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On Cauchy-Frullani Integrals[1]

by A. M. Ostrowski

I. Introduction

1. A beautiful result due essentially to Cauchy [2], [3], but attributed usually to Frullani [2]) is contained in the integral formula

$$\int_0^\infty \frac{f(at)-f(bt)}{t}\,dt=(f(\infty)-f(0))\lg\frac{a}{b} \qquad (a\wedge b>0),$$
$$f(0):=\lim_{x\downarrow 0} f(x), \qquad f(\infty):=\lim_{x\to\infty} f(x). \tag{I,1}$$

$f(x)$ is assumed L integrable in $(0, \infty)$[3]).

It can be expected that this formula remains valid, if the limits $f(0), f(\infty)$ do not exist, but are replaced by appropiate *mean values* in the corresponding neighbourhoods of $x=0$, $x=\infty$. Thus the problem arises to find such definitions of mean values. The problem will be then more or less completely solved, if from the convergence of the left hand integral in (I, 1), a and b varying in some intervals, the existence of these mean values follows.

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[2] G. Frullani, Sopra Gli Integrali Definiti, Ricevuta adi 21 Novembre, Memorie della Società Italiana delle Scienze, Modena, XX, pp. 448–467.

The volume is dated 1828; however, it contains another paper by Frullani with the note: "Ricevuta adi 1. Oktobre 1830". On page 460 of his paper Frullani says: "Io comunicai questo risultato al ch. Plana sino dal 1821. Successivamente, e nel giornale della Scuola Politecnica per l'anno 1823 ne ho veduta una dimostratione dovuta al ch. Cauchy, e dedotta da principi differentissimi dai precedenti". However, Frullani's "proof" is completely illusory.

As to Cauchy, his paper [2] of 1823 contains only the case $f(\infty)=0$ of (I,1). However, Cauchy assumes in his proof implicitly (and unnecessarily) that $\int^\infty [f(x)/x]\,dx$ is convergent, which does not follow from $\lim_{x\to\infty} f(x)=0$. In the article [3] of 1827, Cauchy writes down the formula (I,1)

Note continued on next page

2. Before attacking the general problem observe that introducing in the integral in (I, 1) a new variable of integration, τ, by $t=u\tau$, $u>0$, this integral becomes

$$\int_0^\infty \frac{f(ua\tau)-f(ub\tau)}{\tau}\,d\tau,$$

so that our integral, in the case of convergence, remains convergent and does not change its value if a and b are multiplied by an arbitrary positive number. We can therefore replace in our discussion, putting $\varrho:=a/b$, the integral in (I, 1) with the integral

$$\int_0^\infty \frac{f(\varrho t)-f(t)}{t}\,dt, \quad \varrho>0. \tag{I,2}$$

Further it is useful to deal separately with the neighbourhoods of 0 and ∞, putting

$$f(x)=\pi(x)+\mu(x), \quad \pi(x)=\begin{cases} f(x) & (x\geq 1)\\ 0 & (0<x<1),\end{cases} \quad \mu(x)=\begin{cases} 0 & (x\geq 1)\\ f(x) & (0<x<1).\end{cases} \tag{I,3}$$

Note 2) continued
(as his formula (66)) and goes on to say that this formula "se déduit aisément, ainsi que M. Ostrogradsky en a fait la remarque, de l'équation

$$\int_0^\infty f'(ar)\,dr = \frac{f(\infty)-f(0)}{a}$$

intégrée par rapport à la quantité a. On pourrait au reste établir l'équation (66)... à l'aide de la théorie des intégrales singulières."

Cauchy's formulation is rather vague. But it appears that the complete formula (I,1) was first indicated by Ostrogradsky, although the carrying out of Ostrogradsky's idea of the proof would require some additional assumptions. This was obviously a personal communication of Ostrogradsky, as in Ostrogradsky's Collected Papers the subject is not mentioned anywhere. Cauchy's own proof alluded to above is the usual proof using our identity (I,4) from sec. 2.

[3] If we say that a function is *L integrable* or *bounded* or *absolutely continuous* or that an expression (or series) is *uniformly convergent* "*in an interval J*" which could be finite or infinite, open or closed or half open, this signifies that the corresponding property holds in any closed interval contained in J.

We denote an open interval by (α, β), a closed interval by $\langle\alpha, \beta\rangle$ and the half open intervals by $\langle\alpha, \beta)$ and $(\alpha, \beta\rangle$.

Further we use the notations $A:=B$, $A=:B$, in the sense A means B, A is denoted by B.

The symbol $\exists$ signifies "*exists*" and the logical symbol $\wedge$ is to be read: *as well as*. This symbol has priority with respect to the symbols $=$, $>$, $<$, $\exists$. $\vee$ is to be read: *or*.

On the other hand the following identity is immediately verified:

$$\int_{\varepsilon}^{A} \frac{f(\varrho t)-f(t)}{t}\,dt = \int_{A}^{\varrho A} \frac{f(t)}{t}\,dt - \int_{\varepsilon}^{\varrho \varepsilon} \frac{f(t)}{t}\,dt \quad (0<\varepsilon<A), \tag{I,4}$$

and it follows at once that a necessary and sufficient condition for the convergence of the integral (I,2), that is for the existence of the limit of the left side integral in (I,4) with $A\to\infty$, $\varepsilon\downarrow 0$, is that both right side terms in (I,4) have limits, that is to say that the integrals (I,2) for $\pi(x)$ and $\mu(x)$ exist separately.

Further, by the identity

$$\int_{0}^{\infty} \frac{f(\varrho t)-f(t)}{t}\,dt = \int_{0}^{\infty} \frac{f(\varrho/t)-f(1/t)}{t}\,dt, \tag{I,5}$$

the discussion of $\mu(x)$ is reduced at once to that of $\pi(x)$ and vice versa. It will therefore be sufficient in the discussion of (I,2) to assume that $f(x)=0$ $(0<x\leqq 1)$.

3. A first general solution of our problem has been indicated by K. S. K. Iyengar, 1940, [1], [2]. Iyengar's necessary and sufficient condition for the integral (I,2) being convergent for any ϱ from an interval on the positive t-axis, is the existence of both

$$\int_{1}^{\infty} \frac{f(t)}{t^2}\,dt, \qquad \lim_{x\to\infty} x \int_{x}^{\infty} \frac{f(t)}{t^2}\,dt. \tag{I,6}$$

However Iyengar's proof, although very skilful in the most parts, contains in an essential point a grave mistake which apparently cannot be improved directly. It concerns Iyengar's formula (6.2). Iyengar proves first that for his function $F(u)$ and a certain ϱ_1, $0<\varrho_1<1$, the expression $F(u)-F(\varrho_1 u)/\varrho_1$ tends to L with $u\downarrow 0$. If we write this in the form

$$F(u)-F(\varrho_1 u)/\varrho_1 = L+\eta(u), \qquad \eta(u)\to 0 \ (u\downarrow 0),$$

Iyengar's formula (6.2) can be written as

$$F(u)-\frac{1}{\varrho_1^m} F(u\varrho_1^m) = L\frac{\varrho_1^{-m}-1}{\varrho_1^{-1}-1} + \sum_{r=1}^{m} \frac{\eta(u\varrho_1^{r-1})}{\varrho_1^{r-1}}. \tag{$*$}$$

Now Iyengar assumes that u varies, for a positive u_0, in the interval $u_0 \geqq u \geqq u_0 \varrho_1$, puts $y := u\varrho_1^m$, so that $y \downarrow 0$ is equivalent with $m \to \infty$, and asserts that then the right hand η-sum in $(*)$ tends with $y \downarrow 0$ to 0. This does not of course follow as already the first term of this sum corresponding to $r=1$ is $\eta(u)$, and u remains $\geqq u_0 \varrho_1 > 0$.

4. Nevertheless, Iyengar's assertion concerning the existence of (I,6) is true, as it has been proved 1942 and 1954 by R. P. Agnew [1], [3]. Further, Agnew replaces the interval on the ϱ-axis in Iyengar's discussion by an *arbitrary set of positive measure.*

5. In what follows we give first another solution of the above problem. We prove that necessary and sufficient for the integral (I,2) to be convergent for all ϱ from a set of positive measure on the positive ϱ-axis is the existence of the limit

$$M(f) := \lim_{x \to \infty} \frac{1}{x} \int_1^x f(t)\, dt. \tag{I,7}$$

This condition is of course essentially simpler than the condition concerning (I,6). On the other hand the proof of our result can be carried out in a simpler way than the argumentation of Agnew, since, in order to deal with not necessarily uniform convergence, we use only Osgood's theorem for a convergent sequence of continuous functions, the proof of which does not even make use of the theory of measure.

6. We give in chap. III the proof of our results concerning the integral (I,2) as theorem A, after some preliminary discussions in chap. II. In chap. IV we prove directly that Iyengar's conditions are equivalent with ours[4]). In this way a considerable simplification of the proof of Iyengar's conditions is achieved. In chap. V we formulate the theorem B concerning the integral in (I,1) in its original form and give further some examples, specializing $f(t)$.

7. The main interest of the formula (I,1) consists of course in the fact that it contains an essentially *arbitrary function* $f(t)$. By a variable transformation we can of course always introduce another arbitrary function. We will show in the second part of this paper, that a generalisation of (I,1) is possible, containing *three*, more or less arbitrary, functions and give the value of an integral of the form

$$\int_a^b (\psi'(x)\, g(\psi(x)) - \varphi'(x)\, g(\varphi(x)))\, dx \tag{I,8}$$

[4] Another proof of this equivalence was given by Agnew [2], with reference to Ostrowski [1].

as theorem C. The proof of this *Three Functions Formula* is given in chap. VII after some preliminary discussions in chap. VI, while chap. VIII brings different specialisations of the Three Functions Formula, obtaining in this way in particular different formulas going back to Cauchy and Lerch.

Main results of this paper have been communicated and proofs partially sketched, 1949, in Ostrowski [1].

II. Discussion of $g(q)$

8. In this whole chapter $f(x)$ is a function L integrable in $(0, \infty)$. The letters x, y, q, u, v, w, t denote *positive* numbers. Then it follows by an obvious change of integration variable that

$$\int_x^{ux} \frac{f(wt)}{t}\,dt = \int_{wx}^{uwx} \frac{f(t)}{t}\,dt. \tag{II,1}$$

We can therefore define

$$g(u) := \lim_{x\to\infty} \int_x^{ux} \frac{f(t)}{t}\,dt = \lim_{x\to\infty} \int_{vx}^{vux} \frac{f(t)}{t}\,dt, \tag{II,2}$$

for any positive u for which the right hand limit exists. It follows then, if both $g(u)$ and $g(v)$ exist,

$$g(v)+g(u) = \int_x^{vx} \frac{f(t)}{t}\,dt + \int_{vx}^{uvx} \frac{f(t)}{t}\,dt = \int_x^{uvx} \frac{f(t)}{t}\,dt,$$

$$g(uv) = g(u)+g(v) \qquad (\exists g(u) \wedge g(v)). \tag{II,3}$$

It follows further that

$$g\left(\frac{1}{u}\right) = \lim_{x\to\infty} \int_x^{x/u} \frac{f(t)}{t}\,dt = \lim_{x\to\infty} \int_{ux}^{x} \frac{f(t)}{t}\,dt = -g(u),$$

if $g(u)$ exists, and, further, replacing v in (II,3) with $1/v$,

$$g(u/v) = g(u)-g(v) \qquad (\exists g(u) \wedge g(v)). \tag{II,4}$$

9. Assume now that $g(u)$ exists for all u from the interval

$$0<Q_1<u<Q_2<\infty.$$

Then, using (II,3) and (II,4) repeatedly, it follows that $g(u)$ exists for all positive u.

If we assume even less than that, namely that $g(u)$ exists for all $u\in S$, where S is a set in $(0, \infty)$ of positive measure, then, if $u_1 \wedge u_2 \in S$, $g(u_1/u_2)$ exists too. But, as follows immediately from a well-known theorem by Steinhaus, the set of all quotients u_1/u_2, if $u_1 \wedge u_2 \in S$, contains an interval of positive length. Therefore, in this case too, $g(u)$ exists for all positive u.

In this case we can assume that in (II,3) u and v are arbitrary positive numbers. On the other hand it follows from (II,2), if we let $x\to\infty$ over integers, that $g(u)$ as the limit of a sequence of continuous functions, is measurable for all positive u. Therefore, in virtue of a theorem by Fréchet [1], [2]; Sierpinski [1]; Banach [1], we have

$$g(u)=C\lg u \tag{II,5}$$

for a convenient *constant* C.[5])

10. LEMMA 1. *Assume that* $f(t)$ *is continuous in* $(0, \infty)$ *and* $g(q)$ *in* (II,2) *exists for all* $q>0$. *Assume further that we have*

$$f(qt)-f(t)\to 0 \quad (t\to\infty,\ q>0). \tag{II,6}$$

Then

$$f(t)\to C \quad (t\to\infty), \tag{II,7}$$

where C is the constant from (II,5), *and*

$$g(q)=C\lg q \quad (q>0). \tag{II,8}$$

11. *Proof.* We start from the identity

$$f(x)\lg\frac{q''}{q'}=\int\limits_{q'x}^{q''x}\frac{f(t)}{t}\,dt-\int\limits_{q'}^{q''}\frac{f(qx)-f(x)}{q}\,dq,$$

[5] The theorem in question deals with the functional equation $\varphi(x+y)=\varphi(x)+\varphi(y)$ to which (II,3) is reduced by the substitution $\varphi(x):=g(e^x)$.

which is immediately verified introducing in the second right side integral the new variable of integration, $t:=qx$.

Let $x_\nu\to\infty$ be an x-sequence, for which

$$\lim_{\nu\to\infty} f(x_\nu)=\Gamma,$$

with Γ finite or infinite. Then it follows

$$f(x_\nu)\lg\frac{q''}{q'}=\int_{q'x_\nu}^{q''x_\nu}\frac{f(t)}{t}\,dt-\int_{q'}^{q''}\frac{f(qx_\nu)-f(x_\nu)}{q}\,dq. \tag{II,9}$$

The sequence of continuous functions

$$G_\nu(q):=f(qx_\nu)-f(x_\nu) \tag{II,10}$$

tends to 0 for any positive q. It follows now from a well-known theorem by Osgood, that for each $\eta>0$ there exists a *subinterval* $q'\leqq q\leqq q''$ of $(0,\infty)$, of positive length, and an n_0, such that

$$|G_\nu(q)|\leqq\eta \qquad (\nu>n_0,\ q'\leqq q\leqq q'').$$

12. We apply now (II,9) to the interval $\langle q', q''\rangle$; then, by (II,5), the first right side integral in (II,9) tends to $g(q''/q')=C\lg(q''/q')$, while the modulus of the second integral remains $\leqq\eta\lg(q''/q')$. Therefore Γ is finite and we have

$$|\Gamma-C|\leqq\eta.$$

Since η is here arbitrary small, we have $\Gamma=C$. We see, that any convergent sequence $f(x_\nu)$ with $\nu\to\infty$ has C as limit and it follows $\lim_{x\to\infty}f(x)=C$. Our lemma is proved.

III. Theorem A

13. A. *Assume $f(x)$ L integrable in $\langle 0,\infty)$, and further*

$$f(t)=0 \qquad (0\leqq t\leqq 1). \tag{III,1}$$

Then

$$\frac{1}{\lg \varrho}\int_0^\infty \frac{f(\varrho t)-f(t)}{t}\,dt = M(f) := \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)\,dt \qquad (\varrho>0), \tag{III,2}$$

either if $M(f)$ exists, and then for all $\varrho>0$, or if the left side integral exists for all ϱ from a subset, of positive measure, of $(0,\infty)$.

14. *Proof.* The function

$$F(x)=\frac{1}{x}\int_0^x f(t)\,dt \qquad (x\geqq 0) \tag{III,3}$$

is obviously continuous for all $x>0$ and we have $f(x)=(xF(x))'$ with the exception of a set of measure 0. Therefore, for $u\wedge\varrho>0$:

$$\int_u^{\varrho u}\frac{f(t)}{t}\,dt=\int_u^{\varrho u}\frac{(tF(t))'}{t}\,dt=F(t)\Big|_u^{\varrho u}+\int_u^{\varrho u}\frac{F(t)}{t}\,dt,$$

$$\int_u^{\varrho u}\frac{f(t)}{t}\,dt=(F(\varrho u)-F(u))+\int_u^{\varrho u}\frac{F(t)}{t}\,dt. \tag{III,4}$$

Suppose now that $M(f)=\lim_{x\to\infty}F(x)$ in (III,2) exists. Then for $u\to\infty$ the first right side term in (III,4) tends to 0, while the integral on the right is $=F(\xi)\int_u^{\varrho u}dt/t=F(\xi)\lg\varrho$, where ξ lies between u and ϱu. We have therefore

$$\int_u^{\varrho u}\frac{f(t)}{t}\,dt\to M(f)\lg\varrho \qquad (u\to\infty),$$

and this is (III,2), using (I,4) with $\varepsilon<1$, $\varrho\varepsilon<1$.

15. Assume now that the left side integral in (III,2) exists for all $\varrho\in S$, where S is a subset, of positive measure, of $(0,\infty)$. We can then assume that all elements of S are even $\leqq q$ for a convenient $q>0$. In the identity (I,4), if we assume $\varepsilon<1/q$, the second right side integral becomes 0 and it follows that the first right side integral tends to a limit with $A\to\infty$ for all $\varrho\in S$. But this signifies that $g(\varrho)$ exists on a set of positive measure and therefore, as was mentioned in sec. 9, $g(\varrho)$ exists for all $\varrho>0$.

16. (III,4) can now be written in the form

$$\int_u^{\varrho u} \frac{f(t)}{t}\,dt=(F(\varrho u)-F(u))+\int_u^{\varrho u} \frac{F(t)}{t}\,dt=\left(u\int_u^{\varrho u} \frac{F(t)}{t}\,dt\right)_u'. \tag{III,5}$$

Here, since the integral on the left tends, with $u\to\infty$, to $g(\varrho)$, we can write

$$\left(u\int_u^{\varrho u} \frac{F(t)}{t}\,dt\right)_u'=g(\varrho)+\varepsilon(u,\varrho)\qquad(\varrho>0),$$

where $\lim_{u\to\infty}\varepsilon(\varrho,u)=0$ for all $\varrho>0$.

Integrating this from 0 to $u>0$ we have

$$u\int_u^{\varrho u} \frac{F(t)}{t}\,dt=g(\varrho)\,u+\int_0^u \varepsilon(\varrho,u)\,du$$

and therefore

$$\int_u^{\varrho u} \frac{F(t)}{t}\,dt\to g(\varrho)\qquad(u\to\infty,\ \varrho>0), \tag{III,6}$$

since $(1/u)\int_0^u \varepsilon(\varrho,u)\,du$ tends to 0 with $u\to\infty$ for any $\varrho>0$.

But now it follows from (III,5)

$$\lim_{u\to\infty}(F(\varrho u)-F(u))=0\qquad(\varrho>0),$$

and we see that all assumptions of lemma 1 are satisfied if we replace there $f(t)$ with $F(t)$; it follows that

$$\lim_{x\to\infty}F(x)=:M_f$$

exists. Theorem A is proved.

IV. Equivalence with Iyengar's Conditions

17. We are going now to prove that *Iyengar's* conditions

$$\exists \int_1^\infty \frac{f(t)}{t^2}\,dt, \quad \exists \lim_{x\to\infty} x \int_x^\infty \frac{f(t)}{t^2}\,dt =: L \tag{IV,1}$$

are equivalent with the condition

$$\exists \lim_{x\to\infty} \frac{1}{x}\int_1^x f(t)\,dt =: M(f), \tag{IV,2}$$

and then $M(f)=L$.

Put for $x \geqq 1$:

$$\varphi(x) := \int_1^x f(t)\,dt; \tag{IV,3}$$

then it follows at once, integrating by parts for any $x \geqq 1$:

$$\int_1^x \frac{f(t)}{t^2}\,dt = \frac{\varphi(x)}{x^2} + 2\int_1^x \frac{\varphi(t)}{t^3}\,dt. \tag{IV,4}$$

18. We prove first the

LEMMA 2. *If $\gamma(x)$ is continuous for $x \geqq 1$, then the relation*

$$2x\int_x^\infty \frac{\gamma(t)}{t^2}\,dt - \gamma(x) \to G \quad (x\to\infty) \tag{IV,5}$$

is equivalent with $\gamma(x)\to G$ $(x\to\infty)$.

Indeed, assume first that $\gamma(x)\to G$ $(x\to\infty)$. Then

$$\lim_{x\to\infty} x\int_x^\infty \frac{\gamma(t)}{t^2}\,dt = \lim_{x\to\infty} x \int_0^{1/x} \gamma(1/t)\,dt = \lim_{x\downarrow 0} \frac{1}{x}\int_0^x \gamma(1/t)\,dt = \lim_{x\downarrow 0} \gamma(1/x) = G,$$

and we see that (IV,5) holds indeed.

On the other hand, if (IV,5) holds, the Bernoulli-L'Hospital Rule can be applied to the limit $\lim_{x\to\infty} x^2 \int_x^\infty (\gamma(t)/t^2)\, dt/x$, since the denominator tends to ∞. But the quotient of the derivatives is

$$\frac{2x \displaystyle\int_x^\infty \frac{\gamma(t)}{t^2}\, dt - \gamma(x)}{1}$$

and tends to G. It follows that

$$x \int_0^x \frac{\gamma(t)}{t^2}\, dt \to G \qquad (x \to \infty),$$

and, from (IV,5), $\gamma(x) \to G(x \to \infty)$. Lemma 2 is proved.

19. LEMMA 3. *Assume* $f(x)$ *L integrable in* $\langle 1, \infty)$. *Consider the three conditions, where* $\varphi(x)$ *is defined by* (IV,3):

$$\exists \int_1^\infty \frac{f(t)}{t^2}\, dt =: K; \tag{IV,6}$$

$$\varphi(x)/x^2 \to 0 \;\; (x \to \infty), \qquad \exists \int_1^\infty \frac{\varphi(t)}{t^3}\, dt = K/2; \tag{IV,7}$$

$$x \int_x^\infty \frac{f(t)}{t^2}\, dt = 2x \int_x^\infty \frac{\varphi(t)}{t^3}\, dt - \frac{\varphi(x)}{x} \qquad (x \geqq 1), \tag{IV,8}$$

where both integrals in (IV,8) *are assumed as existing.*

Then (IV,6) *is equivalent with* (IV,7) *and, if these conditions are satisfied,* (IV,8) *follows.*

20. *Proof.* Assume (IV,6) satisfied. Then the Bernoulli-L'Hospital Rule can be applied for $x \to \infty$ to

$$\frac{2x^2\int\limits_1^x \frac{\varphi(t)}{t^3}\,dt}{x^2},$$

as $x^2 \to \infty$ $(x \to \infty)$. The quotient of the derivatives is, by (IV,4),

$$2\int\limits_1^x \frac{\varphi(t)}{t^3}\,dt + \frac{\varphi(x)}{x^2} \to K. \tag{IV,9}$$

Therefore $2\int_1^x (\varphi(t)/t^3)\,dt \to K$ and it follows from (IV,9) that $\varphi(x)/x^2 \to 0$. We see that (IV,7) follows from (IV,6).

On the other hand it is seen immediately from (IV,4) that (IV,6) follows from (IV,7).

21. Assume now that (IV,6) and (IV,7) are satisfied. We obtain with $x \to \infty$ from (IV,4) the formula

$$\int\limits_1^\infty \frac{f(t)}{t^2}\,dt = 2\int\limits_1^\infty \frac{\varphi(t)}{t^3}\,dt.$$

Subtracting from this formula the formula (IV,4) we obtain

$$\int\limits_x^\infty \frac{f(t)}{t^2}\,dt = 2\int\limits_x^\infty \frac{\varphi(t)}{t^3}\,dt - \frac{\varphi(x)}{x^2},$$

and, multiplying by x, the formula (IV,8). Lemma 3 is proved.

22. Assume now that (IV,1) holds. Then (IV,6), (IV,7) and (IV,8) hold too and from the second formula (IV,1) it follows that the right side expression in (IV,8) tends to L. But then the condition (IV,5) of lemma 2 is satisfied with $\gamma(x) := \varphi(x)/x$, $G := L$. By lemma 2 it follows now that $\varphi(x)/x \to L$, that is the condition (IV,2) with $M(f) = L$.

23. Assume on the other hand that (IV,2) holds. Then the condition (IV,7) follows and therefore also the conditions (IV,6) and (IV,8).

But now it follows from (IV,7), if we put $\gamma(x):=\varphi(x)/x$ and $G:=M(f)$, that $\gamma(x)\to G$ $(x\to\infty)$ and therefore the relation (IV,5) in lemma 2. This signifies that the right side expression in (IV,8) tends to $M(f)$ and the second formula in (IV,1) follows with $L=M(f)$.

V. Corollaries from Theorem A

24. LEMMA 4. *Put*

$$m(f):=\lim_{x\downarrow 0} x\int_x^1 \frac{f(t)}{t^2}\,dt, \tag{V,1}$$

if this limit exists. Putting $f(1/t)=:F(t)$, $f(ct)=:g(t)$ $(c>0)$, *and using* (I,7), *the relation holds*

$$m(f)=M(F)=m(g), \tag{V,2}$$

provided $m(f)$ *or* $M(F)$ *exists.*

Proof. Indeed, if $m(f)$ exists we have

$$m(f)=\lim_{x\downarrow 0} x\int_x^1 \frac{F(1/t)}{t^2}\,dt=\lim_{x\downarrow 0} x\int_1^{1/x} F(\tau)\,d\tau=\lim_{x\to\infty}\frac{1}{x}\int_1^x F(\tau)\,d\tau=M(F),$$

while, if the existence of $M(F)$ is assumed, the above transformation can be read from the right to the left.

Further

$$m(g)=\lim_{x\downarrow 0} x\int_x^1 \frac{f(ct)}{t^2}\,dt=\lim_{x\downarrow 0} cx\int_{cx}^c \frac{f(\tau)}{\tau^2}\,d\tau=m(f).$$

25. B. *Assume* $f(x)$ *L integrable in* $(0,\infty)$. *Then we have, if both* $m(f)$ *and* $M(f)$ *exist, for any positive* a, b,

$$\int_0^\infty \frac{f(at)-f(bt)}{t}\,dt=(M(f)-m(f))\lg\frac{a}{b}. \tag{V,3}$$

Conversely, if the integral in (V,3) *is convergent for a set of couples of positive values of a and b, such that a/b runs through a set of positive measure, both M* (*f*) *and m* (*f*) *exist.*

26. *Proof.* If we introduce $\tau := bt$ as a new integration variable and denote a/b by ϱ, the formula (V,3) goes over into the formula

$$\int_0^\infty \frac{f(\varrho t)-f(t)}{t}\,dt=(M(f)-m(f))\lg\varrho \qquad (\varrho>0). \tag{V,4}$$

Define $\pi(x)$ and $\mu(x)$ by (I,3). Then we see that (V,4) holds if both formulas

$$\int_0^\infty \frac{\pi(\varrho t)-\pi(t)}{t}\,dt=M(f)\lg\varrho, \tag{V,5}$$

$$\int_0^\infty \frac{\mu(\varrho t)-\mu(t)}{t}\,dt=-m(f)\lg\varrho \tag{V,6}$$

hold. Further, for any $\varrho>0$, the integral in (V,4) converges, then and only then, when the integrals in (V,5) and (V,6) converge.

27. Clearly

$$M(f)=M(\pi), \qquad m(f)=m(\mu). \tag{V,7}$$

But now, if $M(f)$ exists, the formula (V,5) follows immediately from theorem A, replacing there $f(t)$ with $\pi(t)$, and conversely, if the integral in (V,5) converges for all ϱ from a set of positive measure, $M(f)$ exists.

In order to reduce (V,6) to (V,5), put $\mu(1/t) =: P(t)$. Then, by (I,5)

$$\int_0^\infty \frac{\mu(\varrho t)-\mu(t)}{t}\,dt=\int_0^\infty \frac{P(t/\varrho)-P(t)}{t}\,dt$$

and (V,6) becomes

$$\int_0^\infty \frac{P(\tau/\varrho)-P(\tau)}{\tau}\,d\tau=-m(\mu)\lg\varrho=M(P)\lg\frac{1}{\varrho}. \tag{V,8}$$

This follows from (V,5) if $m(\mu)=M(P)$ exists, while, if the integral in (V,6) converges for a ϱ-set of positive measure, $M(P)=m(f)$ exists. Theorem B is proved.

28. In many cases the value of $M(f)$ can easily be obtained from the

LEMMA 5. *Let p be a positive constant and $f(x)$ L integrable in (x_0, ∞). Assume that with $x\to\infty$:*

$$\frac{1}{p}\int_{x}^{x+p} f(t)\,dt \to \alpha. \tag{V,9}$$

Then $M(f)=\alpha$.

Indeed, putting $\varphi(x):=\int_{x_0}^{x} f(t)\,dt$, (V,9) becomes

$$\frac{\varphi(x+p)-\varphi(x)}{p} \to \alpha.$$

Since $\varphi(x)$ is bounded in $\langle x_0, \infty)$, it follows by a theorem of Cauchy [1] that $\varphi(x)/x\to\alpha$ $(x\to\infty)$.

From lemma 5 follows in particular that $M(f)=f(\infty)$, if $f(\infty):=\lim_{x\to\infty} f(x)$ exists. It follows further, from (V,2): if $f(0):=\lim_{x\downarrow 0} f(x)$ exists, then $m(f)=f(0)$.

29. The most important case is that of an L integrable function $f(x)$ which is *periodic with the period p*. As in this case $\int_x^{x+p} f(t)\,dt/p$ is independent of x, we have then

$$M(f)=\frac{1}{p}\int_{x}^{x+p} f(t)\,dt. \tag{V,10}$$

For such a periodic function our formula (V,3) becomes

$$\int_{0}^{\infty}\frac{f(at)-f(bt)}{t}\,dt=\left(\frac{1}{p}\int_{0}^{p} f(t)\,dt-m(f)\right)\lg\frac{a}{b}, \tag{V,11}$$

assuming that $m(f)$ exists.

30. Using the formula (V,11), the following lemma is useful:

LEMMA 6. *If $f(t)/t$ is integrable into $t=0$.*

$$\exists \int_0^p \frac{f(t)}{t}\,dt, \quad p>0, \tag{V,12}$$

then

$$m(f)=0\ ^{6)}. \tag{V,13}$$

Proof. Put

$$\varphi(t):=f(t)/t, \qquad \psi(x):=\int_x^p \varphi(t)\,dt.$$

Then, by (V,1), $m(f)$ is the limit, with $\varepsilon\downarrow 0$, of

$$\varepsilon\int_\varepsilon^p \frac{\varphi(t)}{t}\,dt \equiv \varepsilon\int_\varepsilon^p \frac{\psi'(t)}{t}\,dt = \frac{\varepsilon\psi(p)}{p} - \psi(\varepsilon) + \varepsilon\int_\varepsilon^p \frac{\psi(t)}{t^2}\,dt,$$

that is $-\psi(0)+m(\psi)$. As $\psi(x)$ is continuous from the right at $t=0$, (V,13) follows.

COROLLARY. *If for a constant A and a $p>0$, $\int_0^p (f(t)-A)/t\,\mathrm{d}t$ exists, then $m(f)=A$.*

31. We give now some examples for the formulas (V,3) and (V,11).

(a) It is known from the theory of the Gamma function, that

$$\frac{2}{\pi}\int_0^{\pi/2} tg^\alpha x\,dx = \frac{1}{\cos\alpha\frac{\pi}{2}} \qquad (|\alpha|<1).$$

[6] The special case of the formula (V,11), where (V,12) holds and therefore $m(f)=0$, has been found independently by Tricomi and published in Tricomi [1].

Further, under the assumption that (V,12) holds (and therefore $m(f)=0$), a formula analogous to (V,3) has been found independently and published in Tricomi [1], however, under more special assumptions about $M(f)$, namely that not only $M(f)=\lim_{x\to\infty}(1/x)\int_p^x f(t)\,dt$ exists, but that the difference $1/x\int_p^x f(t)\,dt - M(f)$ is not only $o(1)$ but even $O(1/x)$.

Therefore

$$\frac{1}{\pi}\int_0^\pi |tg\,x|^\alpha\,dx=\frac{1}{\cos\alpha\frac{\pi}{2}}\qquad(0<\alpha<1)$$

and by (V,11) with $m(f)=0$:

$$\int_0^\infty (|tg\,ax|^\alpha-|tg\,bx|^\alpha)\frac{dx}{x}=\frac{\lg(a/b)}{\cos\alpha\frac{\pi}{2}}\qquad(0<\alpha<1,\ a\wedge b>0),\tag{V,14}$$

since tg$x$ vanishes for $x=0$.

(b) From the well-known integral

$$\int_0^{\pi/2}\lg\cos x\,dx=\frac{\pi}{2}\lg\tfrac{1}{2},\qquad \frac{1}{\pi}\int_0^\pi \lg|\cos x|\,dx=\lg\tfrac{1}{2}$$

it follows by (V,11), since lg|cos$x$| vanishes for $x=0$,

$$\int_0^\infty \lg\left|\frac{\cos ax}{\cos bx}\right|\frac{dx}{x}=\lg\tfrac{1}{2}\lg a/b.\tag{V,15}$$

(c) From the representation of the Bessel function $J(u)=J_0(u)$,

$$J(u)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(u\sin x)\,dx$$

(see for instance Courant-Hilbert [1]), it follows by (V,11), since cos(u sin$x$) becomes 1 for $x=0$,

$$\int_0^\infty \frac{\cos(u\sin ax)-\cos(u\sin bx)}{x}\,dx=(J(u)-1)\lg a/b.\tag{V,16}$$

(d) In the theory of Riemann's Zeta function the following relation is derived (see for instance Titchmarsh [1]):

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,dt \to \zeta(2\sigma) \qquad (T\to\infty,\ \sigma>\tfrac{1}{2},\ \sigma\neq 1).$$

It follows therefore by (V,3)

$$\int_0^\infty \frac{|\zeta(\sigma+iat)|^2-|\zeta(\sigma+ibt)|^2}{t}\,dt=(\zeta(2\sigma)-|\zeta(\sigma)|^2)\lg a/b \qquad (\sigma>\tfrac{1}{2},\ \sigma\neq 1), \tag{V,17}$$

32. Our result can be applied to a type of integrals considered by Lerch [2] and rediscovered independently, in a different form, by Hardy [1]. Consider

$$\int_0^\infty \sum_{\nu=0}^{n} A_\nu F(a_\nu x)\frac{dx}{x^m}, \tag{V,18}$$

where the $n+1$ constants A_ν and $n+1$ positive constants a_ν satisfy the m conditions

$$\sum_{\nu=0}^{n} A_\nu a_\nu^\mu=0 \qquad (\mu=0,1,\dots,m-1). \tag{V,19}$$

Suppose that we have for $F(x)$:

$$F(x)=P(x)+x^{m-1}f(x), \tag{V,20}$$

where $f(x)$ is L integrable in $(0,\infty)$ and $m(f)$ as well as $M(f)=M(F/x^{m-1})$ exist, while $P(x)$ is a polynomial of degree $\leqq m-2$. Then we will prove

$$\int_0^\infty \sum_{\nu=0}^{n} A_\nu F(a_\nu x)\frac{dx}{x^m}=(M(f)-m(f))\sum_{\nu=0}^{n} A_\nu a_\nu^{m-1}\lg a_\nu. \tag{V,21}$$

It follows from (V,19) immediately that

$$\sum_{\nu=0}^{n} A_\nu P(a_\nu x) \equiv 0, \qquad \int_\alpha^\beta \sum_{\nu=0}^{n} A_\nu P(a_\nu x) \frac{dx}{x^m} \equiv 0 \qquad (0 \leqq \alpha < \beta \leqq \infty).$$

We can therefore, proving (V,21), assume without loss of generality that $P(x) \equiv 0$.

Put

$$B_\nu \equiv A_\nu a_\nu^{m-1}, \qquad \sum_{\nu=0}^{n} B_\nu = 0.$$

Then we can write

$$\int_0^\infty \sum_{\nu=0}^{n} A_\nu F(a_\nu x) \frac{dx}{x^m} = \int_0^\infty \sum_{\nu=0}^{n} B_\nu f(a_\nu x) \frac{dx}{x} = \sum_{\nu=1}^{n} B_\nu \int_0^\infty \frac{f(a_\nu x) - f(a_0 x)}{x} dx,$$

and this is, by (V,3),

$$(M(f) - m(f)) \sum_{\nu=1}^{n} B_\nu \lg \frac{a_\nu}{a_0} = (M(f) - m(f)) \sum_{\nu=0}^{n} B_\nu \lg a_\nu,$$

which proves (V,21)[7].

VI. Discussion of the Means $m_a^{\pm}(f)$

33. $m(f(t))$ in (V,1), if it exists, serves to replace $\lim_{x \downarrow 0} f(x)$. In order to find a convenient expression to replace $\lim_{x \downarrow a} f(x)$, we will of course use

$$m(f(t+a)) = \lim_{x \downarrow 0} x \int_x^p \frac{f(t+a)}{t^2} dt = \lim_{x \downarrow 0} x \int_{x+a}^{p+a} \frac{f(t)}{(t-a)^2} dt =$$

$$= \lim_{x \downarrow a} (x-a) \int_x^{p+a} \frac{f(t)}{(t-a)^2} dt =: m_a^+(f(t)) \equiv m_a^+(f). \qquad \text{(VI,1)}$$

[7] The special case of this formula, when $F(x)$ is analytic at $x=0$ and continuous in $(0, \infty)$, while $\int^\infty (F(x)/x^m)\, dx$ is convergent, has been given by Hardy, 1905 [1]. Lerch, 1893 (Lerch [1]), has the corresponding formula in the assumption that $f(x)$ in (V,20) has finite limits for $x \to \infty$ and $x \to 0$, while it is integrable in $(0, \infty)$.

p is here an arbitrary positive number, but such that $f(t)$ is L integrable in $(a, a+p\rangle$.

34. If we have to replace $\lim_{x\uparrow a} f(x)$, we will correspondingly use

$$m(f(a-t)) = \lim_{x\downarrow 0} x \int_x^p \frac{f(a-t)}{t^2}\, dt = -\lim_{x\downarrow 0} x \int_{a-x}^{a-p} \frac{f(t)}{(t-a)^2}\, dt =$$

$$= \lim_{x\uparrow a} (x-a) \int_x^{a-p} \frac{f(t)}{(t-a)^2}\, dt =: m_a^-(f(t)) \equiv m_a^-(f) = m_0^+(f(a-t)). \tag{VI.2}$$

In particular we have obviously

$$m_0^+(f) = m(f), \tag{VI,3}$$

$$m_0^-(f(t)) = m(f(-t)). \tag{VI,4}$$

If in these formulas a is to be replaced by ∞ or $-\infty$, the corresponding definitions will be

$$m_\infty^-(f) := M(f), \qquad m_{-\infty}^+(f) := M(f(-t)) = m_\infty^-(f(-t)). \tag{VI,5}$$

(V,2) can now be written as

$$m_0^+(f) = M(f(1/t)) = m_\infty^-(f(1/t)). \tag{VI,6}$$

Using again (VI,5) and (VI,4), we have further

$$m_0^-(f) = M(f(-1/t)) = m_{-\infty}^+(f(1/t)). \tag{VI,6a}$$

35. In the following part of this chapter we will use ε for $+$ or $-$. $\varepsilon = \text{sign}\,\alpha$, for a real $\alpha \neq 0$ denotes then the sign of α.

If $a \neq b$ are two real numbers, we will use the formula

$$m_a^\varepsilon(f(t)) = m_b^\varepsilon(f(t+a-b)) = m_0^\varepsilon(f(t+a)), \tag{VI,7}$$

valid if one of the three expressions in (VI,7) exists.

To prove this formula, use (VI,1), (VI,2) and (VI,4), and introduce instead of t, and x new variables τ, y defined by

$t-a=:\tau-b, \qquad x-a=:y-b.$

We obtain

$$m_a^\varepsilon(f(t))=\lim_{x\to a+\varepsilon\cdot 0}(x-a)\int_x^{a+\varepsilon p}\frac{f(t)}{(t-a)^2}\,dt=\lim_{y\to b+\varepsilon\cdot 0}(y-b)\int_y^{b+\varepsilon p}\frac{f(\tau+a-b)}{(\tau-b)^2}\,d\tau=$$
$$=m_b^\varepsilon(f(t+a-b))=m_0^\varepsilon(f(t+a)).$$

Further it is easy to prove that always

$$m(f(a-t))=m_a^-(f(t))=m_b^+(f(a+b-t)). \tag{VI,8}$$

Indeed, using the transformations

$t-a=:b-\tau, \qquad x-a=:b-y,$

we obtain from (VI,2)

$$m_a^-(f(t))=\lim_{x\uparrow a}(x-a)\int_x^{a-p}\frac{f(t)}{(t-a)^2}\,dt=$$
$$=\lim_{y\downarrow b}(y-b)\int_y^{b+p}\frac{f(a+b-\tau)}{(\tau-b)^2}\,d\tau=m_b^+(f(a+b-t)).$$

36. LEMMA 7. *Consider two functions, $u(x)$ and $v(x)$, for which in a certain limiting process in x,*

$\varepsilon:=\operatorname{sgn} u(x)=(\operatorname{sgn} v(x)$

remains constant and $v(x)/u(x)$ tends to a positive limit, γ.

If $m_0^\varepsilon(f)$ exists and both $u(x)$ and $v(x)$ tend to 0, then

$$\int_{u(x)}^{v(x)}\frac{f(t)}{t}\,dt\to m_0^\varepsilon(f)\lg\gamma. \tag{VI,9}$$

If $m_{\varepsilon\cdot\infty}^{-\varepsilon}(f)$ exists and both $u(x)$ and $v(x)$ tend to $\varepsilon\cdot\infty$, then

$$\int_{u(x)}^{v(x)} \frac{f(t)}{t}\, dt \to m_{\varepsilon\cdot\infty}^{-\varepsilon}(f) \lg \gamma . \tag{VI,10}$$

37. *Proof.* Using (VI,6) and (VI,6a), it is immediately seen that (VI,9) follows from (VI,10), if we replace in the integral in (VI,9) the integration variable, t, by $\tau := 1/t$.

Using the second formula (VI,5), we see that the case $\varepsilon = -$ follows from the case $\varepsilon = +$ if we introduce a new variable of integration, $\tau := -t$.

If suffices therefore to prove (VI,10) and to assume $\varepsilon = +$, so that the formula to be proved becomes, using the first formula (VI,5),

$$\int_{u(x)}^{v(x)} \frac{f(t)}{t}\, dt \to M(f) \lg \gamma, \qquad u(x) \wedge v(x) \to \infty . \tag{VI,11}$$

Introducing $F(x)$ by (III,3) we obtain, applying partial integration, similarly as in sec. 14,

$$\int_{u(x)}^{v(x)} \frac{f(t)}{t}\, dt = F(v(x)) - F(u(x)) + \int_{u(x)}^{v(x)} \frac{F(t)}{t}\, dt .$$

Since $F(x) \to M(f)$ $(x \to \infty)$, the difference of the two first terms on the right tends to 0, while, applying the mean value theorem, we obtain

$$\int_{u(x)}^{v(x)} \frac{F(t)}{t}\, dt = F(\xi(x)) \int_{u(x)}^{v(x)} \frac{dt}{t} = F(\xi(x)) \lg \frac{v(x)}{u(x)},$$

where $\xi(x)$ lies between $u(x)$ and $v(x)$. But then $\xi(x) \to \infty$, $F(\xi(x)) \to M(f)$ and (VI,11) follows. Lemma 7 is proved.

38. LEMMA 8. *Let*

$$\exists \int_{\alpha}^{\beta} |f(x)|\, dx \tag{VI,12}$$

exist, where $-\infty \leqslant \alpha < \beta \leqslant \infty$. *Then, for any finite* x_0 *with* $\alpha \leqq x_0 \leqq \beta$ *and for any* $q(x)$ *which is measurable and bounded in a neighborhood of* x_0 *in* $\langle \alpha, \beta \rangle$,

$$m_{x_0}^{\varepsilon}((x-x_0)\,q(x)f(x))=0, \qquad \varepsilon=+\vee-. \tag{VI,13}$$

If, on the other hand, $x_0=\varepsilon\cdot\infty$ is α or β, and if, for x going to $\varepsilon\cdot\infty$, $q(x)=O(x^2)$, then

$$m_{\varepsilon\cdot\infty}^{-\varepsilon}(q(x)\,f(x)/x)=0. \tag{VI,14}$$

39. *Proof.* If $|x_0|<\infty$, then, using (VI,7), we can assume $x_0=0$, and using (VI,4), that $\varepsilon=+$. We can then assume, without loss of generality, that $\alpha=0$.

Let p be a positive number $<\beta$ and such that $q(x)$ is bounded and measurable in $\langle 0,p\rangle$. Then we can obviously replace β in (VI,12) with p and $q(x)f(x)$ with $f(x)$, that is to say we can assume in the proof $q(x)\equiv 1$ and we have to prove that then from (VI,12) it follows that $m(xf(x))=0$.

But writing $xf(x)=:\varphi(x)$, (VI,12) can be written as

$$\exists\int_0^p \frac{\varphi(x)}{x}\,dx,$$

and now it follows from lemma 6 in sec. 30 indeed that $m(\varphi(x))=0$.

40. In the case that $x=\varepsilon\cdot\infty$, we can, without loss of generality, assume $\varepsilon=+$ and we have to prove that

$$M(q(x)\,f(x)/x)=0, \qquad \frac{1}{x}\int_p^x q(x)\,f(x)\frac{dx}{x}\to 0,$$

with $x\to\infty$ for a convenient $p>0$.

Assume that $|q(x)|\leqq Cx^2$ $(x\geqq p)$. For an arbitrarily small $\delta>0$ choose $P>p$, so that

$$\int_P^\infty |f(x)|\,dx<\frac{\delta}{C}.$$

Then we can write, if $x>P$,

$$\frac{1}{x}\int_P^x q(t)\,f(t)\frac{dt}{t}\leqq C\int_P^x \frac{t}{x}\,|f(t)|\,dt<C\int_P^x |f(t)|\,dt<\delta,$$

and therefore

$$\overline{\lim}\,\frac{1}{x}\left|\int_{p}^{x} q(t)\,\frac{f(t)}{t}\,dt\right| \leqq \delta + \lim \frac{1}{x}\int_{p}^{P} |q(t)|\,\frac{|f(t)|}{t}\,dt = \delta\,.$$

(VI,14) follows immediately.

41. The assumption (VI,12) of lemma 8 is not necessarily satisfied even in the case of the existence of the integral

$$\exists \int_{\alpha}^{\beta} f(x)\,dx\,. \qquad \text{(VI,15)}$$

In this case the assertions of lemma 8 can still be proved if $q(x)$ satisfies some more special conditions.

LEMMA 9. *Let* (VI,15) *exist with* $-\infty \leqslant \alpha < \beta \leqslant \infty$. *Then the relation* (VI, 13) *holds for any finite* x_0 *with* $\alpha \leqq x_0 \leqq \beta$ *and for any* $q(x)$ *which is totally continuous in* (α, β) *and for which* $Q_0 := \lim q(x)$ *with x going to* x_0 *from* (α, β) *exists and*

$$m_{x_0}^{\varepsilon}((x - x_0)\,|q'(x)|) = 0\,. \qquad \text{(VI,16)}$$

If on the other hand $x_0 = \varepsilon \cdot \infty$, *then the relation* (VI, 14) *holds as soon as* $q(x)$ *is totally continuous in* (α, β) *and*

$$m_{\varepsilon\cdot\infty}^{-\varepsilon}\left(\frac{|q'(x)|}{x}\right) = 0\,. \qquad \text{(VI,17)}$$

42. *Proof.* We prove first the relation (VI,14) under the condition (VI,17). Without loss of generality, we can assume $\varepsilon = +$, so that (VI,17) becomes, with $x \to \infty$,

$$\frac{1}{x}\int_{p}^{x} |q'(t)|\,\frac{dt}{t} \to 0 \quad (x \to \infty), \qquad \text{(VI,18)}$$

and we have to prove that

$$\frac{1}{x}\int_p^x q(t)\,f(t)\frac{dt}{t}\to 0 \quad (x\to\infty) \tag{VI,19}$$

for some positive p.

We begin by proving that

$$q(x)=o(x^2). \tag{VI,20}$$

Put

$$Q(x):=\int_p^x |q'(t)|\,dt.$$

We can then write (VI,18) as

$$R(x):=\int_p^x Q'(t)\frac{dt}{t}=o(x).$$

But then, since $q(x)$ is totally continuous, $q(x)-q(p)=\int_p^x q'(t)\,dt$,

$$|q(x)-q(p)|\leqq\int_p^x \frac{Q'(t)}{t}\,t\,dt=\int_p^x R'(t)\,t\,dt=R(t)\,t\Big|_p^x-\int_p^x R(t)\,dt.$$

Here, the integrated part is, by definition of $R(t)$, $o(x^2)$. As to the right hand integral, the limit of its quotient through x^2 is obtained immediately from the Bernoulli-L'Hospital Rule, as $\lim_{x\to\infty}(1/2x)\,R(t)=0$. (VI,20) is proved.

43. Put now, using (VI, 20),

$$s(x):=q(x)/x=o(x).$$

Then obviously

$$s'(x)=\frac{q'(x)-s(x)}{x}.$$

Put, using (VI,15),

$$\varphi(x) := \int_x^\infty f(t)\,dt, \qquad \varphi(x) \to 0 \quad (x \to \infty).$$

Then we can write the integral in (VI,19) as

$$-\int_p^x s(t)\,\varphi'(t)\,dt = s(t)\,\varphi(t)\Big|_x^p + \int_p^x \varphi(t)\frac{q'(t)}{t}\,dt - \int_p^x \varphi(t)\frac{s(t)}{t}\,dt.$$

Here, the integrated part on the right is obviously $o(x)$ as $s(x) = o(x)$. The modulus of the first integral on the right is

$$\leqq \int_p^x |\varphi(t)|\,\frac{|q'(t)|}{t}\,dt = o(x),$$

in virtue of (VI,18) as $|\varphi(t)|$ is bounded. As to the last integral on the right, by the Bernoulli-L'Hospital Rule its quotient through x has the limit

$$\lim_{x\to\infty} \varphi(x)\frac{s(x)}{x} = 0.$$

The formula (VI,19) and therefore the formula (VI,14) is proved.

44. We consider now the case of a *finite* x_0. We can assume, without loss of generality, that $\varepsilon = +$ and $\alpha = x_0 = 0$. Then our assumptions can be written as

$$\exists \int_0^p f(\tau)\,d\tau, \qquad t\int_t^p |q'(\tau)|\,d\tau/\tau \to 0 \quad (t\downarrow 0), \tag{VI,21}$$

for a positive $p > 0$, with $t\downarrow 0$, and we have to prove that

$$t\int_t^p q(\tau)\,f(\tau)\,d\tau/\tau \to 0. \tag{VI,22}$$

Introducing into the formulas (VI,21) and (VI,22) a new integration variable, $\xi := 1/\tau$, they become, if we write $x := 1/t$, $p_1 := 1/p > 0$, and put $Q(\xi) := q(\tau)$ and $F(\xi) := f(\tau)$, with $x \to \infty$, respectively

$$\exists \int_{p_1}^{\infty} F(\xi)\, d\xi/\xi^2, \qquad \frac{1}{x}\int_{p_1}^{x} \xi\, |Q'(\xi)|\, d\xi \to 0, \qquad \frac{1}{x}\int_{p_1}^{x} Q(\xi)\, F(\xi)\, d\xi/\xi \to 0, \qquad \text{(VI,23)}$$

since $-Q'(\xi) = q'(\tau)/\xi^2$. Putting $G(x) := \int_{p_1}^{x} F(\xi)\, d\xi/\xi^2$ it follows

$$\int_{p_1}^{x} Q(\xi)\, F(\xi)\, d\xi/\xi = \int_{p_1}^{x} \xi Q(\xi)\, G'(\xi)\, d\xi =$$

$$= xQ(x)\, G(x) - \int_{p_1}^{x} G(\xi)\, Q(\xi)\, d\xi - \int_{p_1}^{x} \xi G(\xi)\, Q'(\xi)\, d\xi .$$

This is obviously $= o(x)$ if $Q(\xi) =$ const. Otherwise, putting $Q(\xi) - Q_0 =: Q_1(\xi)$, we have to prove that

$$xQ_1(x)\, G(x) - \int_{p_1}^{x} G(\xi)\, Q_1(\xi)\, d\xi - \int_{p_1}^{x} \xi G(\xi)\, Q_1'(\xi)\, d\xi = o(x). \qquad \text{(VI,24)}$$

But here obviously the first two terms are $o(x)$, while the same follows for the third term from the second relation (VI,23). (VI,24) and lemma 9 are proved.

VII. The Three Functions Formula

45. C. *Consider the open interval, J, between a and b, $a \gtrless b$, where a and b could also have the values $+\infty$ or $-\infty$. Assume in the whole statement of theorem* C *that x only runs through J.*

Consider two functions $\varphi(x)$, $\psi(x)$, absolutely continuous in J and assume that

$$\varphi(x) \wedge \psi(x) \to a' \quad (x \to a), \qquad \text{(VII,1)}$$

$$\varphi(x) \wedge \psi(x) \to b' \quad (x \to b), \qquad \text{(VII,2)}$$

where

$$-\infty \leqq a' < b' \leqq \infty \qquad \text{(VII,3)}$$

and $\varphi(x)$ and $\psi(x)$ only assume values from the open interval (a', b').

Assume further the existence of two positive finite numbers γ^+ and γ^-, defined by

$$0<\gamma^+ := \lim_{x\to a}\begin{cases}\dfrac{\varphi(x)-a'}{\psi(x)-a'} & (a'>-\infty)\\[2ex] \dfrac{\varphi(x)}{\psi(x)} & (a'=-\infty),\end{cases} \tag{VII,4}$$

$$0<\gamma^- := \lim_{x\to b}\begin{cases}\dfrac{\varphi(x)-b'}{\psi(x)-b'} & (b'<\infty)\\[2ex] \dfrac{\varphi(x)}{\psi(x)} & (b'=\infty).\end{cases} \tag{VII,5}$$

Consider $g(x)$, L integrable and bounded in (a', b'), and put

$$G_+(x) := \begin{cases}(x-a')\,g(x) & (a'>-\infty)\\ xg(x) & (a'=-\infty),\end{cases} \tag{VII,6}$$

$$G_-(x) := \begin{cases}(x-b')\,g(x) & (b'<\infty)\\ xg(x) & (b'=\infty).\end{cases} \tag{VII,7}$$

Assume finally that the following mean values exist:

$$\exists m_{a'}^+(G_+)\wedge m_{b'}^-(G_-). \tag{VII,8}$$

Then the following integral converges and has the indicated value:

$$\int_a^b \{\psi'(x)\,g(\psi(x))-\varphi'(x)\,g(\varphi(x))\}\,dx = L_+ - L_-, \tag{VII,9}$$

$$L_+ := m_{a'}^+(G_+)\lg\gamma^+, \qquad L_- := m_{b'}^-(G_-)\lg\gamma^-. \tag{VII,10}$$

46. *Proof.* Assume two numbers A, B from J; then we can write

$$\int_A^B (\psi' g(\psi)-\varphi' g(\varphi))\,dx = \int_A^B \psi' g(\psi)\,dx - \int_A^B \varphi' g(\varphi)\,dx =$$

$$= \int_{\psi(A)}^{\psi(B)} g(y)\,dy - \int_{\varphi(A)}^{\varphi(B)} g(y)\,dy = \int_{\psi(A)}^{\varphi(A)} g(y)\,dy - \int_{\psi(B)}^{\varphi(B)} g(y)\,dy. \tag{VII,11}$$

(VII,9) will be proved if we show that

$$\int_{\psi(x)}^{\varphi(x)} g(y)\,dy \to L_+ \quad (x \to a), \qquad \int_{\psi(x)}^{\varphi(x)} g(y)\,dy \to L_- \quad (x \to b). \tag{VII,12}$$

Consider first the first integral (VII,12). Using (VII,6) it can be written as

$$\int_{\psi(x)}^{\varphi(x)} \frac{G_+(y)}{y-a'}\,dy, \tag{VII,13}$$

where, if $a' = -\infty$, the denominator, $y-a'$, has to be replaced with y. If now $a' = \infty$, then our assertion follows at once from the second part of lemma 7 in sec. 36, replacing there respectively $u(x)$, $v(x)$, $\varepsilon, f(t)$, γ with $\psi(x)$, $\varphi(x)$, $+$, $G_+(t)$ and γ^+.

47. If, on the other hand, a' is *finite*, the integral (VII,13) becomes, introducing $t := y - a'$ as a new integration variable,

$$\int_{\psi(x)-a'}^{\varphi(x)-a'} \frac{G_+(t+a')}{t}\,dt. \tag{VII,14}$$

To this integral lemma 7 can be applied, replacing respectively $u(x)$, $v(x)$, $\varepsilon, f(t)$ with $\psi(x)-a'$, $\varphi(x)-a'$, $+$, $G_+(t+a')$. Then to γ in lemma 7 corresponds, in virtue of (VII,4), γ^+, and the integral (VII,14) has the limit $m_0^+(G_+(a'+t)) \lg \gamma^+$. This is, by (VI,7), just L_+.

The proof of the second formula (VII,12) is completely symmetric to that of the first one. Theorem C is proved.

48. In order to obtain (V,3) from (VII,9) it is sufficient to assume in (VII, 9) $a := 0$, $b := \infty$, and to replace $g(t)$, $\varphi(x)$ and $\psi(x)$ in the formula (VII,9) respectively with $f(t)/t$, bx, ax; then, we obtain $a' = 0$, $b' = \infty$, $\gamma^+ = \gamma^- = b/a$ and $G_+(t) = G_-(t) = f(t)$, while $m_{a'}^+(G_+)$, $m_{b'}^-(G_-)$ become respectively $m(f)$, $M(f)$. We see that theorem B follows, indeed, from theorem C.

49. Applying the Three Functions Formula, in many cases the consideration of two special cases which do not fall completely under the wording of theorem C, can be useful:

LEMMA 10. *If in theorem* C, *the integral*

$$\int_{\beta}^{b'} g(y)\,dy, \qquad \beta<b', \tag{VII,15}$$

exists, the condition (VII,5) *and the second condition* (VII,8) *can be dropped, while the formula* (VII,9) *is valid with* $L_-=0$.

Similarly, if the integral

$$\int_{a'}^{\alpha} g(y)\,dy, \qquad \alpha>a', \tag{VII,16}$$

exists, the conditon (VII,4) *and the first condition* (VII,8) *can be dropped, and the formula* (VII,9) *is valid with* $L_+=0$.

Indeed, in the first case we use, for the second limit in (VII,12),

$$\int_{\psi(x)}^{\varphi(x)} g(y)\,dy=\int_{\psi(x)}^{b'} g(y)\,dy-\int_{\varphi(x)}^{b'} g(y)\,dy\to 0,$$

and we see that in (VII,9) L_- can be replaced with 0.

Similarly, in the second case, it follows, as $x\to a$,

$$\int_{\psi(x)}^{\varphi(x)} g(y)\,dy=\int_{a'}^{\varphi(x)} g(y)\,dy-\int_{a'}^{\psi(x)} g(y)\,dy\to 0=L_+ .$$

50. The special conditions (VII,4) and (VII,5) which are *sufficient* for the convergence of the integral in (VII,9), are also *necessary* if we require, for instance, that the integral in (VII,9) converges for any function $g(t)$ satisfying our conditions. It is even necessary if we restrict ourselves to the functions $g(t):=1/(t-a')$, $g(t):=1/(t-b')$, or, if a' or b' are $\mp\infty$, $g(t):=1/t$.

Indeed, it follows from the decomposition (VII,11) that, for $g(t):=1/(t-a')$, the integral

$$\int_{\psi(A)}^{\varphi(A)} \frac{dt}{t-a'}=\lg\left|\frac{\varphi(A)-a'}{\psi(A)-a'}\right|$$

must have a finite limit, if, with $A \to a$,

$$\varphi(A) \wedge \psi(A) \to a'.$$

But this is only possible if the corresponding condition (VII,4) is satisfied. The argument is obviously the same in the other cases.

VIII. Special Cases of the Three Functions Formula

51. A. If $f(x)$ is L integrable in (0,1) and $m(f)$ as well as the following integrals exist:

$$\exists \int_0^1 f(x)\,dx, \qquad \exists \int_{1/2}^1 \frac{f(x)}{x}\,dx, \tag{VIII,1}$$

we have

$$\int_0^{\pi/2} \frac{\cos x\, f(\sin x) - f\left(\operatorname{tg}\frac{x}{2}\right)}{\sin x}\,dx = m(f) \lg 2. \tag{VIII,2}$$

To prove (VIII,2) put in (VII,9)

$$g(t) := f(t)/t, \qquad \psi(x) := \sin x, \qquad \varphi(x) := \operatorname{tg}\frac{x}{2}, \qquad a=0,\ b=\frac{\pi}{2}.$$

Since then

$$a'=0,\ b'=1,\ \psi'(x)=\cos x, \qquad \frac{\varphi'(x)}{\operatorname{tg}\frac{x}{2}} = \frac{1}{2\cos^2\frac{x}{2}\operatorname{tg}\frac{x}{2}} = \frac{1}{\sin x},$$

the integral in (VII,9) becomes that in (VIII,2).

As to γ^+ we obtain, by the Bernoulli-L'Hospital Rule,

$$\gamma^+ = \lim_{x \downarrow 0} \frac{\sin x}{\operatorname{tg}\frac{x}{2}} = 2.$$

But now the formula (VIII,2) follows immediately from the lemma 10.

52. B. If we take in (VII,9) $\psi(t)\equiv t$, this formula becomes

$$\int_a^b [g(t)-\varphi'(t)\,g(\varphi(t))]\,dt=$$

$$=m_a^+((x-a)\,g(x))\lg\varphi'(a)-m_b^-((x-b)\,g(x))\lg\varphi'(b), \qquad \text{(VIII,3)}$$

if $a<b$ are finite and $\varphi'(a)\wedge\varphi'(b)$ exist and are positive.

Indeed, in this case we have $a'=a$, $b'=b$, and γ^+, γ^- become resp. $\varphi'(a)$, $\varphi'(b)$[8]).

If $b=\infty$, we have in the last right hand term of (VIII,3), instead of $\varphi'(b)$, $\lg\lim_{x\to\infty}(\varphi(x)/x)$, assuming that $0<\lim_{x\to\infty}(\varphi(x)/x)<\infty$. If $\varphi'(\infty):=\lim_{x\to\infty}\varphi'(x)$ exists and is positive, then, by the Bernoulli-L'Hospital Rule, we can replace $\varphi'(b)$ with $\varphi'(\infty)$. The procedure is similar if $a=-\infty$.

Take, for instance, in (VIII,3)

$$\varphi(x):=\gamma\sqrt[3]{x^3+3x+1}\,.$$

Since here $\varphi'(0)=\varphi'(\infty)=\gamma$, we obtain, taking $a=0$,

$$\int_0^\infty\left[g(t)-\frac{\gamma(t^2+1)}{\sqrt[3]{t^3+3t+1}^{\,2}}\,g(\sqrt[3]{t^3+3t+1})\right]dt=\gamma\,[m(xg(x))-M(xg(x))]\,.$$

$g(x)$ is here assumed to be L integrable and bounded in (a, b).

53. C. Consider the functions $\alpha(x)$, $\beta(x)$, totally continuous in (0, 1) and such that

$$\lim_{x\downarrow 0}\alpha(x)=\lim_{x\downarrow 0}\beta(x)=0, \qquad \lim_{x\uparrow 1}\alpha(x)=\lim_{x\uparrow 1}\beta(x)=1,$$

$$\alpha(x)\wedge\beta(x)\in(0,1) \qquad (0<x<1).$$

Assume further that

$$\gamma:=\lim_{x\uparrow 1}\frac{\beta(x)-1}{\alpha(x)-1} \qquad \text{(VIII,4)}$$

[8] This formula is due to Lerch [2], under the assumption that the limits of $(x-a)\,g(x)$ $(x\downarrow a)$, $(x-b)\,g(x)$ $(x\uparrow b)$ exist. Observe that in the case of Lerch from the formula (VIII,3) a special case of the Three Functions Formula is obtained by subtraction.

exists and is positive, and finally that $f(x)$ is L integrable and bounded in $(0,1)$, while the following expressions exist:

$$\exists \int_0^p f(x)\,dx, \quad 0<p<1; \quad \exists m_1^-(f). \tag{VIII,5}$$

Then the following formula holds:

$$\int_0^1 \left[\frac{\alpha'(t) f(\alpha(t))}{1-\alpha(t)} - \frac{\beta'(t) f(\beta(t))}{1-\beta(t)}\right] dt = m_1^-(f) \lg \gamma. \tag{VIII,6}$$

54. Indeed, if we take

$$\varphi(x) := 1-\beta(x), \quad \psi(x) := 1-\alpha(x), \quad g(x) := \frac{f(1-x)}{x}, \quad a=1, \quad b=0,$$

it follows in the theorem C and the lemma 10, $a'=0$, $b'=1$, and the corresponding condition (VII,4) is satisfied with $\gamma^+ := \gamma$.

On the other hand, as $1/x$ is monotonic and bounded between p and 1, the integral $\int_p^1 (f(1-x)/x)\,dx$ exists. Therefore the condition (VII,15) of lemma 10 is satisfied in this case, so that we can take $L_- = 0$ and the formula (VIII,6) follows.

If we assume that

$$\exists f(1) := \lim_{x \uparrow 1} f(x), \qquad \exists \alpha'(1) \wedge \beta'(1)$$

in the assumption $\alpha(1) := 1$, $\beta(1) := 1$ and $\beta'(1)/\alpha'(1) > 0$, the formula (VIII,6) becomes

$$\int_0^1 \left[\frac{\alpha'(t) f(\alpha(t))}{1-\alpha(t)} - \frac{\beta'(t) f(\beta(t))}{1-\beta(t)}\right] dt = f(1) \lg \frac{\beta'(1)}{\alpha'(1)} \text{ [9]}. \tag{VIII,7}$$

55. D. Assume, in the general hypotheses of theorem C, $a'=0$, $b'=\infty$, and assume, instead of the assumption (VII,5), that, putting $g(x) := f(x)/x$,

$$\exists \int_1^\infty \frac{f(x)}{x}\,dx. \tag{VIII,8}$$

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[9] See Cauchy [2].

Then we can apply lemma 10 and obtain the formula

$$\int_a^b \left[\frac{\psi'(t)}{\psi(t)} f(\psi(t)) - \frac{\varphi'(t)}{\varphi(t)} f(\varphi(t))\right] dt = m(f) \lg \lim_{t\to a} \frac{\varphi(t)}{\psi(t)}. \tag{VIII,9}$$

If in particular f, φ, ψ are continuous in a, and $\varphi'(a) \wedge \psi'(a)$ exist with $\varphi'(a)\,\psi'(a) > 0$, the formula (VIII,9) becomes

$$\int_a^b \left[\frac{\psi'(t)}{\psi(t)} f(\psi(t)) - \frac{\varphi'(t)}{\varphi(t)} f(\varphi(t))\right] dt = f(a) \lg \frac{\varphi'(a)}{\psi'(a)} \text{ [10]}. \tag{VIII,10}$$

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