (see the proof of (5.5)). Note that the sum $-\sum a_k I_0(0)/\lambda_k$ cancels out in this range.

By Lemma 15, we obtain, as in (5.5),

$$W_{6,2} = O(x^{(n-1)/2} \log(1+x)), \quad \text{if} \quad 0 < \varepsilon \leqslant n-1. \tag{5.9}$$

Finally, by Lemma 20 we obtain, as in (5.6),

$$W_{6,3} = O(x^{(n/2)-(\varepsilon/n)}). \tag{5.10}$$

Hence

$$W_6 = O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\varepsilon/n)}), \qquad 0 < \varepsilon \leqslant n-1. \tag{5.11}$$

To estimate W_3, we note that the principal term in the expansion given by Lemma 21 is a constant multiple of

$$\sum_{\mu \leqslant nx^{1-(1/n)+\varepsilon'}} a_k \exp(2\pi i \eta \varphi(x))\, \lambda_k^a x^a \int_{\lambda_N}^{\lambda_{N+1}} \beta(t) \exp(2\pi i \eta (t-x)\varphi'(x))\, dt,$$

together with a similar term with ψ in place of φ. It follows that

$$W_3 \ll \sum_{\mu \leqslant (n+\delta)x^{1-(1/n)}} a_k \cdot \lambda_k^a \cdot x^a + \sum_{(n+\delta)x^{1-(1/n)} < \mu \leqslant nx^{1-(1/n)+\varepsilon'}} \left(\frac{a_k \lambda_k^a x^a}{\frac{1}{n}\mu x^{(1/n)-1} - 1}\right)$$
$$\ll x^{(n-1)/2} \log(1+x), \tag{5.12}$$

since $0 < \varepsilon' < 1/n \leqslant (n-1)/n$. Note that the estimate of the first sum is similar to that of (5.3), and that of the second similar to (5.5).

By Lemma 4, we have

$$W_4 = c_4 \sum_{\mu \leqslant nx^{1-(1/n)}} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i m_k^{1/(n-1)} \cdot q)\} + O(x^{(n/2)-1}), \tag{5.13}$$

where c_4 is a non-zero constant, $m_k = \lambda_k/2\pi\eta$, $q=(n-1)2^{r_1/(n-1)}$. Since $-1 < \{(2-n)/2(n-1)\} \leq 0$, for $n \geq 2$, we have only to use (5.4). Note that $-\sum (a_k/\lambda_k) I_0(0)$ cancels out $+\sum \Delta(1)/\Delta(0) \cdot a_k/\lambda_k$.

Combining (5.1), (5.2), (5.7), (5.11), (5.12), and (5.13), we get

$$\sum_{\mu \leq nx^{1-(1/n)}} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i m_k^{1/(n-1)} \cdot q)\} - c \sum_{k=1}^{N} a_k \exp(2\pi i \eta \lambda_k)$$
$$= O(x^{(n/2)-1}) + O(1) + O(x^{-q'}) + O(x^{(n-1)/2}) + O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\varepsilon/n)}), \quad (5.14)$$

where c is a constant, $n \geq 2$, $0 < \varepsilon \leq n-1$, $0 < \varepsilon' < 1/n$, q' is an arbitrarily given positive integer. If we choose $\varepsilon/n = \frac{1}{2}$, as we may, we get for $x = \lambda_N$, $n \geq 3$,

$$\sum_{\lambda_k \leq c_0 \lambda_N^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(x^{(n-1)/2} \log(1+x)), \quad (5.15)$$

where $H = q(2\pi\eta)^{-1/(n-1)}$, since

$$\sum_{k=1}^{N} a_k \exp(2\pi i \eta \lambda_k) = O(x) = O(x^{(n-1)/2}),$$

for $n \geq 3$. On the other hand, we have, for any $\varepsilon_1 > 0$, and $x = \lambda_N$,

$$\sum_{c_0 \lambda_N^{n-1} < \lambda_k < c_0 \lambda_{N+1}^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} \ll x^{(2-n)/2} \sum_{c_0 \lambda_N^{n-1} < \lambda_k < c_0 \lambda_{N+1}^{n-1}} a_k$$
$$\ll x^{(2-n)/2+\varepsilon_1} (\lambda_{N+1}^{n-1} - \lambda_N^{n-1})$$
$$\ll x^{n-2+1-(n/2)+\varepsilon_1} \ll x^{(n-1)/2},$$

since $\lambda_{N+1} \leq x + B$, so that $\lambda_{N+1}^{n-1} - \lambda_N^{n-1} \ll x^{n-2}$.

Thus it follows from (5.15) that

$$\sum_{\lambda_k \leq c_0 y^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(y^{(n-1)/2} \log(1+y)),$$

for $\lambda_N \leq y < \lambda_{N+1}$. From this we deduce that

$$\sum_{\lambda_k \leq c_0 y} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(y^{1/2} \log(1+y)),$$

and this implies, by partial summation, that

$$\sum_{\lambda_k \leq y} a_k \exp\{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O_\eta(y^{1-(1/2(n-1))} \log(1+y)),$$

where $H = q/(2\pi\eta)^{1/(n-1)}$. Note that c_0 involves η, where η is arbitrary and positive. By taking conjugates, if necessary, we obtain the theorem.

Remark. It is possible to prove a corresponding result for the coefficient sums of Dirichlet series satisfying a functional equation of the type treated in [1, 2]. The proof requires no new ideas. In particular, the result is valid for the zeta-function of an ideal class and Hecke's zeta-function with Grössencharacters.

We note that if $n = 2$, (5.14) reduces to the result given in [4].

We also note that considered as a result valid for *all* $n \geqslant 2$, the theorem is a "best possible" one, after Walfisz [6, p. 566], though the estimate can be improved for particular values of n. For $n \geqslant 7$, for example, partial summation will suffice.

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E.T.H. Zurich
and
The University of Chicago, Chicago, Ill., U.S.A.

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