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Exponential sums associated with the Dedekind zeta-function

by K. CHANDRASEKHARAN AND RAGHAVAN NARASIMHAN

(To Carl Ludwig Siegel for his eightieth birthday)

§1. Introduction

Let K be an algebraic number field of degree n , and let $\zeta_K(s)$ be the Dedekind zeta-function associated with it. For $\operatorname{Re} s > 1$, $\zeta_K(s) = \sum_{k=1}^{\infty} a_k k^{-s}$, where a_k denotes the number of integral ideals in K with norm k . The function $\zeta_K(s)$ is meromorphic in the complex s -plane with a simple pole at $s = 1$. If r_1 denotes the number of real conjugates of K , and $2r_2$ the number of imaginary conjugates, and D the discriminant, $\zeta_K(s)$ satisfies the functional equation $\xi(s) = \xi(1-s)$, where

$$\xi(s) = \Gamma^{r_1}(\tfrac{1}{2}s) \Gamma^{r_2}(s) B^{-s} \zeta_K(s),$$

with $B = 2^{r_2} \pi^{n/2} (|D|)^{-1/2}$, $r_1 + 2r_2 = n$. It is known that $a_k = O(k^\varepsilon)$, for every $\varepsilon > 0$ [2, p. 55], while

$$\sum_{k \leq x} a_k = \lambda x + O(x^{(n-1)/(n+1)}),$$

where λ is a positive constant determined by the field K [5, §2.6].

Our object is to prove the following

THEOREM. *If $n \geq 3$, η real, $\eta \neq 0$, then*

$$\sum_{k \leq x} a_k \exp(2\pi i k^{1/(n-1)} \eta) = O(x^{1-(1/2(n-1))} \log x),$$

for $x \geq 2$, the ‘ O ’ depending on η .

The case $n = 2$ was considered by us in two previous papers [3, 4]. In [3] we showed that there was a connexion between the order of magnitude of the corresponding sums and the existence of an infinity of zeros of the associated zeta-function on the critical line. In [4] we proved an approximate reciprocity

formula for the sum $\sum_{k \leq x} a_k \exp(2\pi i k \eta)$. For $n \geq 3$ we begin in the same way as before, and relate the sum $\sum_{k \leq x} a_k \exp(2\pi i k \eta)$ to the sum

$$\sum_{k \leq c_0 x^{n-1}} k^{(2-n)/2(n-1)} a_k \exp(2\pi i k^{1/(n-1)} \cdot \eta^{-1/(n-1)} \cdot H),$$

where H is a constant. The difference between the two sums can be expressed as a sum of terms for each of which we determine the asymptotic behaviour, which is different in different ranges of k . The principal terms in the asymptotic expansions give the required result with x^θ , for any positive θ , instead of $\log x$. Replacing the asymptotic expansion, in some places, by a direct estimate, which is slightly more sophisticated, we get the stated theorem.

§2. Preliminaries

It is known that the functional equation for $\zeta_K(s)$ implies for $\rho \geq 0$, ρ integral, the identity [1, (4.23)]

$$\frac{1}{\Gamma(\rho+1)} \sum'_{\lambda_k \leq x} a_k (x - \lambda_k)^\rho = Q_\rho(x) + \sum_{k=1}^{\infty} a_k \lambda_k^{-1-\rho} I_\rho(\lambda_k x), \quad (2.1)$$

for $x > 0$, provided that $\rho > \frac{1}{2}(n-1)$. The dash in the summation on the left-hand side of (2.1) indicates that if $\rho = 0$, and $x = \lambda_k$, the last term should be halved. Here $\lambda_k = B \cdot k$, B being defined as in §1, and

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{B^{-s} \zeta_K(s) \Gamma(s)}{\Gamma(s + \rho + 1)} x^{s+\rho} ds,$$

where $\mathcal{C}$ is a curve which encloses all the singularities of the integrand. Clearly

$$Q_\rho(x) = cx^{1+\rho} + \sum_{j=0}^p c_j x^{\rho-j},$$

where c equals the residue of $\zeta_K(s)$ at $s = 1$ multiplied by $\{B\Gamma(\rho+2)\}^{-1}$. The function I_ρ in (2.1) is defined, for $x > 0$, by

$$I_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}'_\rho} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(\rho+2-s)\Delta(1-s)} x^{1+\rho-s} ds, \quad \Delta(s) = \Gamma^{r_1}(\tfrac{1}{2}s) \Gamma^{r_2}(s), \quad (2.2)$$

where $\mathcal{C}'_\rho$ denotes the path of integration extending from $c_\rho - i\infty$ to $c_\rho - iR$, thence to $c_\rho + r - iR$, $c_\rho + r + iR$, $c_\rho + iR$, and $c_\rho + i\infty$, with r and R chosen suitably large, and with $c_\rho = \frac{1}{2} + (\rho/n) - \varepsilon$, $0 < \varepsilon < 1/2n$.

The following asymptotic formula plays a key rôle here:

$$I_\rho(x) = \sum_{\nu=0}^m e'_\nu x^{\omega_\rho - (\nu/n)} \cos(hx^{1/n} + \pi_\nu) + O(x^{\omega_\rho - (m+1/n)}), \quad (2.3)$$

where $\omega_\rho = \frac{1}{2} - (1/2n) + \rho(1 - (1/n))$, $h = n2^{(n-r_2)/n}$, $\pi_\nu = \pi_\nu(\rho) = -\pi((n/2) + (\rho/2) + \frac{1}{4}(r_1 + 3) - (\nu/2))$. It was proved in [2, Lemma 1] for $\rho \geq 0$. But the formula and the proof are, in fact, valid for all real ρ .

Since $I_0(x)$ is continuous at $x = 0$, with the value $I_0(0) = \Delta(1)/\Delta(0)$,

$$\int_0^\varepsilon I'_0(x) dx \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0+, \quad (2.4)$$

where I'_0 stands for the derivative of I_0 . Clearly $I'_0(x) = I_{-1}(x)$, for $x > 0$.

§3. Some basic lemmas

We shall obtain an expression for the sum $\sum_{k \leq x} a_k \exp(2\pi i \lambda_k \eta)$, with an arbitrary $\eta > 0$, and $\lambda_k = B \cdot k$, as a sum of three sums of integrals, and then estimate those integrals separately.

Let $a_0 = 0$, $\lambda_0 = 0$, and

$$A(x) = \sum'_{\lambda_k \leq x} a_k, \quad A(x) = 0, \quad \text{for} \quad 0 \leq x < \lambda_1. \quad (3.1)$$

Let $f(t) = \exp(2\pi i \eta t)$, $\eta > 0$, $t \geq 0$. Let β be a smoothing function defined as follows: $\beta(t) \in C^\infty(-\infty, \infty)$, $\beta(t) = 1$ in a neighbourhood of $t \leq \lambda_N$ (where N is a fixed positive integer), $\beta(t) = 0$ in a neighbourhood of $t \geq \lambda_{N+1}$, and $0 \leq \beta(t) \leq 1$ everywhere. Since $\lambda_{N+1} - \lambda_N$ is bounded below by a positive constant, β can be so chosen that all its derivatives are bounded in $(-\infty, \infty)$. We then have

$$\sum_{k=0}^{\infty} a_k f(\lambda_k) \beta(\lambda_k) = \int_0^\infty f(t) \beta(t) dA(t). \quad (3.2)$$

Integrating this by parts r times, where r is an integer so large that the infinite series in (2.1) converges absolutely, and uniformly, for $x > 0$, and $\rho \geq r > 0$, we

obtain

$$\int_0^\infty f(t)\beta(t) dA(t) = (-1)^r \int_0^\infty A^r(t) \cdot (f(t)\beta(t))^{(r+1)} dt, \quad (3.3)$$

where

$$A^r(t) = \frac{1}{\Gamma(r+1)} \sum_{\lambda_k \leq t} a_k (t - \lambda_k)^r.$$

Writing $F(t) = f(t)\beta(t)$, we have, from (2.1),

$$\begin{aligned} (-1)^r \int_0^\infty A^r(t) \cdot F^{(r+1)}(t) dt &= (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) dt \\ &\quad + (-1)^r \int_0^{\lambda_{N+1}} \left(\sum_{k=1}^\infty a_k \lambda_k^{-1-r} I_r(\lambda_k t) \right) \cdot F^{(r+1)}(t) dt. \end{aligned} \quad (3.4)$$

Consider that part of the last integral given by

$$(-1)^r \sum_{\lambda_k > y} a_k \lambda_k^{-1-r} \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt.$$

This is

$$\begin{aligned} &\ll \sum_{\lambda_k > y} a_k \lambda_k^{-1-r} \int_0^{\lambda_{N+1}} \lambda_k^{\omega_r} (1+t)^{\omega_r} dt \quad (\text{see (2.3)}) \\ &\ll \sum_{\lambda_k > y} a_k \lambda_k^{-((1/2)+(1/2n)+(r/n))} \int_0^{\lambda_{N+1}} (1+t)^{(1/2)+r(1-(1/n))-(1/2n)} dt \\ &\ll \sum_{\lambda_k > y} a_k \lambda_k^{-((1/2)+(1/2n)+(r/n))} \cdot \lambda_{N+1}^{(3/2)+r(1-(1/n))-(1/2n)} \\ &\ll y^{(1/2)-(1/2n)-(r/n)} \lambda_{N+1}^{(3/2)+r(1-(1/n))-(1/2n)}. \end{aligned} \quad (3.5)$$

Choose N and y such that

$$\lambda_N \leq x < \lambda_{N+1}, \quad y = c_0 x^{n-1+\varepsilon_0}, \quad \varepsilon_0 > 0. \quad (3.6)$$

Then (3.5) is

$$\begin{aligned} &\ll x^{(3/2)+r(1-(1/n))-(1/2n)-(r/n)(n-1+\varepsilon_0)+n((1/2)-(1/2n))}, \quad \text{if } \varepsilon_0 < 1, \\ &\ll x^{1+(n/2)-(r\varepsilon_0/n)} \\ &\ll x^{-q}, \end{aligned}$$

for a given $q > 0$, if r is chosen large enough. Hence, for any $q > 0$,

$$(-1)^r \sum_{\lambda_k > y} a_k \lambda_k^{-1-r} \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt = O(x^{-q}) \quad (3.7)$$

for $r = r(q)$. Using (3.7) in (3.3) and (3.4), we get

$$\begin{aligned} \int_0^\infty f(t) \beta(t) dA(t) &= (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) dt + O(x^{-q}) \\ &\quad + (-1)^r \sum_{\lambda_k \leq y} a_k \lambda_k^{-1-r} \int_0^{\lambda_{N+1}} I_r(\lambda_k t) \cdot F^{(r+1)}(t) dt. \end{aligned} \quad (3.8)$$

The last sum of integrals equals

$$\begin{aligned} &\sum_{\lambda_k \leq y} a_k \lambda_k^{-1} \int_0^{\lambda_{N+1}} F'(t) \cdot I_0(\lambda_k t) dt \\ &= - \sum_{\lambda_k \leq y} a_k \int_0^{\lambda_{N+1}} I_{-1}(\lambda_k t) \cdot F(t) dt - \sum_{\lambda_k \leq y} \frac{a_k}{\lambda_k} \cdot I_0(0) \\ &= - \sum_{\lambda_k \leq y} a_k \int_0^{\lambda_N} I_{-1}(\lambda_k t) f(t) dt - \sum_{\lambda_k \leq y} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t) \cdot f(t) \cdot \beta(t) dt \\ &\quad - \sum_{\lambda_k \leq y} \frac{a_k}{\lambda_k} \cdot I_0(0). \end{aligned} \quad (3.9)$$

From (3.9), (3.8), (3.6), and (3.2), we have

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k f(\lambda_k) &= \sum_{k=0}^N a_k f(\lambda_k) \\ &= \sum_{k=1}^N a_k \exp(2\pi i \eta \lambda_k) \\ &= (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) dt + O(x^{-q}) \\ &\quad - \sum_{\lambda_k \leq y = c_0 x^{n-1+\varepsilon_0}} a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &\quad - \sum_{\lambda_k \leq y = c_0 x^{n-1+\varepsilon_0}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t) \cdot f(t) \cdot \beta(t) dt \\ &\quad - \sum_{\lambda_k \leq y} \frac{a_k}{\lambda_k} \cdot I_0(0). \end{aligned} \quad (3.10)$$

Now let

$$c_0 = \left(\frac{2\pi n \eta}{h} \right)^n, \quad (3.11)$$

where h is defined as in (2.3). Then $\lambda_k \leq c_0 x^{n-1+\varepsilon_0}$ implies that

$$\mu \leq nx^{1-(1/n)+\varepsilon'}, \quad \text{where} \quad \mu = \frac{h}{2\pi\eta} \cdot \lambda_k^{1/n}, \quad \varepsilon' = \frac{\varepsilon_0}{n}. \quad (3.12)$$

Then the first sum on the right-hand side of (3.10) equals

$$\begin{aligned} & \sum_{\lambda_k \leq c_0 x^{n-1+\varepsilon_0}} a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= \sum_{\lambda_k \leq c_0 x^{n-1}} + \sum_{c_0 x^{n-1} < \lambda_k \leq c_0 x^{n-1+\varepsilon_0}} \\ &= \left[\sum_{\mu \leq nx^{1-(1/n)}} + \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} \right] a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_0^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &\quad - \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_{\lambda_N}^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &\quad + \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt, \end{aligned} \quad (3.13)$$

provided that the integral from 0 to ∞ converges, which we shall prove (in Lemma 2). Combining (3.13) with (3.10), we obtain the following

LEMMA 1. *If ε' is arbitrary with $0 < \varepsilon' < (1/n)$, and $\lambda_N \leq x < \lambda_{N+1}$, then*

$$\begin{aligned} \sum_{k=1}^N a_k \exp(2\pi i \eta \lambda_k) &= (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) dt + O(x^{-q}) \\ &\quad - \sum_{\mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t) \exp(2\pi i \eta t) \cdot \beta(t) \cdot dt \\ &\quad - \sum_{\mu \leq nx^{1-(1/n)+\varepsilon'}} \frac{a_k}{\lambda_k} I_0(0) \\ &\quad - \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_0^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \end{aligned}$$

$$\begin{aligned}
& + \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_{\lambda_N}^{\infty} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\
& - \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt,
\end{aligned}$$

where η is real, $\eta > 0$, $\beta(t) \in C^\infty(-\infty, \infty)$, β defined as in (3.2), $F(t) = \exp(2\pi i \eta t) \beta(t)$, and $\mu = (h/2\pi\eta) \cdot \lambda_k^{1/n}$ (μ defined as in (3.12) and h as in (2.3)).

LEMMA 2. *The integral*

$$\int_0^\infty \exp(i\eta t) I_{-1}(t) dt$$

converges for $\eta > 0$.

Proof. We first observe that

$$\int_1^\infty t^a \exp(i(\eta t \pm ht^{1/n})) dt$$

converges if $a < 0$. For, if $u(t) = (\eta t \pm ht^{1/n})$, then

$$\left| \frac{du}{dt} \right| \geq \eta - \frac{h}{n} t^{(1/n)-1} \geq \frac{1}{2}\eta, \quad \text{if } t \geq \left(\frac{2h}{n\eta} \right)^{n/(n-1)},$$

so that, if R and R' are large enough, we have, by the second mean-value theorem,

$$\left| \int_R^{R'} t^a \cos \{u(t)\} dt \right| \leq \frac{2R^a}{\eta}, \quad \left| \int_R^{R'} t^a \sin \{u(t)\} dt \right| \leq \frac{2R^a}{\eta},$$

and, in particular,

$$\int_R^{R'} t^a \exp(iu(t)) dt = o(1), \quad \text{as } R, R' \rightarrow \infty.$$

Now from (2.3) we have

$$I_{-1}(t) = \sum_{\nu=0}^m e'_\nu t^{\omega_{-1}-(\nu/n)} \cos(ht^{1/n} + \pi_\nu) + O(t^{\omega_{-1}-(m+1)/n}),$$

and

$$\int_1^\infty t^{\omega_{-1} - (1/n)(m+1)} dt < \infty,$$

if m is large, while $\omega_{-1} - \nu/n < 0$ for $\nu \geq 0$. Hence

$$\int_1^\infty \exp(i\eta t) I_{-1}(t) dt$$

converges. On the other hand, the limit

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0+} \int_\varepsilon^1 \exp(i\eta t) I_{-1}(t) dt \\ = \lim_{\varepsilon \rightarrow 0+} \{(I_0(1) \exp(i\eta) - I_0(\varepsilon) \exp(i\eta\varepsilon)) - i\eta \int_\varepsilon^1 I_0(t) \exp(i\eta t) dt\} \end{aligned}$$

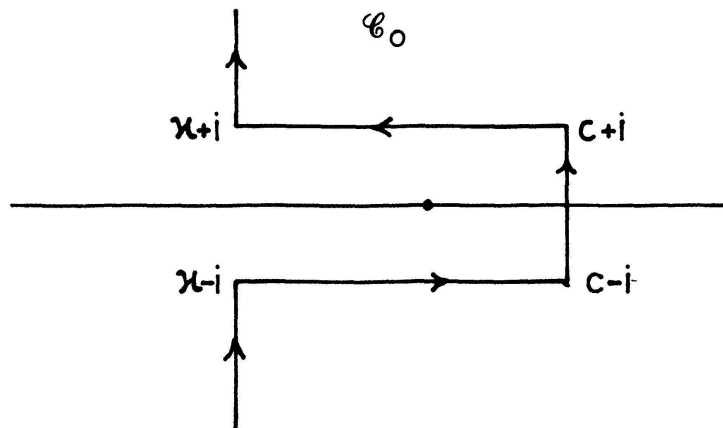
exists, and is finite, since $I_0(t)$ is continuous at $t=0$ (as remarked in (2.4)).

We shall now express the integral in Lemma 2 as a contour integral in the complex s -plane.

LEMMA 3. For $\xi > 0$, we have

$$\int_0^\infty \exp(i\xi t) I_{-1}(t) dt = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot \left(\frac{\xi}{i}\right)^{s-1} ds,$$

where $\Delta(s)$ is defined as in (2.2), and $\mathcal{C}_0$ denotes the path of integration indicated in the diagram, with $0 < c < 1$, and κ a large, negative number.



Proof. Because of Lemma 2,

$$\begin{aligned} \int_0^\infty \exp(i\xi t) I_{-1}(t) dt &= \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \exp(i\xi t) I_{-1}(t) \exp(-\varepsilon t) dt \\ &= \lim_{z \rightarrow -i\xi} \int_0^\infty \exp(-zt) I_{-1}(t) dt, \end{aligned} \quad (3.14)$$

where $z = \varepsilon - i\xi$, $\varepsilon > 0$. Now

$$I_{-1}(t) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)}{\Delta(1-s)} t^{-s} ds,$$

since $\Delta(s)/\Delta(1-s)$ has no singularities in $\operatorname{Re} s > 0$. If κ is large, and negative, the integral from $\kappa + i$ to $\kappa + i\infty$, and the integral from $\kappa - i\infty$ to $\kappa - i$ converge absolutely, since

$$\left| \frac{\Delta(s)}{\Delta(1-s)} \right| \ll (1 + |\tau|)^{n\sigma - (n/2)},$$

where $s = \sigma + i\tau$. Moreover $\operatorname{Re} s \leq c < 1$, everywhere on $\mathcal{C}_0$. Hence

$$\begin{aligned} \int_0^\infty \exp(-zt) I_{-1}(t) dt &= \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)}{\Delta(1-s)} ds \int_0^\infty \exp(-zt) t^{-s} dt, \quad \operatorname{Re} z > 0 \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot z^{s-1} ds. \end{aligned} \quad (3.15)$$

Let $z = |z| \cdot \exp(i\theta)$. Then $|z^{s-1}| = |z|^{\sigma-1} \cdot \exp(-\theta\tau)$. Since $\operatorname{Re} z > 0$, we have $|\theta| < \frac{1}{2}\pi$, so that $|z^{s-1}| \leq |z|^{\sigma-1} \exp(\frac{1}{2}\pi|\tau|)$. On the other hand,

$$\left| \frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)} \right| \sim \sqrt{2\pi} \cdot \exp(-\tfrac{1}{2}\pi|\tau|) \cdot |\tau|^{n\sigma - (n/2) + (1/2) - \sigma}, \quad \text{as } |\tau| \rightarrow \infty.$$

Hence the integrand in (3.15) is $O(|\tau|^{\sigma(n-1) - ((n-1)/2)}) = O(|\tau|^{-2})$, if κ is large and negative, and $|\tau|$ large. Thus the integral in (3.15) converges uniformly for $\operatorname{Re} z > 0$.

Now let $z \rightarrow -i\xi$, $\xi > 0$. Then

$$\lim_{z \rightarrow -i\xi} \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} z^{s-1} ds = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot \left(\frac{\xi}{i}\right)^{s-1} ds.$$

Comparing this with (3.14) and (3.15), we get the lemma.

§4. Some asymptotic expansions

LEMMA 4. *We have*

$$\begin{aligned} & \int_0^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= \frac{1}{2\eta} \sum_{\nu=0}^m m_k^{((2-n)/2(n-1)) - (\nu/(n-1))} \\ & \quad \times \{c_\nu \cos(qm_k^{1/(n-1)} + \pi\omega' + \frac{1}{2}\pi\nu) - id_\nu \sin(qm_k^{1/(n-1)} + \pi\omega' + \frac{1}{2}\pi\nu)\} \\ & \quad - \frac{1}{\lambda_k} \sum_{1 \leq l < ((n+2m)/2(n-1))} \frac{\Delta(l) \cdot (-1)^{l-1}}{\Delta(1-l) \cdot (l-1)!} \cdot \left(\frac{-2\pi i \eta}{\lambda_k}\right)^{l-1} + O(m_k^{-((n+2m)/2(n-1))}), \end{aligned}$$

where $\eta > 0$, $m_k = \lambda_k/2\pi\eta$, $q = (n-1)2^{r_1/(n-1)}$, $c_0 = d_0 = \pi^{-1}(n-1)^{-1/2}2^{r_1/2(n-1)}$, $\omega' = -\frac{1}{2} + r_1/4$, l integral.

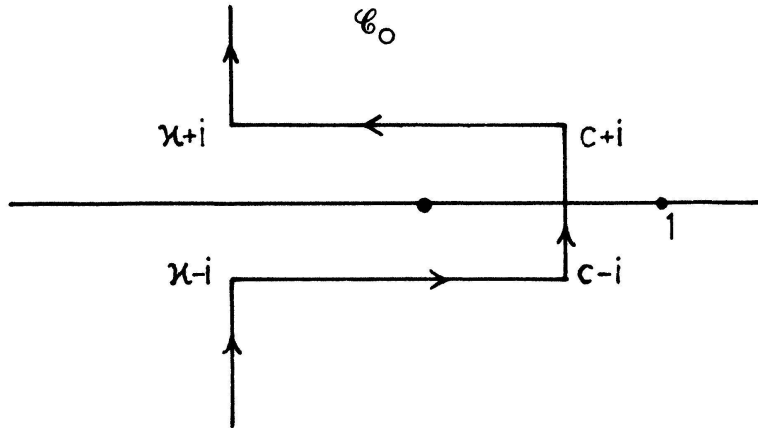
Proof. Let

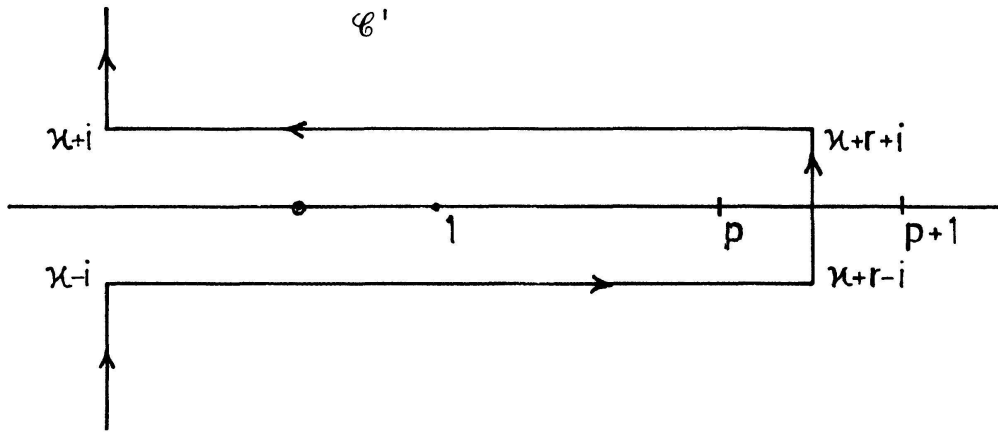
$$J = \int_0^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt, \quad m_k = \frac{\lambda_k}{2\pi\eta}, \quad \eta > 0.$$

Then

$$J = \frac{1}{\lambda_k} \int_0^\infty \exp(it/m_k) I_{-1}(t) dt = \frac{1}{\lambda_k} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot \left(\frac{1}{im_k}\right)^{s-1} ds,$$

after Lemma 3. Deform the path of integration $\mathcal{C}_0$ into $\mathcal{C}'$, by choosing p to be a sufficiently large integer, and $p < \kappa + r < p+1$, as indicated in the diagram.





We then have

$$\begin{aligned}
 J &= \frac{1}{\lambda_k} \left[\frac{1}{2\pi i} \int_{C'} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot \left(\frac{1}{im_k} \right)^{s-1} ds - \sum_{l=1}^p \frac{\Delta(l) \cdot (-1)^{l-1}}{\Delta(1-l) \cdot (l-1)!} \left(\frac{-2\pi i \eta}{\lambda_k} \right)^{l-1} \right] \\
 &= \left(\frac{i}{2\pi \eta} \right) \cdot \frac{1}{2\pi i} \int_{C'} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot m_k^{-s} \left(\cos \frac{\pi s}{2} - i \sin \frac{\pi s}{2} \right) ds \\
 &\quad - \frac{1}{\lambda_k} \left[\sum_{l=1}^p \frac{\Delta(l)(-1)^{l-1}}{\Delta(1-l) \cdot (l-1)!} \cdot \left(\frac{-2\pi i \eta}{\lambda_k} \right)^{l-1} \right].
 \end{aligned} \tag{4.2}$$

We seek an expansion for

$$J_1 = \left(\frac{i}{2\pi \eta} \right) \cdot \frac{1}{2\pi i} \int_{C'} \frac{\Delta(s)\Gamma(1-s)}{\Delta(1-s)} \cdot m_k^{-s} \left(\cos \frac{\pi s}{2} - i \sin \frac{\pi s}{2} \right) ds. \tag{4.3}$$

Since

$$\cos \frac{\pi s}{2} = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}, \quad \sin \frac{\pi s}{2} = \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)},$$

we have

$$J_1 = \left(\frac{i}{2\pi \eta} \right) \cdot \frac{1}{2\pi i} \int_{C'} \{V_0(s) - iV_1(s)\} \cdot \pi \cdot m_k^{-s} ds, \tag{4.4}$$

where

$$V_0(s) = \frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}, \quad V_1(s) = \frac{\Gamma(1-s)\Delta(s)}{\Delta(1-s)\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)}.$$

Choose

$$U_0(s) = -\frac{1}{\pi} \cdot b^{-s} \Gamma(S) \sin \left\{ \pi \left(\frac{1}{2} S + \omega' \right) \right\},$$

$$U_1(s) = \frac{1}{\pi} \cdot b^{-s} \Gamma(S) \cos \left\{ \pi \left(\frac{1}{2} S + \omega' \right) \right\},$$

where

$$S = (n-1)s - \frac{1}{2}n + 1, \quad \omega' = -\frac{1}{2} + \frac{r_1}{4}, \quad b = (n-1)^{n-1} \cdot 2^{r_1}.$$

By comparing the Stirling expansions of V_0 and U_0 on the one hand, and of V_1 , U_1 on the other, we get (as in [2, Lemma 1])

$$\begin{aligned} V_0(s) &= b^{1/2} U_0(s) \left\{ 1 + \sum_{\nu=1}^m \frac{e_\nu}{s^\nu} + O(|s|^{-m-1}) \right\} \\ &= -\frac{b^{(1/2)-s}}{\pi} \cdot \Gamma(S) \sin \left(\frac{\pi S}{2} + \pi \omega' \right) \cdot \left\{ 1 + \sum_{\nu=1}^m \frac{e_\nu}{s^\nu} + O(|s|^{-m-1}) \right\}, \end{aligned}$$

and

$$\begin{aligned} V_1(s) &= b^{1/2} U_1(s) \left\{ 1 + \sum_{\nu=1}^m \frac{e'_\nu}{s^\nu} + O(|s|^{-m-1}) \right\} \\ &= \frac{b^{(1/2)-s}}{\pi} \cdot \Gamma(S) \cos \left(\frac{\pi S}{2} + \pi \omega' \right) \left\{ 1 + \sum_{\nu=1}^m \frac{e'_\nu}{s^\nu} + O(|s|^{-m-1}) \right\}. \end{aligned}$$

Now, following the same procedure as in [2, Lemma 1], we can get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}'} V_0(s) x^{-s} ds &= - \sum_{\nu=0}^m d_\nu x^{((2-n)/2(n-1)) - \nu/(n-1)} \sin \left(qx^{1/(n-1)} + \pi \omega' + \frac{\pi \nu}{2} \right) \\ &\quad + O(x^{-((n+2m)/2(n-1))}), \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}'} V_1(s) x^{-s} ds &= \sum_{\nu=0}^m c_\nu x^{((2-n)/2(n-1)) - (\nu/(n-1))} \cos \left(qx^{1/(n-1)} + \pi \omega' + \frac{\pi \nu}{2} \right) \\ &\quad + O(x^{-((n+2m)/2(n-1))}), \end{aligned} \quad (4.6)$$

where

$$q = b^{1/(n-1)} = (n-1)2^{r_1/(n-1)}, \quad c_0 = \frac{q^{-(n/2-1)}}{n-1} \cdot \frac{b^{1/2}}{\pi} = d_0,$$

provided that

$$p \geq \frac{n+2m}{2(n-1)}.$$

From (4.4) we have

$$J_1 = \frac{1}{2\eta} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}'} \{V_1(s) + iV_0(s)\} m_k^{-s} ds.$$

From (4.5) and (4.6) we therefore obtain

$$\begin{aligned} 2\eta \cdot J_1 &= \sum_{\nu=0}^m m_k^{((2-n)/2(n-1))-\nu/(n-1)} \\ &\times \left\{ c_\nu \cos \left(qm_k^{1/(n-1)} + \pi\omega' + \frac{\pi\nu}{2} \right) - id_\nu \sin \left(qm_k^{1/(n-1)} + \pi\omega' + \frac{\pi\nu}{2} \right) \right\} \\ &\quad + O(m_k^{-((n+2m)/2(n-1))}), \end{aligned} \quad (4.7)$$

where $c_0 = d_0$. However, from (4.2) we have

$$J = J_1 - \frac{1}{\lambda_k} \sum_{l=1}^p \frac{\Delta(l) \cdot (-1)^{l-1}}{\Delta(1-l) \cdot (l-1)!} \cdot \left(\frac{-2\pi i \eta}{\lambda_k} \right)^{l-1}, \quad (4.8)$$

for a suitably chosen integer p . Now (4.8) and (4.7) give us the required result.

LEMMA 5. *Let a be a real number, $a \neq 0$, $\mu = (h/2\pi\eta)\lambda_k^{1/n}$, $\eta > 0$, and $h = n2^{(n-r_2)/n}$. Let $\varphi(t) = \varphi(t, \mu) = t - \mu t^{1/n}$, $F_0(t) = F_0(t, \mu) = t^a/\varphi'(t)$, $F_{l+1}(t) = F_{l+1}(t, \mu) = 1/\varphi'(t) \cdot (d/dt)F_l(t)$, for $l = 0, 1, 2, \dots$. Then*

$$F_l(t) = \frac{t^{a-l}}{(\varphi'(t))^{l+1}} \sum_{k=0}^l c_{l,k} \frac{\left(\frac{1}{n} \mu t^{(1/n)-1} \right)^k}{\left(1 - \frac{1}{n} \mu t^{(1/n)-1} \right)^k},$$

with $c_{0,0} = 1$, $c_{l,-1} = 0$, $c_{l,l+1} = 0$, and

$$c_{l+1,k} = c_{l,k} \left\{ (a-l) - k \left(1 - \frac{1}{n} \right) \right\} - c_{l,k-1} \left\{ (l+1) \left(1 - \frac{1}{n} \right) + (k-1) \left(1 - \frac{1}{n} \right) \right\}$$

for $k = 0, 1, \dots, l+1$.

Analogously, if

$$\psi(t) = \psi(t, \mu) = t + \mu t^{1/n}, \quad G_0(t) = G_0(t, \mu) = \frac{t^a}{\psi'(t)},$$

$$G_{l+1}(t) = G_{l+1}(t, \mu) = \frac{1}{\psi'(t)} \cdot \frac{d}{dt} G_l(t),$$

for $l = 0, 1, 2, \dots$, then

$$G_l(t) = \frac{t^{a-l}}{(\psi'(t))^{l+1}} \sum_{k=0}^l d_{l,k} \frac{\left(\frac{1}{n} \mu t^{(1/n)-1} \right)^k}{\left(1 + \frac{1}{n} \mu t^{(1/n)-1} \right)^k},$$

with $d_{0,0} = 1$, $d_{l,-1} = 0$, $d_{l,l+1} = 0$, and

$$d_{l+1,k} = d_{l,k} \left\{ (a-l) + k \left(1 - \frac{1}{n} \right) \right\} + d_{l,k-1} \left\{ (l+1) \left(1 - \frac{1}{n} \right) + (k-1) \left(1 - \frac{1}{n} \right) \right\}.$$

The proof follows by induction on l .

LEMMA 6. Let $a = a(\nu) = \omega_{-1} - \nu/n$, for $\nu = 0, 1, 2, \dots$. If $\delta > 0$, δ sufficiently small, and $0 < \eta_0 < \eta$, and $\mu \leq (n - \delta)x^{1-(1/n)}$, we have

$$\int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt = \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a-L-1}),$$

where the 'O' depends on L and δ , but is uniform in μ , and $F_{l,\nu}$ is the function F_l of Lemma 5, with $a = a(\nu)$.

Analogously we have

$$\int_x^\infty t^a \exp(2\pi i \eta \psi(t)) dt = \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a-L-1}),$$

where $G_{l,\nu}$ is the function G_l of Lemma 5, with $a = a(\nu)$.

Proof. We note that $\omega_{-1} = (1/2n) - \frac{1}{2}$, and that x is bounded below. Since $t \geq x$, we have

$$1 - \frac{\mu}{n} t^{(1/n)-1} \geq 1 - \frac{\mu}{n} x^{(1/n)-1} \geq \frac{\delta}{n},$$

and

$$\frac{\mu}{n} t^{(1/n)-1} \leq \frac{\mu}{n} x^{(1/n)-1} < 1.$$

Hence, from Lemma 5, we have

$$F_{l,\nu}(t) = O(t^{a-l}), \quad t \geq x, \quad \mu \leq (n - \delta)x^{1-(1/n)},$$

where the 'O' depends on a , l , and δ , but *not* on t , x , or μ . By partial integration, we have

$$\begin{aligned} \int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt &= \int_x^\infty \frac{t^a}{\varphi'(t)} \cdot \exp(2\pi i \eta \varphi(t)) \cdot \varphi'(t) dt \\ &= \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + \frac{(-1)^{L+1}}{(2\pi i \eta)^{L+1}} \\ &\quad \times \int_x^\infty F_{L+1,\nu}(t) \cdot \varphi'(t) \cdot \exp(2\pi i \eta \varphi(t)) dt \\ &= \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a-L}). \end{aligned}$$

Replacing L by $L + 1$, we get the lemma.

LEMMA 7. *Under the same conditions as in Lemma 6, we have*

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt &= \sum_{\nu=0}^{n(m+1)} \lambda_k^{\omega_{-1} - (\nu/n)} \sum_{l=0}^{m+1} \\ &\quad \times (\exp(2\pi i \eta \varphi(x)) \cdot b_{l,\nu} F_{l,\nu}(x) + \exp(2\pi i \eta \psi(x)) b'_{l,\nu} G_{l,\nu}(x)) + O(x^{\omega_{-1}-m}), \end{aligned}$$

where $\omega_{-1} = (1/2n) - \frac{1}{2}$, and $F_{l,\nu}$, $G_{l,\nu}$ stand for the functions F_l , G_l defined in Lemma 5, with $a = a(\nu) = \omega_{-1} - (\nu/n)$, $\nu = 0, 1, 2, \dots$.

Proof. On using the asymptotic expansion for $I_{-1}(t)$, we have

$$\int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt = \sum_{\nu=0}^m e'_\nu \int_x^\infty (\lambda_k t)^{\omega_{-1}-(\nu/n)} \times \exp(2\pi i \eta t) \cos(h\lambda_k^{1/n} t^{1/n} + \pi_\nu) dt + O_m((\lambda_k x)^{\omega_{-1}-((m+1)/n)+1}),$$

where $h = n2^{(n-r_2)/n}$, and $\pi_\nu = \pi_\nu(-1) = -\pi(\frac{1}{2}n + (r_1/4 + \frac{1}{4}) - (\nu/2))$. Thus

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt &= \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \int_x^\infty t^{\omega_{-1}-(\nu/n)} \exp(2\pi i \eta t - h\lambda_k^{1/n} t^{1/n}) dt \right. \\ &\quad \left. + b'_\nu \int_x^\infty t^{\omega_{-1}-(\nu/n)} \exp(2\pi i \eta t + h\lambda_k^{1/n} t^{1/n}) dt \right) + O_m((\lambda_k x)^{\omega_{-1}-((m+1)/n)+1}). \end{aligned}$$

If we now use Lemma 6, we get

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt &= \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left\{ b_\nu \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right. \\ &\quad \left. + b'_\nu \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{\omega_{-1}-(\nu/n)-L-1}) \right\} \\ &\quad + O_m(x^{\omega_{-1}-((m+1)/n)+1}). \end{aligned}$$

If $L = [(m+1)/n]$, then $L+1 > (m+1)/n$, hence

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt &= \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(\exp(2\pi i \eta \varphi(x)) \sum_{0 \leq l \leq (m+1)/n} b_{l,\nu} F_{l,\nu}(x) \right. \\ &\quad \left. + \exp(2\pi i \eta \psi(x)) \sum_{0 \leq l \leq (m+1)/n} b'_{l,\nu} G_{l,\nu}(x) \right) + O_m(x^{\omega_{-1}-((m+1)/n)+1}). \end{aligned}$$

Replacing m by $n(m+1)$, we get the lemma.

Our treatment of the estimates for the range $\mu \geq (n+\delta)x^{1-(1/n)}$ is based on the next two lemmas.

LEMMA 8. *For a fixed ξ , such that $0 < \xi_0 \leq \xi \leq \xi_1$, we have, as $\mu \rightarrow \infty$, the following asymptotic expansion (in decreasing powers of μ):*

$$F_\nu(\xi, \mu) = \frac{1}{\mu^{\nu+1}} \left(\sum_{l=0}^L \frac{d_{l,\nu}}{\mu^l} + o(\mu^{-L}) \right),$$

where the $d_{l,\nu}$ are continuous functions of ξ for $0 < \xi_0 \leq \xi \leq \xi_1$.

Proof. We have only to use the formula

$$F_\nu(\xi, \mu) = \frac{\xi^{a-\nu}}{(\varphi'(\xi))^{\nu+1}} \sum_{m=0}^{\nu} c_{\nu, m} \frac{\left(\frac{1}{n} \mu \xi^{(1/n)-1}\right)^m}{\left(1 - \frac{1}{n} \mu \xi^{(1/n)-1}\right)^m},$$

(see Lemma 5) and the Binomial Theorem.

LEMMA 9. Suppose for a fixed integer $K \geq 0$, we have

$$f(\mu) = f(x, \mu) = \sum_{k=0}^K \mu^{n_k} (a_k(\xi) + b_k(\xi) \exp(i\mu\xi) + b'_k(\xi) \exp(-i\mu\xi)) + o(\mu^{n_K}),$$

as $\mu \rightarrow \infty$, where $n_0 > n_1 > n_2 > \dots \rightarrow -\infty$, for fixed x in a compact set in $\mathbf{R}_1$, and this holds uniformly in ξ for $a \leq \xi \leq b$, where $a_k(\xi)$ and $b_k(\xi)$ are continuous in ξ , and the function $f(x, \mu)$ is independent of ξ , then $a_k(\xi)$ is a constant, say a_k , and $b_k(\xi) = b'_k(\xi) = 0$ for all $k \leq K$.

Proof. By hypothesis,

$$\mu^{-n_0} f(\mu) = a_0(\xi) + b_0(\xi) \exp(i\mu\xi) + b'_0(\xi) \exp(-i\mu\xi) + o(1),$$

as $\mu \rightarrow \infty$, uniformly in ξ . Integrating this with respect to ξ , we get

$$(\beta - a) \mu^{-n_0} f(\mu) = \int_a^\beta a_0(\xi) d\xi + o(1),$$

as $\mu \rightarrow \infty$, for $a \leq \xi \leq \beta < b$, because of the Riemann–Lebesgue Lemma. Hence the limit $\lim_{\mu \rightarrow \infty} \mu^{-n_0} f(\mu)$ exists, and equals, say, c which is independent of ξ . It follows that $a_0(\xi) = c$. Hence

$$\lim_{\mu \rightarrow \infty} \{b_0(\xi) \exp(i\mu\xi) + b'_0(\xi) \exp(-i\mu\xi)\} = 0,$$

or $\lim_{\mu \rightarrow \infty} b_0(\xi) \exp(2i\mu\xi) + b'_0(\xi) = 0$. Again, by integration, we see that $b'_0(\xi) \equiv 0$, hence also $b_0(\xi) \equiv 0$. A repetition of the argument leads to the lemma.

COROLLARY. If

$$f(\mu) = \sum_{k=0}^K \mu^{n_k} \{a_k(\xi) + b_k(\xi) \cos(\mu^\alpha \xi) + b'_k(\xi) \sin(\mu^\alpha \xi)\} + o(\mu^{n_K}),$$

for $\alpha > 0$, uniformly in ξ for $a \leq \xi \leq b$, as $\mu \rightarrow \infty$, then $a_k(\xi)$ is independent of ξ , and $b_k(\xi) \equiv b'_k(\xi) \equiv 0$.

LEMMA 10. If $\delta > 0$, and sufficiently small, and $0 < \eta_0 \leq \eta$, and $\mu \geq (n + \delta)x^{1-(1/n)}$, we have

$$\begin{aligned} \int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ = -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ \left. + b'_\nu \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right) + O(\mu^{-L-1}) + O(\lambda_k^{\omega_0-(m/n)-1}), \end{aligned}$$

where $\omega_0 = \frac{1}{2} - (1/2n)$ (as in (2.3)), and $\omega_{-1} = (1/2n) - \frac{1}{2}$. If $L = m$, the term $O(\mu^{-L-1})$ can be dropped.

Proof. Let ξ be a number such that $\frac{1}{4}\lambda_1 \leq \xi \leq \frac{1}{2}\lambda_1$, say. Then

$$\int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt = \int_0^\xi + \int_\xi^x.$$

If $a = a(\nu) = \omega_{-1} - (\nu/n) = (1/2n) - \frac{1}{2} - (\nu/n)$, for $\nu = 0, 1, 2, \dots$, we obtain by partial integration,

$$\begin{aligned} \int_\xi^x t^a \exp(2\pi i \eta \varphi(t)) dt \\ = \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} - \exp(2\pi i \eta \varphi(\xi)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} \\ + (-1)^{L+1} \int_\xi^x \frac{F_{L+1,\nu}(t, \mu) \varphi'(t)}{(2\pi i \eta)^{L+1}} \exp(2\pi i \eta \varphi(t)) dt. \end{aligned}$$

To estimate the last integral, we use, after Lemma 5,

$$\varphi'(t) F_k(t) = \frac{t^{a-k}}{\{\varphi'(t)\}^k} \sum_{l=0}^k c_{k,l} \left(\frac{\frac{1}{n} \mu t^{(1/n)-1}}{1 - \frac{1}{n} \mu t^{(1/n)-1}} \right)^l.$$

Here $t \leq x$, and $\mu \geq (n + \delta)x^{1-(1/n)}$, so that

$$\frac{1}{n} \mu t^{(1/n)-1} \geq \frac{1}{n} \mu x^{(1/n)-1} \geq \frac{n + \delta}{n} = 1 + \frac{\delta}{n},$$

while

$$\frac{t^{-1}}{\varphi'(t)} = \frac{1}{t\left(1 - \frac{1}{n} \mu t^{(1/n)-1}\right)} = \frac{1}{t - \frac{1}{n} \mu t^{1/n}} = \frac{-nt^{-1/n}}{\mu - nt^{1-(1/n)}}.$$

For $1 \leq \xi \leq t \leq x$, the numerator is bounded, while

$$\mu - nt^{1-(1/n)} \geq \mu - nx^{1-(1/n)} \geq \mu - \frac{n\mu}{n+\delta} = \frac{\delta\mu}{n+\delta},$$

since $nx^{1-(1/n)} \leq n\mu/(n+\delta)$. Hence

$$\left| \frac{t^{-1}}{\varphi'(t)} \right| \leq \frac{n(n+\delta)}{\delta} \cdot \frac{\xi^{-1/n}}{\mu},$$

so that

$$\left(\frac{t^{-1}}{\varphi'(t)} \right)^k = O(\mu^{-k}),$$

provided that $k \geq 1$, $\frac{1}{4}\lambda_1 \leq \xi \leq \frac{1}{2}\lambda_1$. On the other hand,

$$\frac{\frac{1}{n} \mu t^{(1/n)-1}}{1 - \frac{1}{n} \mu t^{(1/n)-1}} = \frac{u}{1-u} = -\left(1 + \frac{1}{u-1}\right), \quad \text{with } u \geq 1 + \frac{\delta}{n},$$

hence

$$\left| \frac{\frac{1}{n} \mu t^{(1/n)-1}}{1 - \frac{1}{n} \mu t^{(1/n)-1}} \right| < C, \text{ say.}$$

Thus $\varphi'(t)F_k(t) = O(t^a \mu^{-k})$, for $k \geq 1$. It follows that

$$\begin{aligned} (-1)^{L+1} \int_{\xi}^x \frac{F_{L+1, \nu}(t, \mu)}{(2\pi i \eta)^{L+1}} \cdot \varphi'(t) \cdot \exp(2\pi i \eta \varphi(t)) dt &= O\left(\int_{\xi}^x t^a \mu^{-(L+1)} dt\right) \\ &= O(\mu^{-(L+1)}) \cdot (x^{a+1} - \xi^{a+1}) \\ &= O(x^{\omega_{-1}+1} \mu^{-(L+1)}) \\ &= O(\mu^{n(\omega_{-1}+1)/(n-1)-(L+1)}), \quad \omega_{-1} = \frac{1}{2n} - \frac{1}{2}. \end{aligned}$$

Thus we obtain (after replacing L by a larger L' if necessary),

$$\begin{aligned} \int_{\xi}^x t^a \exp(2\pi i \eta \varphi(t)) dt &= \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \\ &\quad - \exp(2\pi i \eta \varphi(\xi)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} + O(\mu^{-L-1}), \end{aligned} \quad (4.9)$$

for $\mu \geq (n + \delta)x^{1-(1/n)}$, where the 'O' does *not* depend on ξ .

If we replace φ by ψ , we obtain an analogue with $G_{l,\nu}(x, \mu)$ instead of $F_{l,\nu}(x, \mu)$. It follows that

$$\begin{aligned} &\int_{\xi}^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_{\nu}'' \exp(2\pi i \eta \varphi(x, \mu)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ &\quad \left. - b_{\nu}'' \exp(2\pi i \eta \varphi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} \right) + O(\mu^{-L-1}) \\ &\quad + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_{\nu}''' \exp(2\pi i \eta \psi(x, \mu)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ &\quad \left. - b_{\nu}''' \exp(2\pi i \eta \psi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} \right) + O(\mu^{-L-1}) \\ &\quad + O(\lambda_k^{\omega_{-1}-(m+1)/n}). \end{aligned}$$

On the other hand

$$\begin{aligned} &\int_0^{\xi} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= \frac{I_0(\lambda_k \xi) \exp(2\pi i \eta \xi)}{\lambda_k} - \frac{I_0(0)}{\lambda_k} - \frac{2\pi i \eta}{\lambda_k} \int_0^{\xi} \exp(2\pi i \eta t) I_0(\lambda_k t) dt, \end{aligned}$$

where

$$\begin{aligned} \int_0^{\xi} \exp(2\pi i \eta t) I_0(\lambda_k t) dt &= \left[\frac{I_1(\lambda_k t) \cdot \exp(2\pi i \eta t)}{\lambda_k} \right]_0^{\xi} \\ &\quad - \int_0^{\xi} \frac{2\pi i \eta}{\lambda_k} \cdot I_1(\lambda_k t) \cdot \exp(2\pi i \eta t) dt \\ &= \sum_{\nu=1}^m \frac{c_{\nu}(\xi) I_{\nu}(\lambda_k \xi)}{\lambda_k^{\nu}} + \frac{(-1)^m}{\lambda_k^m} \int_0^{\xi} (2\pi i \eta)^m \cdot I_m(\lambda_k t) \\ &\quad \times \exp(2\pi i \eta t) dt. \end{aligned}$$

Since

$$I_m(x) = O((1+x)^{\omega_0+m(1-(1/n))}),$$

and ξ is bounded, we have

$$\begin{aligned} -\frac{2\pi i\eta}{\lambda_k} \int_0^\xi \exp(2\pi i\eta t) I_0(\lambda_k t) dt &= \sum_{\nu=1}^m \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O\left(\frac{1}{\lambda_k^{m+1}}\right) \cdot \int_0^\xi |I_m(\lambda_k t)| dt \\ &= \sum_{\nu=1}^m \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O(\lambda_k^{\omega_0-(m/n)-1}). \end{aligned}$$

Thus altogether we have

$$\int_0^\xi \exp(2\pi i\eta t) I_{-1}(\lambda_k t) dt = \sum_{\nu=0}^m \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} - \frac{I_0(0)}{\lambda_k} + O(\lambda_k^{\omega_0-(m/n)-1}), \quad (4.10)$$

the 'O' being independent of ξ .

Combining (4.10) and (4.9), we get

$$\begin{aligned} &\int_0^x \exp(2\pi i\eta t) I_{-1}(\lambda_k t) dt \\ &= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^m \frac{c'_\nu(\xi) I_\nu(\lambda_k \xi)}{\lambda_k^{\nu+1}} + O(\lambda_k^{\omega_0-(m/n)+1}) \\ &\quad + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b''_\nu \exp(2\pi i\eta\varphi(x, \mu)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i\eta)^{l+1}} \right. \\ &\quad \quad \quad \left. - b''_\nu \exp(2\pi i\eta\varphi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i\eta)^{l+1}} \right) \\ &\quad + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b'''_\nu \exp(2\pi i\eta\psi(x, \mu)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i\eta)^{l+1}} \right. \\ &\quad \quad \quad \left. - b'''_\nu \exp(2\pi i\eta\psi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(\xi, \mu)}{(2\pi i\eta)^{l+1}} \right) \\ &\quad + O(\mu^{-L-1}) + O(\lambda_k^{\omega_{-1}-(m/n)-(1/n)}), \end{aligned}$$

if $\mu \geq (n+\delta)x^{1-(1/n)}$, $\delta > 0$, δ sufficiently small, and $\mu = (h/2\pi\eta) \cdot \lambda_k^{1/n}$.

If we now use the asymptotic expansion (2.3) for I_ν , $0 \leq \nu \leq m$, and apply Lemmas 8 and 9, we get the required result.

LEMMA 11. If $(n - \delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)}$, $\delta > 0$, $0 < 2\varepsilon/n < 1$, then

$$\begin{aligned} \int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt \\ = \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l, \nu}(x)}{(2\pi i \eta)^{l+1}} + O_L(x^{a-L+((L+1)2\varepsilon/n)}), \end{aligned}$$

and, for all $\mu > 0$,

$$\int_x^\infty t^a \exp(2\pi i \eta \psi(t)) dt = \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} G_{l, \nu}(x)}{(2\pi i \eta)^{l+1}} + O_L(x^{a-L}),$$

where, as before, $a = a(\nu) = \omega_{-1} - (\nu/n)$; $\nu = 0, 1, 2, \dots$

If $\varepsilon > 0$ (and not necessarily $2\varepsilon < n$), we have

$$\int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt \ll |F_{0,0}(x)|.$$

Proof. By partial integration we have

$$\begin{aligned} \int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt &= \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l, \nu}(x)}{(2\pi i \eta)^{l+1}} \\ &\quad + \frac{(-1)^{L+1}}{(2\pi i \eta)^{L+1}} \int_x^\infty F_{L+1, \nu}(t) \cdot \varphi'(t) \cdot \exp(2\pi i \eta \varphi(t)) dt. \end{aligned}$$

In order to estimate the last integral, in the range under consideration, we have to estimate $F_{k, \nu}(t)$ anew. We have

$$\left(\frac{\frac{1}{n} \mu t^{(1/n)-1}}{1 - \frac{1}{n} \mu t^{(1/n)-1}} \right) = \left(\frac{\frac{1}{n} \mu}{t^{1-(1/n)} - \frac{1}{n} \mu} \right) \leq \frac{x^{1-(1/n)}}{\frac{1}{n} x^{1-(1/n)-\varepsilon/n}} \leq nx^{\varepsilon/n},$$

since $\mu \leq nx^{1-(1/n)}$, and $t^{1-(1/n)} - (\mu/n) \geq t^{1-(1/n)} - x^{1-(1/n)} + (1/n)x^{1-(1/n)-(\varepsilon/n)} \geq (1/n)x^{1-(1/n)-(\varepsilon/n)}$, for $t \geq x$. On the other hand,

$$\begin{aligned} \frac{t^{a-k}}{(\varphi'(t))^{k+1}} &= \frac{t^a}{\left(t - \frac{\mu}{n} t^{1/n}\right)^k} \cdot \frac{1}{\left(1 - \frac{\mu}{n} t^{(1/n)-1}\right)} \\ &\leq \frac{t^a}{\left(t^{1/n} \cdot \frac{1}{n} x^{1-(1/n)-(\varepsilon/n)}\right)^k \cdot \left|1 - \frac{\mu}{n} t^{(1/n)-1}\right|}, \end{aligned}$$

since

$$t - \frac{\mu}{n} t^{1/n} = t^{1/n} \left(t^{1-(1/n)} - \frac{\mu}{n} \right) \geq t^{1/n} \cdot \frac{1}{n} x^{1-(1/n)-(\varepsilon/n)}.$$

Therefore we have

$$|F_{k, \nu}(t)| \ll \frac{x^{k\varepsilon/n} \cdot t^a}{t^{k/n} (x^{1-(1/n)-(\varepsilon/n)})^k \cdot |\varphi'(t)|},$$

on using the formula for $F_k(t)$ given in Lemma 5. It follows that

$$\begin{aligned} \frac{(-1)^{L+1}}{(2\pi i \eta)^{L+1}} \int_x^\infty F_{L+1, \nu}(t) \cdot \varphi'(t) \cdot \exp(2\pi i \eta \varphi(t)) dt \\ \ll \frac{x^{(L+1)(\varepsilon/n)}}{x^{(L+1)(1-(1/n)-(\varepsilon/n))}} \int_x^\infty t^{a-(L+1)/n} dt \\ \ll x^{a-L+((L+1)2\varepsilon/n)}, \end{aligned}$$

hence the first part of the lemma.

The proof of the second part is similar. We use the fact that

$$\frac{\frac{1}{n} \mu t^{(1/n)-1}}{1 + \frac{1}{n} \mu t^{(1/n)-1}} \leq 1, \quad \text{and} \quad \frac{t^{a-k}}{\{\psi'(t)\}^{k+1}} \leq t^{a-k}.$$

To prove the final remark, we note that

$$\begin{aligned} \int_x^\infty t^a \exp(2\pi i \eta \varphi(t)) dt = \int_x^{x(1+\gamma)} \frac{t^a}{\varphi'(t)} \cdot \exp(2\pi i \eta \varphi(t)) \cdot \varphi'(t) dt \\ + \int_{x(1+\gamma)}^\infty t^a \exp(2\pi i \eta \varphi(t)) dt. \end{aligned}$$

Since $t^a/\varphi'(t)$ is decreasing, we may apply the second mean-value theorem to the first term (that is, separately to the real and imaginary part). To the second term we may apply Lemma 6.

LEMMA 12. If $(n-\delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)}$, $0 < 2\varepsilon/n < 1$, $\delta > 0$, δ sufficiently small, then we have

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt = \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \varphi(x, \mu)) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l, \nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ \left. + b'_\nu \exp(2\pi i \eta \psi(x, \mu)) \sum_{l=0}^L \frac{(-1)^{l+1} G_{l, \nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right) + O_m(x^{\omega_{-1}-((m+1)/n)+1}), \end{aligned}$$

provided that $L = [(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$ (and not necessarily $2\varepsilon < n$), we have

$$\int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. We use the asymptotic expansion for $I_{-1}(\lambda_k t)$, and then Lemma 11 gives the required result.

LEMMA 13. If $nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leq \mu < (n+\delta)x^{1-(1/n)}$, and $0 < \gamma < 1$, $0 < 2\varepsilon/n < 1$, then

$$\begin{aligned} \int_{x(1-\gamma)}^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ = \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ \left. + b'_\nu \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right) \\ - \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \varphi\{x(1-\gamma)\}) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x(1-\gamma), \mu)}{(2\pi i \eta)^{l+1}} \right. \\ \left. + b'_\nu \exp(2\pi i \eta \psi\{x(1-\gamma)\}) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x(1-\gamma), \mu)}{(2\pi i \eta)^{l+1}} \right) \\ + O(x^{\omega_{-1}-((m+1)/n)+1}), \end{aligned}$$

provided that $L = [(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$, then we have

$$\int_{x(1-\gamma)}^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. We shall first estimate

$$\int_{x(1-\gamma)}^x t^a \exp(2\pi i \eta \varphi(t)) dt, \quad \text{with } a = a(\nu) = \omega_{-1} - \frac{\nu}{n}.$$

Here $t \leq x$, $(\mu/n) - t^{1-(1/n)} \geq (\mu/n) - x^{1-(1/n)} \geq x^{1-(1/n)-(\varepsilon/n)}/n$, so that

$$\left| \frac{\frac{\mu}{n} t^{(1/n)-1}}{1 - \frac{\mu}{n} t^{(1/n)-1}} \right| = \left| \frac{\frac{\mu}{n}}{t^{1-(1/n)} - \frac{\mu}{n}} \right| = \left| \frac{-\frac{\mu}{n}}{\frac{\mu}{n} - t^{1-(1/n)}} \right| \leq \frac{x^{1-(1/n)}}{\frac{1}{n} x^{1-(1/n)-(\varepsilon/n)}} \leq nx^{\varepsilon/n}.$$

Hence

$$|F_{L+1, \nu}(t) \cdot \varphi'(t)| \ll \frac{x^{(L+1)(\varepsilon/n)} \cdot t^a}{\left| t^{1-(1/n)} - \frac{\mu}{n} \right|^{L+1} \cdot t^{(L+1)/n}} \\ \ll x^{(L+1)(\varepsilon/n)} \cdot t^{a-((L+1)/n)} \cdot x^{-(L+1)(1-(1/n)-(\varepsilon/n))}$$

It follows that

$$\int_{x(1-\gamma)}^x F_{L+1, \nu}(t) \cdot \varphi'(t) \cdot \exp(2\pi i \eta \varphi(t)) dt \ll x^{(L+1)(2\varepsilon/n) - (L+1)(1-(1/n))} \\ \times \int_{x(1-\gamma)}^x t^{a-(L+1)/n} dt \\ \ll x^{a-L+(L+1)(2\varepsilon/n)}.$$

Hence

$$\int_{x(1-\gamma)}^x t^a \exp(2\pi i \eta \varphi(t)) dt = \exp(2\pi i \eta \varphi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l, \nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \\ - \exp(2\pi i \eta \varphi(x(1-\gamma))) \sum_{l=0}^L \frac{(-1)^l F_{l, \nu}(x(1-\gamma), \mu)}{(2\pi i \eta)^{l+1}} \\ + O(x^{a-L+(L+1)(2\varepsilon/n)}).$$

An analogous result holds with ψ in place of φ , and $G_{l, \nu}$ in place of $F_{l, \nu}$. If we use the asymptotic formula for $I_{-1}(\lambda_k t)$, and choose $L = [(m+1)/(n-2\varepsilon)]$, $0 < 2\varepsilon/n < 1$, we get the first part of the lemma.

If $2\varepsilon/n \geq 1$, we simply use the asymptotic expansion for $I_{-1}(\lambda_k t)$, and observe that while

$$\left| \int_{x(1-\gamma)}^x t^a \exp(2\pi i \eta \psi(t)) dt \right| \ll |G_{0,0}(x)| \ll |F_{0,0}(x)|,$$

the second mean-value theorem may be applied to the integral

$$\int_{x(1-\gamma)}^x t^a \exp(2\pi i \eta \varphi(t)) dt.$$

LEMMA 14. *If $nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leq \mu < (n+\delta)x^{1-(1/n)}$, $0 < \gamma < 1$, $\delta > 0$,*

$\varepsilon > 0$, then

$$\begin{aligned} & \int_0^{x(1-\gamma)} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \phi(x(1-\gamma))) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x(1-\gamma))}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu \exp(2\pi i \eta \psi(x(1-\gamma))) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x(1-\gamma))}{(2\pi i \eta)^{l+1}} \right) + O(\lambda_k^{\omega_0-(m/n)-1}). \end{aligned}$$

Proof. Since

$$\frac{\mu}{\{x(1-\gamma)\}^{1-(1/n)}} \geq \frac{n}{(1-\gamma)^{1-(1/n)}} > (n + \delta_1), \quad \text{say,}$$

an application of Lemma 10 gives the result. Note that $\omega_0 - (m/n) - 1 = \omega_{-1} - (m+1)/n$.

LEMMA 15. If $nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leq \mu < (n + \delta)x^{1-(1/n)}$, $0 < \gamma < 1$, $\delta > 0$, $0 < 2\varepsilon/n < 1$, then

$$\begin{aligned} & \int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= -\frac{I_0(0)}{\lambda_k} + \sum_{\nu=0}^m \lambda_k^{\omega_{-1}-(\nu/n)} \left(b_\nu \exp(2\pi i \eta \phi(x)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + b'_\nu \exp(2\pi i \eta \psi(x)) \right. \\ & \quad \left. \times \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right) + O(x^{\omega_{-1}-((m+1)/n)+1}) + O(\lambda_k^{\omega_{-1}-((m+1)/n)}), \end{aligned}$$

where $L = [(m+1)/(n-2\varepsilon)]$.

If $\varepsilon > 0$, then

$$\frac{I(0)}{\lambda_k} + \int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \ll \lambda_k^{\omega_{-1}} |F_{0,0}(x)|.$$

Proof. This results from combining Lemmas 14 and 13.

LEMMA 16. If $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}$, and $0 < \gamma < 1$, $\varepsilon > 0$, then

$$\begin{aligned} & \int_{x(1+\gamma)}^\infty t^a \exp(2\pi i \eta \phi(t)) dt \\ &= \exp(2\pi i \eta \phi(x(1+\gamma))) \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x(1+\gamma))}{(2\pi i \eta)^{l+1}} + O(x^{a-L-1}), \end{aligned}$$

where $a = a(\nu) = \omega_{-1} - (\nu/n)$; $\nu = 0, 1, 2, \dots$.

Proof. Here we have

$$\frac{\mu}{\{x(1+\gamma)\}^{1-(1/n)}} \leq \frac{n}{(1+\gamma)^{1-(1/n)}} \leq n - \delta_1, \quad \text{say,}$$

so that the required result follows from Lemma 6.

LEMMA 17. *If $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}$, $0 < \gamma < 1$, γ sufficiently small, and $\varepsilon > 0$, then*

$$\begin{aligned} & \int_x^{x(1+\gamma)} t^a \exp(2\pi i \eta \varphi(t)) dt \\ &= \frac{1}{2} n x^{a+1} \left\{ \sum_{p=0}^{N-1} \frac{\{A_p(1+\gamma) \exp(2\pi i \eta \varphi(x(1+\gamma))) - B_p(1) \exp(2\pi i \eta \varphi(x))\}}{(2\pi x \eta)^{p+1}} \right. \\ & \quad \left. + \sum_{m=0}^{N-1} \frac{\alpha'_m \exp(2\pi i x \eta P(v_0))}{(2\pi x \eta)^{m+(1/2)}} \right. \\ & \quad \left. - \sum_{m=0}^{N-1} \frac{\beta'_m \exp(2\pi i x \eta P(v_0))}{(2\pi x \eta)^m} \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds \right\} + O(x^{a-N}), \end{aligned}$$

where $P(v) = v^n - \alpha v$, $\alpha = \mu x^{(1/n)-1}$, $v_0 = (\alpha/n)^{1/(n-1)}$, $\tau = P(1) - P(v_0)$, and

$$\int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds = O(x^{-1/2}),$$

uniformly in τ . The terms A_p , B_p , α'_m , β'_m depend on α , and, for fixed p or m , are continuous functions of α in a neighbourhood of the point $\alpha = n$.

Proof. We have

$$\begin{aligned} & \int_x^{x(1+\gamma)} t^a \exp(2\pi i \eta \varphi(t)) dt \\ &= x^{a+1} \int_1^{1+\gamma} u^a \exp(2\pi i x \eta (u - \mu x^{(1/n)-1} \cdot u^{1/n})) du, \quad (t = ux) \\ &= x^{a+1} \int_1^{1+\gamma} u^a \exp(2\pi i x \eta (u - \alpha u^{1/n})) du \\ &= n x^{a+1} \int_1^{(1+\gamma)^{1/n}} \exp(2\pi i x \eta P(v)) v^{an+n-1} dv. \quad (u = v^n) \end{aligned}$$

Now $\partial P/\partial v = 0$, for $v = (\alpha/n)^{1/(n-1)} = v_0$. In order to make the substitution $P(v) - P(v_0) = s$, consider the expansion:

$$P(v) - P(v_0) = (f(v - v_0))^2,$$

where $f(z)$ is a power series of the form $(z/\sqrt{2})\sqrt{P''(v_0)} + \dots$. (Note that $P'(v_0) = 0$, while $P'(v) < 0$ for $v < v_0$, and $P'(v) > 0$ for $v > v_0$. Further $P''(v_0)$ is close to $n(n-1)$ if x is large, in the range of μ under consideration.) We then have $f(v - v_0) = s^{1/2}$. Since there exists an F , such that $F[f(z)] = z$ with

$$F(z) = \frac{\sqrt{2}}{\sqrt{P''(v_0)}} z + \dots,$$

we have

$$v - v_0 = F(s^{1/2}),$$

where $F(s)$ is holomorphic in a neighbourhood of the origin, say in $|s| < 2R$, which is independent of α . The above integral therefore equals

$$\begin{aligned} & \frac{1}{2} n x^{a+1} \cdot \exp(2\pi i x \eta P(v_0)) \\ & \times \int_{s=P(1)-P(v_0)}^{s=P((1+\gamma)^{1/n})-P(v_0)} (v_0 + F(s^{1/2}))^{an+n-1} s^{-1/2} F'(s^{1/2}) \exp(2\pi i x \eta s) ds, \end{aligned}$$

provided that γ is so small that $P((1+\gamma)^{1/n}) - P(v_0) < R$, say. Now set

$$g(s) = (v_0 + F(s))^{an+n-1} F'(s),$$

$$\tau = P(1) - P(v_0), \quad C = P((1+\gamma)^{1/n}) - P(v_0).$$

Note that $\tau = O(x^{-2\varepsilon/n})$, while C is bounded below by a positive constant. We then have

$$\begin{aligned} & \int_x^{x(1+\gamma)} t^a \exp(2\pi i \eta \varphi(t)) dt \\ & = \frac{1}{2} n x^{a+1} \exp(2\pi i x \eta P(v_0)) \int_{\tau}^C g(s^{1/2}) s^{-1/2} \exp(2\pi i x \eta s) ds. \end{aligned}$$

Now $g(s) = \sum_{m=0}^{\infty} g_m s^m$, uniformly for $|s| \leq C < R$. For $N \geq 1$, let

$$g(s) = \sum_{m \leq 2N} g_m s^m + \sum_{m=2N+1}^{\infty} g_m s^m = \sum_{m \leq 2N} g_m s^m + \tilde{g}_N(s), \quad \text{say.}$$

Then the function $H_N(s) = \tilde{g}_N(s) s^{-1/2}$ is N times continuously differentiable in

$s \geq 0$, and the first $N-1$ derivatives vanish at $s=0$. Hence

$$\int_0^C H_N(s) \exp(2\pi i x \eta s) ds = \sum_{p=0}^{N-1} \frac{(-1)^p \exp(2\pi i x \eta C) H_p(C)}{(2\pi i x \eta)^{p+1}} + O(x^{-N-1}).$$

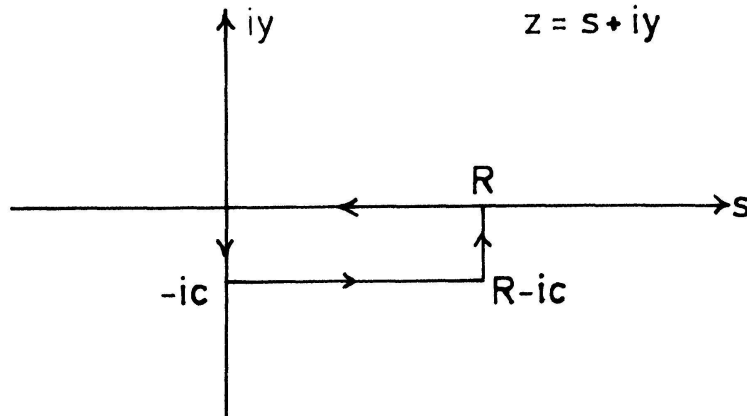
Now

$$\int_0^C s^{(m-1)/2} \exp(2\pi i x \eta s) ds = \begin{cases} \sum_{p=0}^{((m-1)/2)-1} \frac{a_p(C) \exp(2\pi i x \eta C)}{(2\pi i x \eta)^{p+1}} + \frac{(-1)^{(m-1)/2} \left(\frac{m-1}{2}\right)! (\exp(2\pi i x \eta C) - 1)}{(2\pi i x \eta)^{((m-1)/2)-1}}, & \text{if } m \text{ is an odd integer } \geq 1; \\ \sum_{p=0}^{(m-1)/2} \frac{a'_p(C) \exp(2\pi i x \eta C)}{(2\pi i x \eta)^{p+1}} + \frac{(-1)^{m/2} \left(\frac{m-1}{2}\right) \left(\frac{m-3}{2}\right) \cdots \frac{1}{2}}{(2\pi i x \eta)^{m/2}} & \times \int_0^C s^{-1/2} \exp(2\pi i x \eta s) ds, \\ & \text{if } m \text{ is an even integer } \geq 0. \end{cases}$$

On the other hand, as $x \rightarrow \infty$,

$$\int_0^C s^{-1/2} \exp(2\pi i x \eta s) ds = \frac{b_0}{(2\pi x \eta)^{1/2}} + \sum_{r=0}^{N-1} \frac{b_r(C) \exp(2\pi i x \eta C)}{(2\pi x \eta)^{r+1}} + O(x^{-N-1}).$$

To prove this, one has only to consider the contour integral $\int z^{-1/2} \exp(-xz) dz$, for $x > 0$, in the complex z -plane, taken along the rectangle with corners at the points $z = 0, -iC, R-iC$, and R , with $R > 0$, and then let $R \rightarrow \infty$. We then get



$$\begin{aligned} \int_0^C y^{-1/2} \exp(ixy) dy \\ = i^{-3/2} \left[\int_0^\infty s^{-1/2} \exp(-xs) ds - \exp(ixC) \int_0^\infty (s-iC)^{-1/2} \exp(-xs) ds \right]. \end{aligned}$$

If we note that the first integral equals $\sqrt{\pi}x^{-1/2}$, and integrate the second integral by parts a sufficient number of times, we get the result.

Putting the above results together, we obtain

$$\begin{aligned} \int_0^C g(s^{1/2})s^{-1/2} \exp(2\pi i x \eta s) ds &= \sum_{p=0}^{N-1} \frac{\alpha_p(C) \exp(2\pi i x \eta C)}{(2\pi i x \eta)^{p+1}} \\ &+ \sum_{m=0}^N \frac{\alpha'_m}{(2\pi x \eta)^{m+(1/2)}} + \sum_{q=0}^{N-1} \frac{\alpha''_q}{(2\pi x \eta)^{q+1}} + O(x^{-N-1}). \end{aligned}$$

In the same way we see that

$$\begin{aligned} \int_0^\tau g(s^{1/2})s^{-1/2} \exp(2\pi i x \eta s) ds &= \sum_{p=0}^{N-1} \frac{\beta_p(\tau) \exp(2\pi i x \eta \tau)}{(2\pi i x \eta)^{p+1}} + \sum_{q=0}^{N-1} \frac{\alpha''_q}{(2\pi x \eta)^{q+1}} \\ &+ \sum_{m=0}^N \frac{\beta'_m}{(2\pi x \eta)^m} \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds + O(x^{-N-1}). \end{aligned}$$

Thus

$$\begin{aligned} \int_\tau^C g(s^{1/2})s^{-1/2} \exp(2\pi i x \eta s) ds &= \sum_{p=0}^{N-1} \frac{\alpha_p(C) \exp(2\pi i x \eta C) - \beta_p(\tau) \exp(2\pi i x \eta \tau)}{(2\pi i x \eta)^{p+1}} \\ &+ \sum_{m=0}^N \frac{\alpha'_m}{(2\pi x \eta)^{m+(1/2)}} - \sum_{m=0}^N \frac{\beta'_m}{(2\pi x \eta)^m} \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds + O(x^{-N-1}). \end{aligned}$$

Hence

$$\begin{aligned} &\int_x^{x(1+\gamma)} t^a \exp(2\pi i \eta \varphi(t)) dt \\ &= \frac{1}{2} n x^{a+1} \left\{ \sum_{p=0}^{N-1} \frac{(A_p(1+\gamma) \exp(2\pi i \eta \varphi(x(1+\gamma))) - B_p(1) \exp(2\pi i \eta \varphi(x)))}{(2\pi x \eta)^{p+1}} \right. \\ &\quad + \sum_{m=0}^N \frac{\alpha'_m \exp(2\pi i x \eta P(v_0))}{(2\pi x \eta)^{m+(1/2)}} - \sum_{m=0}^N \beta'_m \exp(2\pi i x \eta P(v_0)) \\ &\quad \times \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds \Big\} + O(x^{a-N}), \end{aligned}$$

since $xP((1+\gamma)^{1/n}) = \varphi(x(1+\gamma))$, and $xP(1) = \varphi(x)$. (The constants in the above formula are independent of N .) We have $xP(v_0) = \mu^{n/(n-1)} \cdot n^{-1/(n-1)}$, $\mu = (h/2\pi\eta) \cdot \lambda_k^{1/n}$. Note also that if

$$K(\tau, x) = \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds,$$

then

$$K(\tau, x) = x^{1/2} \int_0^{\tau x} y^{-1/2} \exp(2\pi i \eta y) \cdot x^{-1} dy = x^{-1/2} \int_0^{\tau x} y^{-1/2} \exp(2\pi i \eta y) dy.$$

Since $K(b) = \int_0^b s^{-1/2} \exp(2\pi i s) ds$ tends to zero as $b \rightarrow 0+$, and $K(b)$ tends to a finite limit as $b \rightarrow +\infty$, $K(b)$ is bounded everywhere, hence

$$K(\tau, x) = O(x^{-1/2})$$

uniformly in τ .

LEMMA 18. If $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}$, $\varepsilon > 0$, then

$$\int_x^\infty t^a \exp(2\pi i \eta \psi(t)) dt = \exp(2\pi i \eta \psi(x)) \sum_{l=0}^L \frac{(-1)^{l+1} G_{l, \nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a-L-1}),$$

where $a = a(\nu) = \omega_{-1} - (\nu/n)$.

The proof here is simpler than in the case of $\varphi(t)$, since $\psi(t) = t + \mu t^{1/n}$, so that $\psi'(t)$ is bounded both above and below for $t > 0$. (See Lemma 6.)

LEMMA 19. If $nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}$, $\varepsilon > 0$, then

$$\begin{aligned} \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt &= \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u_{\nu, m} x^{a-(\nu/n)+1} \cdot \lambda_k^{a-(\nu/n)} \frac{\exp(2\pi i \eta \varphi(x))}{(2\pi x \eta)^{m+1}} \\ &+ \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u'_{\nu, m} x^{a-(\nu/n)+1} \cdot \lambda_k^{a-(\nu/n)} \frac{\exp(2\pi i \eta x P(v_0))}{(2\pi x \eta)^{m+(1/2)}} \\ &+ \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u''_{\nu, m} x^{a-(\nu/n)+1} \cdot \lambda_k^{a-(\nu/n)} \cdot \frac{\exp(2\pi i \eta x P(v_0))}{(2\pi x \eta)^m} \\ &\times \int_0^\tau s^{-1/2} \exp(2\pi i x \eta s) ds \\ &+ \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} \frac{(-1)^l G_{m, \nu}(x)}{(2\pi i \eta)^{m+1}} \cdot \lambda_k^{a-(\nu/n)} + O(x^{a-N}). \end{aligned}$$

The coefficients $u_{\nu, m}$, $u'_{\nu, m}$, $u''_{\nu, m}$ are continuous functions of α for fixed ν and m .

The proof follows upon putting together Lemmas 16, 17, and 18, after using the asymptotic expansion for $I_{-1}(\lambda_k t)$, and then noting that the integral on the

left-hand side is independent of γ , hence in the expansion for it (obtained with the help of those lemmas) in decreasing powers of x , all the terms involving γ should vanish (by the kind of argument used in the proof of Lemma 10).

LEMMA 20. If $nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)} + x^{1-(1/n)-\varepsilon'}$, $0 < \varepsilon' < 1$, then

$$\int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt$$

has the same expansion as in Lemma 19, but for the additional summand $-I_0(0)/\lambda_k$.

The proof follows upon considering the integrals $\int_0^{x(1-\gamma)}$ and $\int_{x(1-\gamma)}^x$, with $0 < \gamma < 1$. The first integral can be treated as in Lemma 10, and the second as in Lemma 17.

LEMMA 21. If $\lambda_N \leq x < \lambda_{N+1}$, then

$$\begin{aligned} \int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) \beta(t) dt &= \exp(2\pi i \eta \varphi(x)) \sum c_{j, p, q, \nu} \lambda_k^{a-j} x^{a-j} (\varphi^{(p)}(x))^q \\ &\quad \times \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l \cdot \beta(t) \cdot \exp(2\pi i \eta (t-x) \varphi'(x)) dt \\ &\quad + \exp(2\pi i \eta \psi(x)) \sum c'_{j, p, q, \nu} \lambda_k^{a-j} x^{a-j} (\psi^{(p)}(x))^q \\ &\quad \times \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l \cdot \beta(t) \cdot \exp(2\pi i \eta (t-x) \psi'(x)) dt \\ &\quad + O(\lambda_k^{1/n} x^{(1/n)-M-1}), \end{aligned}$$

where $a = \omega_{-1} - (\nu/n)$, and the summation on the right-hand side extends over: $0 \leq \nu < n(M+1)$, $0 \leq j \leq M$, $0 \leq q \leq M$, $2 \leq p \leq M$, $j + pq = l$, all integers, and β is the function defined as in §3.

Proof. From the asymptotic expansion of $I_{-1}(t)$, we have

$$\begin{aligned} \int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) \cdot \beta(t) dt &= \sum_{\nu=0}^m e'_{\nu} \lambda_k^{\omega_{-1} - (\nu/n)} \int_{\lambda_N}^{\lambda_{N+1}} t^{\omega_{-1} - (\nu/n)} \cdot \exp(2\pi i \eta \varphi(t)) \\ &\quad \times \beta(t) dt + \sum_{\nu=0}^m e'_{\nu} \lambda_k^{\omega_{-1} - (\nu/n)} \int_{\lambda_N}^{\lambda_{N+1}} t^{\omega_{-1} - (\nu/n)} \cdot \exp(2\pi i \eta \psi(t)) \\ &\quad \times \beta(t) dt + O(x^{\omega_{-1} - (m+1/n)}). \end{aligned}$$

Now $t^a = x^a + \sum_{j \geq 1} c_j \cdot x^{a-j}(t-x)^j$, for $\lambda_N \leq t \leq \lambda_{N+1}$, and $(t-x)^j = O(1)$, for $j \geq 1$. Further

$$\varphi(t) = \varphi(x) + \sum_{1 \leq p \leq M} \frac{(t-x)^p}{p!} \cdot \varphi^{(p)}(x) + O(\lambda_k^{1/n} \cdot x^{(1/n)-M-1}).$$

Hence

$$\begin{aligned} & \int_{\lambda_N}^{\lambda_{N+1}} t^a \exp(2\pi i \eta \varphi(t)) \cdot \beta(t) \cdot dt \\ &= \sum_{j=0}^M c_j x^{a-j} \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^j \exp(2\pi i \eta \varphi(t)) \cdot \beta(t) dt + O(x^{-M-1}) \\ &= \exp(2\pi i \eta \varphi(x)) \sum_{j=0}^M c_j x^{a-j} \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^j \exp(2\pi i \eta (t-x) \varphi'(x)) \cdot \beta(t) \\ & \quad \times \exp \left\{ (2\pi i \eta) \left(\sum_{2 \leq p \leq M} \frac{(t-x)^p}{p!} \varphi^{(p)}(x) \right) \right\} dt + O(x^{-M-1}) + O(\lambda_k^{1/n} x^{(1/n)-M-1}) \\ &= \exp(2\pi i \eta \varphi(x)) \sum_{j, p, q} c'_{j, p, q} x^{a-j} (\varphi^{(p)}(x))^q \\ & \quad \times \int_{\lambda_N}^{\lambda_{N+1}} (t-x)^l \cdot \beta(t) \cdot \exp(2\pi i \eta (t-x) \varphi'(x)) dt + O(\lambda_k^{1/n} x^{(1/n)-M-1}), \end{aligned}$$

the last sum extending over: $0 \leq j \leq M$, $2 \leq p \leq M$, $q \geq 0$, $j + pq = l$, all integers. Note that the last integral equals

$$\int_{\lambda_N - x}^{\lambda_{N+1} - x} t^{j+pq} \cdot \beta(x+t) \cdot \exp(2\pi i \eta t \varphi'(x)) dt = O(1),$$

and, if we integrate by parts, we see that this is also $\ll 1/|\varphi'(x)|$.

A similar result holds with ψ in place of φ . If we choose $m+1 \geq n(M+1)$, we get the lemma.

LEMMA 22. *For the first integral in Lemma 1, we have the estimate:*

$$(-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) \cdot dt = O(1).$$

Proof. By partial integration, we see that

$$\begin{aligned}
 & (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) \cdot Q_r(t) dt \\
 &= \int_0^{\lambda_{N+1}} F'(t) \cdot Q_0(t) dt = -Q_0(0) - \int_0^{\lambda_{N+1}} F(t) \cdot Q_{-1}(t) \cdot dt \\
 &= -Q_0(0) - R_0 \left\{ \int_0^{\lambda_N} \exp(2\pi i \eta t) dt + \int_{\lambda_N}^{\lambda_{N+1}} \exp(2\pi i \eta t) \beta(t) dt \right\} \\
 &= O(1),
 \end{aligned}$$

since $\lambda_{N+1} - \lambda_N$, and β are bounded; and

$$Q_{-1}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} B^{-s} \zeta_K(s) x^{s-1} ds, \quad x > 0,$$

so that $Q_{-1}(x)$ equals B^{-1} times the residue of $\zeta_K(s)$ at $s = 1$, and $Q_0(0)$ (which is defined by continuity) equals $\zeta_K(0)$.

§5. Proof of theorem

If $0 < \varepsilon' < 1/n$, $\lambda_N \leq x < \lambda_{N+1}$, and $\eta > 0$, we have, by Lemma 1,

$$\sum_{k=1}^N a_k \exp(2\pi i \eta \lambda_k) = W_1 + W_2 + W_3 + W_4 + W_5 + W_6, \quad (5.1)$$

where

$$W_1 = (-1)^r \int_0^{\lambda_{N+1}} F^{(r+1)}(t) Q_r(t) dt,$$

$$W_2 = O(x^{-q'}), \quad \text{for any } q' > 0,$$

$$W_3 = - \sum_{\mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_{\lambda_N}^{\lambda_{N+1}} I_{-1}(\lambda_k t) \cdot \exp(2\pi i \eta t) \cdot \beta(t) dt,$$

$$W_4 = - \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_0^{\infty} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt - \sum_{\mu \leq nx^{1-(1/n)}} \frac{a_k I_0(0)}{\lambda_k},$$

$$W_5 = \sum_{\mu \leq nx^{1-(1/n)}} a_k \int_{\lambda_N}^{\infty} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt,$$

$$W_6 = - \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_0^{\lambda_N} \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt - \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} \frac{a_k I_0(0)}{\lambda_k}.$$

We suppose, to begin with, that $\lambda_N = x$. By Lemma 22, we have

$$W_1 = O(1). \quad (5.2)$$

To estimate W_5 we write it as a sum of three separate sums, namely

$$\begin{aligned} W_5 &= W_{5,1} + W_{5,2} + W_{5,3} \\ &= \sum_{\mu \leq (n-\delta)x^{1-(1/n)}} + \sum_{(n-\delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)} - x^{1-(1/n)-(\epsilon/n)}} + \sum_{nx^{1-(1/n)} - x^{1-(1/n)-(\epsilon/n)} < \mu \leq nx^{1-(1/n)}}. \end{aligned}$$

Now, by Lemma 7, we have

$$\begin{aligned} W_{5,1} &= \sum_{\mu \leq (n-\delta)x^{1-(1/n)}} a_k \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &\ll x^a \sum_{\lambda_k \leq (c_0 - \delta_1)x^{n-1}} a_k \lambda_k^a \ll x^{(n/2)-(1/2)}, \end{aligned} \quad (5.3)$$

if we note that, in this range, $F_{0,0}(x) = O(x^a)$, $G_{0,0}(x) = O(x^a)$, (where $a = (1/2n) - \frac{1}{2}$), and use the fact that for $-1 < \alpha \leq 0$, we have

$$\sum_{\lambda_k \leq y} a_k \lambda_k^\alpha \sim cy^{1+\alpha}, \quad \text{as } y \rightarrow \infty, \quad (5.4)$$

since $\sum_{\lambda_k \leq y} a_k \sim \lambda' y$.

By the last part of Lemma 12, we have

$$\begin{aligned} W_{5,2} &\ll \sum_{(c_0 - \delta_1)x^{n-1} < \lambda_k < c_0 x^{n-1} - c_1 x^{n-1-\epsilon/n}} \left(\frac{a_k \lambda_k^a x^a}{1 - \frac{1}{n} \mu x^{(1/n)-1}} \right) \\ &\ll x^{a+a(n-1)} \sum_{(1-\delta_2)y < \lambda_k < y-z} \frac{a_k}{1 - \lambda_k^{1/n} y^{-1/n}}, \end{aligned}$$

with $y = c_0 x^{n-1}$, $z = c_1 x^{n-1-\epsilon/n}$. (Here, as elsewhere, we take $\delta_1 > 0$, $\delta_2 > 0$, and sufficiently small. We adopt the usual convention that an empty sum is zero and an empty product is 1.) Since $A(t) = \sum_{\lambda_k \leq t} a_k = \lambda' t + R(t)$, where $R(t) = O(t^{1-2/(n+1)})$, we have

$$\begin{aligned} \sum_{(1-\delta_2)y < \lambda_k < y-z} \frac{a_k}{1 - \lambda_k^{1/n} y^{-1/n}} &\ll \int_{(1-\delta_2)y}^{y-z} \frac{dA(t)}{1 - (t/y)^{1/n}} \\ &= \lambda' \int_{(1-\delta_2)y}^{y-z} \frac{dt}{1 - (t/y)^{1/n}} + \int_{(1-\delta_2)y}^{y-z} \frac{dR(t)}{1 - (t/y)^{1/n}}. \end{aligned}$$

Now

$$\begin{aligned} c \int_{(1-\delta_2)y}^{y-z} \frac{dt}{1-(t/y)^{1/n}} &= \lambda' y \int_{(1-\delta_2)}^{1-z/y} \frac{(1-s)}{(1-s^{1/n})} \frac{1}{(1-s)} ds \\ &= O\left(y \log \frac{y}{z}\right) = O(x^{n-1} \log(1+x)). \end{aligned}$$

Similarly

$$\int_{(1-\delta_2)y}^{y-z} \frac{dR(t)}{1-(t/y)^{1/n}} = O\left(y^{1-n} \cdot \frac{y}{z}\right) = O(x^{n-1}), \quad \text{if } 0 < \varepsilon \leq n-1.$$

Thus we have altogether

$$W_{5,2} = O(x^{(n-1)/2} \log(1+x)), \quad \text{if } 0 < \varepsilon \leq n-1. \quad (5.5)$$

To handle $W_{5,3}$ we use Lemma 17, and obtain

$$\begin{aligned} W_{5,3} &= \sum_{nx^{1-(1/n)} - x^{1-(1/n)-(\varepsilon/n)} < \mu \leq nx^{1-(1/n)}} a_k \int_x^\infty \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &\ll \sum_{c_0 x^{n-1} - c_2 x^{n-1-\varepsilon/n} < \lambda_k \leq c_0 x^{n-1}} a_k \cdot \lambda_k^a \cdot x^{a+(1/2)} \quad \left(a = \frac{1}{2n} - \frac{1}{2}\right) \\ &\ll x^{1/2n} \sum a_k \lambda_k^a \ll x^{(1/2n)+a(n-1)} \sum_{c_0 x^{n-1} - c_2 x^{n-1-\varepsilon/n} < \lambda_k \leq c_0 x^{n-1}} a_k, \quad \text{since } \lambda_k \ll x^{n-1} \\ &\ll x^{(1/2n)+a(n-1)+(n-1-(\varepsilon/n))} \ll x^{(n/2)-(\varepsilon/n)}, \end{aligned} \quad (5.6)$$

since $\sum_{y-z < \lambda_k \leq y} a_k = c_5 z + O(y^{1-(1/n)})$, say, for $0 < z < y$. Now (5.6), (5.5), and (5.3) give

$$W_5 = O(x^{(n-1)/2}) + O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\varepsilon/n)}), \quad (5.7)$$

if $\varepsilon \leq n-1$.

The estimation of W_6 is similar. We first split it up into three parts:

$$\begin{aligned} W_6 &= W_{6,1} + W_{6,2} + W_{6,3} \\ &= - \sum_{(n+\delta)x^{1-(1/n)} \leq \mu \leq nx^{1-(1/n)+\varepsilon}} - \sum_{nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)} \leq \mu < (n+\delta)x^{1-(1/n)}} \\ &\quad - \sum_{nx^{1-(1/n)} < \mu \leq nx^{1-(1/n)} + x^{1-(1/n)-(\varepsilon/n)}}, \end{aligned}$$

in each of which there are two summands. We use Lemma 10 in $W_{6,1}$ and obtain

$$\begin{aligned} W_{6,1} &= - \sum_{(n+\delta)x^{1-(1/n)} \leq \mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \int_0^x \exp(2\pi i \eta t) I_{-1}(\lambda_k t) dt \\ &= O\left(\sum a_k \lambda_k^a \cdot |F_{0,0}(x)|\right) = O\left[\sum a_k \cdot \lambda_k^a \cdot x^a \left(\frac{1}{n} \mu x^{(1/n)-1} - 1\right)^{-1}\right] \\ &= O(x^{(n-1)/2} \log(1+x)), \quad \text{if } 0 < \varepsilon \leq n-1, \end{aligned} \quad (5.8)$$

as in the case of $W_{5,2}$ (see the proof of (5.5)). Note that the sum $-\sum a_k I_0(0)/\lambda_k$ cancels out in this range.

By Lemma 15, we obtain, as in (5.5),

$$W_{6,2} = O(x^{(n-1)/2} \log(1+x)), \quad \text{if } 0 < \varepsilon \leq n-1. \quad (5.9)$$

Finally, by Lemma 20 we obtain, as in (5.6),

$$W_{6,3} = O(x^{(n/2)-(\varepsilon/n)}). \quad (5.10)$$

Hence

$$W_6 = O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\varepsilon/n)}), \quad 0 < \varepsilon \leq n-1. \quad (5.11)$$

To estimate W_3 , we note that the principal term in the expansion given by Lemma 21 is a constant multiple of

$$\sum_{\mu \leq nx^{1-(1/n)+\varepsilon'}} a_k \exp(2\pi i \eta \varphi(x)) \lambda_k^a x^a \int_{\lambda_N}^{\lambda_{N+1}} \beta(t) \exp(2\pi i \eta(t-x)\varphi'(x)) dt,$$

together with a similar term with ψ in place of φ . It follows that

$$\begin{aligned} W_3 &\ll \sum_{\mu \leq (n+\delta)x^{1-(1/n)}} a_k \cdot \lambda_k^a \cdot x^a + \sum_{(n+\delta)x^{1-(1/n)} < \mu \leq nx^{1-(1/n)+\varepsilon'}} \left(\frac{a_k \lambda_k^a x^a}{\left(\frac{1}{n} \mu x^{(1/n)-1} - 1\right)} \right) \\ &\ll x^{(n-1)/2} \log(1+x), \end{aligned} \quad (5.12)$$

since $0 < \varepsilon' < 1/n \leq (n-1)/n$. Note that the estimate of the first sum is similar to that of (5.3), and that of the second similar to (5.5).

By Lemma 4, we have

$$W_4 = c_4 \sum_{\mu \leq nx^{1-(1/n)}} a_k \lambda_k^{(2-n)/2(n-1)} \exp\{-(2\pi i m_k^{1/(n-1)} \cdot q)\} + O(x^{(n/2)-1}), \quad (5.13)$$

where c_4 is a non-zero constant, $m_k = \lambda_k/2\pi\eta$, $q = (n-1)2^{r_1/(n-1)}$. Since $-1 < \{(2-n)/2(n-1)\} \leq 0$, for $n \geq 2$, we have only to use (5.4). Note that $-\sum (a_k/\lambda_k)I_0(0)$ cancels out $+\sum \Delta(1)/\Delta(0) \cdot a_k/\lambda_k$.

Combining (5.1), (5.2), (5.7), (5.11), (5.12), and (5.13), we get

$$\begin{aligned} & \sum_{\mu \leq nx^{1-(1/n)}} a_k \lambda_k^{(2-n)/2(n-1)} \exp \{-(2\pi i m_k^{1/(n-1)} \cdot q)\} - c \sum_{k=1}^N a_k \exp(2\pi i \eta \lambda_k) \\ &= O(x^{(n/2)-1}) + O(1) + O(x^{-q'}) + O(x^{(n-1)/2}) \\ & \quad + O(x^{(n-1)/2} \log(1+x)) + O(x^{(n/2)-(\epsilon/n)}), \end{aligned} \quad (5.14)$$

where c is a constant, $n \geq 2$, $0 < \epsilon \leq n-1$, $0 < \epsilon' < 1/n$, q' is an arbitrarily given positive integer. If we choose $\epsilon/n = \frac{1}{2}$, as we may, we get for $x = \lambda_N$, $n \geq 3$,

$$\sum_{\lambda_k \leq c_0 \lambda_N^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} \exp \{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(x^{(n-1)/2} \log(1+x)), \quad (5.15)$$

where $H = q(2\pi\eta)^{-1/(n-1)}$, since

$$\sum_{k=1}^N a_k \exp(2\pi i \eta \lambda_k) = O(x) = O(x^{(n-1)/2}),$$

for $n \geq 3$. On the other hand, we have, for any $\epsilon_1 > 0$, and $x = \lambda_N$,

$$\begin{aligned} \sum_{c_0 \lambda_N^{n-1} < \lambda_k < c_0 \lambda_{N+1}^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} &\ll x^{(2-n)/2} \sum_{c_0 \lambda_N^{n-1} < \lambda_k < c_0 \lambda_{N+1}^{n-1}} a_k \\ &\ll x^{(2-n)/2+\epsilon_1} (\lambda_{N+1}^{n-1} - \lambda_N^{n-1}) \\ &\ll x^{n-2+1-(n/2)+\epsilon_1} \ll x^{(n-1)/2}, \end{aligned}$$

since $\lambda_{N+1} \leq x + B$, so that $\lambda_{N+1}^{n-1} - \lambda_N^{n-1} \ll x^{n-2}$.

Thus it follows from (5.15) that

$$\sum_{\lambda_k \leq c_0 y^{n-1}} a_k \lambda_k^{(2-n)/2(n-1)} \exp \{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(y^{(n-1)/2} \log(1+y)),$$

for $\lambda_N \leq y < \lambda_{N+1}$. From this we deduce that

$$\sum_{\lambda_k \leq c_0 y} a_k \lambda_k^{(2-n)/2(n-1)} \exp \{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O(y^{1/2} \log(1+y)),$$

and this implies, by partial summation, that

$$\sum_{\lambda_k \leq y} a_k \exp \{-(2\pi i \lambda_k^{1/(n-1)} \cdot H)\} = O_\eta(y^{1-(1/2(n-1))} \log(1+y)),$$

where $H = q/(2\pi\eta)^{1/(n-1)}$. Note that c_0 involves η , where η is arbitrary and positive. By taking conjugates, if necessary, we obtain the theorem.

Remark. It is possible to prove a corresponding result for the coefficient sums of Dirichlet series satisfying a functional equation of the type treated in [1, 2]. The proof requires no new ideas. In particular, the result is valid for the zeta-function of an ideal class and Hecke's zeta-function with Grössencharacters.

We note that if $n = 2$, (5.14) reduces to the result given in [4].

We also note that considered as a result valid for *all* $n \geq 2$, the theorem is a "best possible" one, after Walfisz [6, p. 566], though the estimate can be improved for particular values of n . For $n \geq 7$, for example, partial summation will suffice.

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