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# The Cut Locus of Noncompact Finitely Connected Surfaces Without Conjugate Points

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## Introduction

In this paper we obtain a characterization and derive some implications of the condition that a complete two dimensional Riemannian manifold without conjugate points have finitely generated fundamental group. The characterization in terms of fundamental domains is a classical result in the case of Gaussian curvature  $K \equiv -1$ . See for example [7] and [11].

Let  $M$  denote an arbitrary complete surface without conjugate points along any geodesic,  $H$  the universal Riemannian covering of  $M$  and  $D$  the deckgroup of the covering. Given a point  $p$  in  $H$  we define the *canonical fundamental domain* for  $D$  with center  $p$  to be the set  $R_p \subseteq H$  given by

$$R_p = \bigcap_{\substack{\phi \in D \\ \phi \neq 1}} E^+(p, \phi p)$$

where  $E^+(p, \phi p) = \{q \in H : d(p, q) \leq d(\phi p, q)\}$ . The set  $E(p, \phi p) = \{q \in H : d(p, q) = d(\phi p, q)\}$  is a *bounding side* for  $R_p$  if  $R_p$  is a proper subset of  $\bigcap_{\substack{\psi \in D \\ \psi \neq \phi, \psi \neq 1}} E^+(p, \psi p)$ .

**THEOREM A.** *Let  $M = H/D$  be a complete nonsimply connected surface without conjugate points. Then the following are equivalent.*

- 1)  $\pi_1(M)$  is finitely generated.
- 2) For some  $p \in H$  the fundamental domain  $R_p$  has only a finite number of bounding sides.
- 3) For every  $p \in H$  the fundamental domain  $R_p$  has only a finite number of bounding sides.

As a corollary we obtain

**THEOREM B.** *Let  $M$  be a complete nonsimply connected surface without conjugate points. For each  $p \in M$  let  $G_2(p)$  be the set of points  $q$  in  $M$  for which there are exactly two shortest geodesics from  $p$  to  $q$ . Then  $G_2(p)$  is nonempty for each  $p$ , and each connected component of  $G_2(p)$  is an open differentiable arc. Moreover the following statements are equivalent.*

- 1)  $\pi_1(M)$  is finitely generated.
- 2) For some  $p \in M$ ,  $G_2(p)$  has a finite number of connected components.
- 3) For every  $p \in M$ ,  $G_2(p)$  has a finite number of connected components.

I am grateful to the referee for pointing out that the attempt to generalize theorems A and B to arbitrary dimensions fails in dimension 3 by theorem 1 of [8]. See also page 410 of [12].

A further consequence of theorem A is

**THEOREM C.** *Let  $M$  be a complete nonsimply connected surface without conjugate points and with finitely generated fundamental group. Then for each point  $p$  in  $M$  there are at most a finite number of points  $q$  in  $M$  for which there exist three or more shortest geodesics joining  $p$  to  $q$ .*

We do not know if the converse to theorem C is true. We remark that there exists at least one shortest geodesic joining any two distinct points  $p, q$  of  $M$  since  $M$  is complete. If  $q$  lies in the cutlocus of  $p$ , then there are at least two but at most a finite number of shortest connecting geodesics since  $M$  has no conjugate points.

In each of the theorems A, B and C it suffices to consider the case that  $M = H/D$  is noncompact. If  $M$  is compact then  $\pi_1(M)$  is finitely generated and each fundamental domain  $R_p \subseteq H$  has only a finite number of bounding sides. A proof of the second assertion is contained in the discussion in section two. The first assertion follows from the second in view of the proof of the statement 2)  $\rightarrow$  1) in theorem A.

The paper is organized as follows. Section 1 contains basic definitions and notation. For convenience we assume that all manifolds  $M$  and Riemannian metrics  $g$  are  $C^\infty$ . Section 2 contains the statements of basic properties of the fundamental domains  $R_p$ . The proofs of these statements form the hardest part of the paper and because of their length are found in the appendix, section 4. In section 3 we prove theorems A, B and C. Theorems B and C follow quickly from theorem A and the facts from section two. The proof of theorem A is reminiscent of the method used by Marden to prove theorem 2 of [11]. In fact, theorem A can also be derived from that result in the orientable case. See the remark at the end of section 3.

## §1. Preliminaries

In this section we establish notation and list some basic facts.  $M$  will always denote a complete connected Riemannian manifold with Riemannian structure  $\langle \cdot, \cdot \rangle$ , Riemannian metric  $d(\cdot, \cdot)$  and sectional curvature  $K$ . Let  $TM$  denote the tangent

bundle of  $M$  and  $T_p(M)$  the tangent space to  $M$  at  $p$ . If  $v \in TM$  is given let  $\gamma_v: \mathbf{R} \rightarrow M$  be the geodesic such that  $\gamma'_v(0) = v$ . The map  $\exp_p: T_p(M) \rightarrow M$  given by  $\exp_p(v) = \gamma_v(1)$  is the exponential map at  $p$ . In this paper all geodesics are assumed to have unit speed and to be defined on the entire real line unless otherwise indicated. A *geodesic segment* is a geodesic defined on a compact interval  $[a, b]$ . A *geodesic ray* is a geodesic defined on  $[0, \infty]$ .

A manifold  $M$  is said to have no *conjugate points* if there exists no nontrivial Jacobi vector field that vanishes twice on some geodesic  $\gamma$  of  $M$ . If  $M$  is simply connected, then there is a unique geodesic joining any two distinct points of  $M$ . In the sequel  $H$  will denote a simply connected and  $M$  an arbitrary complete two dimensional manifold without conjugate points.  $M$  can be written as a quotient surface  $H/D$ , where  $H$  is the universal Riemannian cover of  $M$  and  $D$  is a freely acting, properly discontinuous group of isometries of  $H$ .  $D$  will always denote such a group. Each non-identity element of  $D$  has infinite order since  $Z_p$  does not act properly discontinuously on  $\mathbf{R}^n$  for any prime  $p$  and any integer  $n \geq 1$  [10].

DEFINITION 1.1. If  $p$  and  $q$  are distinct points of  $H$ , let  $\gamma_{pq}$  denote the unique geodesic such that  $\gamma_{pq}(0) = p$  and  $\gamma_{pq}(a) = q$ , where  $a = d(p, q)$ . Let  $V(p, q)$  denote the unit vector  $\gamma'_{pq}(0)$ .

Since  $H \cong \mathbf{R}^2$  is two dimensional one may define the left and right half planes determined by a geodesic  $\gamma$  of  $H$ . Since  $H$  is orientable we may assign an orientation to each tangent space  $T_p(H)$  that varies continuously with  $p$ . A basis  $\{v_1, v_2\}$  of  $T_p(H)$  is *positively oriented* if the orientation of  $T_p(H)$  that it determines agrees with the given orientation for  $T_p(H)$  and is *negatively oriented* otherwise.  $H - \gamma$  consists of two connected components for any maximal geodesic  $\gamma$  of  $H$ . Each of these components is convex in the sense that it contains the unique geodesic segment between any two of its points.

DEFINITION 1.2. Let  $\gamma$  be a maximal geodesic of  $H$ . A point  $p$  in  $H - \gamma$  lies to the *right* (left) of  $\gamma$  if for some number  $t$  the pair of unit vectors  $\{V(\gamma t, p), \gamma'(t)\}$  is positively (negatively) oriented relative to the fixed orientation of  $H$ . The points lying to the right (left) of  $\gamma$  constitute the right (left) *half plane* determined by  $\gamma$ .

Note that the orientation of the pairs  $\{V(\gamma t, p), \gamma'(t)\}$  is continuous in  $t$  hence constant.

DEFINITION 1.3. An *end* of a Hausdorff space  $X$  is a function  $\varepsilon$  that assigns to each compact subset  $K'$  of  $X$  a connected component  $\varepsilon(K')$  of  $X - K'$  with the further requirement that  $\varepsilon(K') \supseteq \varepsilon(L)$  if  $K' \subseteq L$ . A subset  $U$  of  $X$  is a *neighborhood* of an end  $\varepsilon$  if  $U$  contains  $\varepsilon(K')$  for some  $K'$ . A sequence of points  $p_n$  converges to an end  $\varepsilon$  if each neighborhood of  $\varepsilon$  contains  $p_n$  for sufficiently large  $n$ .



In a noncompact Hausdorff space  $X$  a divergent curve  $\gamma: [0, \infty) \rightarrow X$  determines an end  $\varepsilon$  of  $X$ ; for each compact subset  $K$  of  $M$  define  $\varepsilon(K)$  to be the connected component of  $X - K$  that contains a terminal segment of  $\gamma$ . A curve  $\gamma$  is *divergent* if for any compact set  $C \subseteq X$  there exists a number  $T = T(C) > 0$  such that  $\gamma t \in X - C$  for  $t > T$ .

If  $M$  is a noncompact surface with finitely generated fundamental group, then it is known [1], [9] that  $M$  is homeomorphic to a compact surface with a finite number of points removed. Each end of  $M$  corresponds to one of these missing points and has a neighborhood  $U$  homeomorphic to a punctured disk or equivalently a half cylinder  $S^1 \times (0, \infty)$ .

## §2. Fundamental Domains

In this section we define for every point  $p$  in  $H$  and for every freely acting, properly discontinuous group  $D$  of isometries of  $H$ , a *canonical fundamental domain* for  $D$  with center  $p$ . We derive basic properties of fundamental domains that are well known if  $H$  is the hyperbolic plane but which require more discussion in this general case. We also relate the fundamental domain with center  $p$  to the cut locus of  $\pi(p)$  in the quotient surface  $M = H/D$ .

**DEFINITION 2.1.** Let  $D$  be a freely acting, properly discontinuous group of isometries of  $H$ . For any point  $p$  in  $H$  the *canonical fundamental domain* for  $D$  with center  $p$ , denoted  $R_p = \{q \in H : d(p, q) \leq d(\phi p, q) \text{ for all } \phi \text{ in } D\}$ .

It is easy to see that the interior of  $R_p$ , denoted  $\text{Int}(R_p) = \{q \in R_p : d(p, q) < d(\phi p, q) \text{ for every } \phi \neq 1 \text{ in } D\}$ . Also  $\partial R_p$ , the boundary of  $R_p$ ,  $= \{q \in R_p : d(p, q) = d(\phi p, q) \text{ for some } \phi \neq 1 \text{ in } D\}$ . Hence  $\partial R_p$  is contained in the union of the equidistant sets  $E(p, \phi p)$ ,  $\phi \in D$ , where  $E(p, \phi p) = \{q \in H : d(p, q) = d(\phi p, q)\}$ . Now, for each  $\phi$  in  $D$  and each point  $p$  in  $H$  define  $E^+(p, \phi p)$  to be  $\{q \in H : d(p, q) \leq d(\phi p, q)\}$ . By definition then

$$R_p = \bigcap_{\substack{\phi \in D \\ \phi \neq 1}} E^+(p, \phi p).$$

We remark that  $R_p$  is starshaped relative to  $p$ ; that is if  $q \in R_p$  then the geodesic segment  $\gamma_{pq}$  is contained in  $R_p$ . This assertion follows from the fact that for each  $\phi \in D$  the function  $r \rightarrow d(p, r) - d(\phi p, r)$  is nondecreasing on geodesics starting at  $p$ , which implies that each set  $E^+(p, \phi p)$  is starshaped relative to  $p$ .

**DEFINITION 2.2.** We say that an equidistant set  $E(p, \phi p)$  is a *bounding side* for  $R_p$  if  $R_p$  is a proper subset of

$$\bigcap_{\substack{\psi \neq \phi \\ \psi \in D, \psi \neq 1}} E^+(p, \psi p)$$

The definitions and discussion so far apply to a manifold  $H$  of arbitrary dimension. The next definition is motivated by the fact that in dimension two the sets  $E(p, \phi p)$  are differentiable curves in  $H$  that meet transversally if at all.

**DEFINITION 2.3.** A point  $\tilde{q}$  in  $R_p$  is a *vertex* of  $R_p$  if it lies on the intersection of two distinct bounding sides of  $R_p$ .

The proper discontinuity of  $D$  implies that only finitely many of the sets  $E(p, \phi p)$ ,  $\phi \in D$ , meet any given compact subset of  $H$ . Since two distinct sets  $E(p, \phi p)$  and  $E(p, \psi p)$  intersect in at most one point by Proposition 2.8 below, it follows that only finitely many vertices of  $R_p$  lie in any given compact subset of  $H$ .

We next briefly describe the cut locus at a point  $p$  of an arbitrary complete Riemannian manifold  $M$ . If  $M$  has no conjugate points, then we relate the cut locus at  $p$  to the canonical fundamental domain for  $D$  with center  $\tilde{p}$  in  $H$ , where  $M = H/D$  and  $\pi\tilde{p} = p$ ,  $\pi: H \rightarrow M$ .

Let  $M$  be a complete Riemannian manifold of arbitrary dimension, and let  $p$  be a point of  $M$ . If  $S(p)$  denotes the sphere of unit vectors in  $T_p(M)$  let  $f: S(p) \rightarrow [0, \infty]$  be given by  $f(v) = \sup \{t \geq 0: d(p, \exp_p(tv)) = t\}$ . The function  $f$  is known to be continuous on the extended real numbers, and hence it has a positive lower bound on  $S(p)$  [3].

The cut locus at  $p$ , denoted  $C(p)$ , is defined to be  $\{\exp_p(f(v) \cdot v): v \in S(p) \subseteq T_p(M)\}$ . The cut locus at  $p$  is a closed subset of  $M$ , and  $f(v)$  measures the distance from  $p$  to  $C(p)$  in the direction  $v$ .

**DEFINITION 2.3'.** A point  $q \in C(p)$  is a *vertex* of  $C(p)$  if there are at least three distinct shortest geodesics from  $p$  to  $q$ .

We shall show later in corollary 2.15 that a point  $\tilde{q} \in R_p$  is a vertex of  $R_p$  if and only if  $q = \pi\tilde{q} \in C(\pi p)$  and  $q$  is a vertex of  $C(\pi p)$ . If  $q \in C(p)$ , then it is known [3] that either there are at least two distinct shortest geodesics from  $p$  to  $q$ , or  $q$  is conjugate to  $p$  along some shortest geodesic segment from  $p$  to  $q$ . If  $M$  has no conjugate points, then the second case does not occur, and furthermore there are only finitely many shortest geodesics from  $p$  to  $q$ . If  $M = H/D$ , where  $H$  is the universal Riemannian cover of  $M$  and  $D$  the deckgroup of  $M$ , then it is straightforward to verify the following assertions.

- 1) For any point  $p$  in  $H$ , a point  $q$  lies in the interior of  $R_p$  if and only if there is a unique shortest geodesic in  $M$  from  $\pi p$  to  $\pi q$ .
- 2)  $\pi: H \rightarrow M$  maps the interior of  $R_p$  onto its image in a one-one fashion.
- 3)  $\pi: H \rightarrow M$  maps  $R_p$  onto  $M$  and maps  $\partial R_p$  onto  $C(\pi p)$ .
- 4) If  $q \in R_p$  is a vertex of  $R_p$ , then  $\pi q$  lies in  $C(\pi p)$ , and  $\pi q$  is a vertex of  $C(\pi p)$ .

For a more refined study of the cut locus of a compact surface with curvature  $K \leq 0$  see [2].

We return to a study of the fundamental domain  $R_p$ , especially its boundary. To do this we will need to establish certain properties of the equidistant sets  $E(p, q) = \{r \in H : d(p, r) = d(q, r)\}$  for any pair of distinct points  $p$  and  $q$  in  $H$ . The set  $E(p, q)$  is a geodesic if  $H$  is the hyperbolic plane. In the general case  $E(p, q)$  is no longer a geodesic but retains some properties of a geodesic. First,  $E(p, q)$  is a  $C^\infty$  one dimensional submanifold of  $H$  since it is the zero level set of the function  $\bar{g}(r) = d(p, r) - d(q, r)$ , which is  $C^\infty$  on  $H - \{p \cup q\}$ . Note that the gradient of  $\bar{g}$  is nonzero at any point  $r$  in  $E(p, q)$  since the gradients of  $r \rightarrow d(p, r)$  and  $r \rightarrow d(q, r)$  point radially outward from  $p$  and  $q$  respectively if  $r \neq p$  and  $r \neq q$ . Precisely, these gradients are  $-V(r, p)$  and  $-V(r, q)$ .

In the remainder of this section we omit the proofs of the results to make reading easier. The proofs may be found in section 4, the appendix.

We first define the *canonical parametrization* of  $E(p, q)$ . Actually there are two such parametrizations; if  $\alpha$  is one then  $\alpha^*: t \rightarrow \alpha(-t)$  is the other. This parametrization has also been used in [6].

**PROPOSITION 2.4.** *Let  $p$  and  $q$  be distinct points in  $H$ . Then there exists a continuous, one-one map  $\alpha: \mathbf{R} \rightarrow H$  such that  $\alpha(\mathbf{R}) = E(p, q)$ ,  $\alpha(0)$  is the midpoint of the segment  $\gamma_{pq}$  and  $d(p, \alpha t) = |t| + t_0/2$  for every  $t \in \mathbf{R}$ , where  $t_0 = d(p, q)$ .*

**PROPOSITION 2.5.** *The canonical parametrization  $\alpha$  of an equidistant set  $E(p, q)$  is  $C^\infty$  at every number  $t \neq 0$ .*

We now describe some of the properties of geodesics of  $H$  that are retained by the equidistant sets  $E(p, q)$ .

**PROPOSITION 2.6.** *Let  $p$  and  $q$  be distinct points in  $H$ . Then  $H - E(p, q)$  consists of two connected components. The components containing  $p$  and  $q$  are starshaped relative to  $p$  and  $q$  respectively. Any maximal geodesic containing  $p$  or  $q$  meets  $E(p, q)$  at most once.*

**PROPOSITION 2.7.** *Let  $p$  and  $q$  be distinct points in  $H$  and let  $\alpha$  be the canonical parametrization of  $E(p, q)$ . Then  $\lim_{t \rightarrow \infty} V(p, \alpha t)$  and  $\lim_{t \rightarrow -\infty} V(p, \alpha t)$  exist and are distinct. If  $\gamma_1$  and  $\gamma_2$  are the geodesics whose initial velocities are these limits, then the maximal geodesics  $\gamma_1$  and  $\gamma_2$  do not intersect  $E(p, q)$ . The same assertions hold if  $p$  is replaced by  $q$ .*

**PROPOSITION 2.8.** *Let  $p, q$  and  $r$  be distinct points in  $H$ . Then  $E(p, q) \cap E(q, r)$  contains at most one point.*

The results above prepare one to study the properties of the bounding sides of a fundamental domain  $R_p$  in  $H$  with center  $p$ , relative to a freely acting, properly discontinuous group  $D$  of isometries of  $H$ .

**PROPOSITION 2.9.** *The boundary of a fundamental domain  $R_p$  for  $D$  with center  $p$  is contained in the union of the bounding sides.*

**COROLLARY 2.10.**  *$R_p$  is the intersection of those sets  $E^+(p, \phi p)$  such that  $E(p, \phi p)$  is a bounding side for  $R_p$ .*

**PROPOSITION 2.11.** *Let  $E(p, \phi p)$  be a bounding side of  $R_p$ . Then  $E(p, \phi p) \cap R_p$  is nonempty and consists of a subarc, finite or infinite, of the arc  $E(p, \phi p)$ . If  $q$  is an interior point of  $E(p, \phi p) \cap R_p$ , then  $q$  is not a vertex of  $R_p$ . If  $E(p, \phi p) \cap R_p$  is nonempty for some  $\phi \neq 1$  such that  $E(p, \phi p)$  is not a bounding side, then  $E(p, \phi p) \cap R_p$  is a single point.*

The next result shows that the bounding sides of a canonical fundamental domain may be identified in pairs.

**PROPOSITION 2.12.** *Let  $E(p, \phi p)$  be a bounding side for  $R_p$ . Then  $R_p \cap E(p, \phi^{-1}p) = \phi^{-1} \{R_p \cap E(p, \phi p)\}$ . In particular  $E(p, \phi^{-1}p)$  is also a bounding side for  $R_p$ .*

The next results characterize the vertices of  $R_p$ .

**PROPOSITION 2.13.** *If  $q \in \partial R_p$  is a vertex of  $R_p$ , then  $q$  lies on the intersection of exactly two bounding sides of  $R_p$ .*

**PROPOSITION 2.14.** *A point  $q \in R_p$  is a vertex of  $R_p$  if  $q$  lies in the intersection of any two distinct equidistant sets  $E(p, \phi p)$  and  $E(p, \psi p)$  that are not necessarily bounding sides of  $R_p$ .*

**COROLLARY 2.15.** *Let  $q \in \partial R_p$  be a point such that  $\pi(q)$  is a vertex of  $C(\pi p)$ , the cut locus at  $\pi(p)$  in  $M = H/D$ . Then  $q$  is a vertex of  $R_p$ .*

These last results show that there exists an element  $\phi \neq 1$  in  $D$  such that  $E(p, \phi p) \cap R_p$  is a single point  $q$  if and only if for some point  $q$  in  $R_p$  there are at least four distinct shortest geodesics from  $\pi(p)$  to  $\pi(q)$  in  $M = H/D$ . If there exist at least four shortest geodesics from  $\pi(p)$  to  $\pi(q)$  in  $M$ , then there exist at least three distinct, nonidentity elements  $\phi_1, \phi_2$  and  $\phi_3$  in  $D$  such that  $q \in E(p, \phi_i p)$  for  $i = 1, 2, 3$ . One of these equidistant sets cannot be a bounding side of  $R_p$  by proposition 2.13, and therefore it intersects the set  $R_p$  in exactly the point  $q$ . Conversely if  $E(p, \phi p) \cap R_p$  is a single point  $q$  for some  $\phi \neq 1$  in  $D$ , then  $q \in \partial R_p$  and  $q$  lies in some bounding side  $E(p, \psi p)$  by proposition 2.9. By proposition 2.14  $q$  is a vertex of  $R_p$  and since  $E(p, \phi p)$  is not a bounding side of  $R_p$  there exists by proposition 2.13 a third element  $\xi \neq 1$  in  $D$  such that  $E(p, \xi p)$  is a bounding side and  $q \in E(p, \xi p)$ . Therefore there are at least four shortest geodesics from  $\pi p$  to  $\pi q$  in  $M = H/D$ . For a discussion of this possibility in the case that  $M$  is compact with curvature  $K \equiv -1$  see [2].

### § 3. The Main Results

In this section we prove the theorems A, B, and C stated in the introduction. For the proof of theorem A we shall need the following result in which we make no assumption about conjugate points.

**LEMMA.** *Let  $M$  be a complete, noncompact Riemannian manifold of dimension two. Let  $U$  be an unbounded open set in  $M$  that is homeomorphic to  $S^1 \times (0, \infty)$ , and let  $p \in M - \bar{U}$  be given. Let  $p_n$  be a divergent sequence of points contained in  $U$  for large  $n$  for which there exist distinct shortest geodesics  $\gamma_n$  and  $\sigma_n$  joining  $p$  to  $p_n$ . Then infinitely many of the loops at  $p$  given by  $\alpha_n = \sigma_n^{-1} \gamma_n$  are homotopic.*

*Proof.* Passing to a subsequence we may assume that  $p_n \in U$  for all  $n$  and there exist geodesics  $\gamma$  and  $\sigma$  (possibly equal) such that  $\gamma'_n(0) \rightarrow \gamma'(0)$  and  $\sigma'_n(0) \rightarrow \sigma'(0)$  as  $n \rightarrow \infty$ . The geodesics  $\gamma$  and  $\sigma$  start at  $p$  and are distance minimizing on  $[0, \infty)$ . Denote  $\partial U = S^1 \times \{0\}$  by  $C$ . We may assume that  $C$  is parametrized as a nonsingular  $C^\infty$  curve; merely replace  $U$  by  $S^1 \times (1, \infty)$  and replace  $S^1 \times \{1\}$  by a nonsingular  $C^\infty$  curve from the same homotopy class that lies in  $S^1 \times (0, \infty)$ . Since  $C$  is compact we may define  $t_0 > 0 = \sup \{t > 0 : \gamma t \in C\}$  and  $s_0 > 0 = \sup \{t > 0 : \sigma t \in C\}$ . By further altering  $C$  if necessary we may assume that  $\gamma$  and  $\sigma$  meet  $C$  transversally at  $t_0$  and  $s_0$  respectively. Letting  $c_n = d(p, p_n)$  we define  $t_n = \sup \{0 < t \leq c_n : \gamma_n t \in C\}$  and  $s_n = \sup \{0 < t \leq c_n : \sigma_n t \in C\}$ . Since  $\gamma$  and  $\sigma$  meet  $C$  transversally it follows that  $t_n \rightarrow t_0$  and  $s_n \rightarrow s_0$  as  $n \rightarrow \infty$ . Moreover  $\gamma_n t \in U$  for  $t_n < t \leq c_n$  and  $\sigma_n t \in U$  for  $s_n < t \leq c_n$  since  $p_n \in U$ . Note that  $c_n \rightarrow +\infty$  since  $p_n$  is a divergent sequence. Finally,  $\gamma t \in U$  for  $t > t_0$  and  $\sigma t \in U$  for  $t > s_0$ .

Parametrize  $C$  on  $[0, 2\pi]$  and let  $a_n, b_n$  be those numbers in  $[0, 2\pi]$  such that  $\gamma_n(t_n) = C(a_n)$  and  $\sigma_n(s_n) = C(b_n)$ . The points  $\gamma_n(t_n)$  and  $\sigma_n(s_n)$  are distinct for large  $n$  since  $\gamma_n$  and  $\sigma_n$  are minimizing on  $[0, c_n]$ . By passing to a subsequence and relabeling if necessary we may assume that  $a_n < b_n$  for every  $n$ . Let  $C_n$  denote the restriction of  $C$  to  $[a_n, b_n]$ . Let  $\gamma_n^*, \bar{\gamma}_n$  denote the restrictions of  $\gamma_n$  to  $[0, t_n]$  and  $[t_n, c_n]$  respectively. Let  $\sigma_n^*, \bar{\sigma}_n$  denote the restrictions of  $\sigma_n$  to  $[0, s_n]$  and  $[s_n, c_n]$ . Let  $\gamma^*$  and  $\sigma^*$  denote the restrictions of  $\gamma$  and  $\sigma$  to  $[0, t_0]$  and  $[0, s_0]$ . We may write the loop  $\alpha_n = \sigma_n^{-1} \gamma_n$  as a product of two loops at  $p$ ,  $\alpha_n = [(\sigma_n^*)^{-1} B_n \sigma_n^*] A_n$ , where  $A_n = (\sigma_n^*)^{-1} C_n \gamma_n^*$  and  $B_n = (\bar{\sigma}_n)^{-1} (\bar{\gamma}_n) C_n^{-1}$ . The curve  $B_n$  is a simple closed curve since  $\bar{\gamma}_n$  and  $\bar{\sigma}_n$  intersect  $C$  only at  $t_n$  and  $s_n$  by the definition of these numbers and intersect each other only at  $p_n = \gamma_n(c_n) = \sigma_n(c_n)$  since  $\gamma_n$  and  $\sigma_n$  are minimizing on  $[0, c_n]$ . Since  $B_n$  lies in  $\bar{U}$ , a closed half cylinder, and is a simple closed curve an application of the Jordan curve theorem shows that  $B_n$  is homotopic either to a point or to the curve  $C^{-1}$ , which wraps around the cylinder exactly once. Passing to a subsequence the loops  $(\sigma_n^*)^{-1} B_n \sigma_n^*$  are homotopic either to a point for all  $n$  or to the loop  $(\sigma^*)^{-1} C^{-1} \sigma^*$  for all  $n$ . For large  $n$  the loop  $A_n$  is homotopic to the loop  $(\sigma^*)^{-1} C^* \gamma^*$ , where  $C^*$  is a point if



$\gamma = \sigma$  and is the restriction of  $C$  to  $[a, b]$  if  $\gamma \neq \sigma$ , where  $C(a) = \gamma(t_0)$  and  $C(b) = \sigma(s_0)$ . Therefore the loops  $\alpha_n$  are homotopic to each other for large  $n$ .

We now begin the proof of theorem A. The assertion 3)  $\rightarrow$  2) is obvious. We show that 2)  $\rightarrow$  1). Let  $S = \{\phi \in D : \phi(R_p) \cap R_p \text{ is nonempty}\}$ , where  $R_p$  is a fixed fundamental domain in  $H$  with a finite number of bounding sides. We assert that  $S$  is a finite set, and assuming that this has been proved we apply theorem 29.4 (i) of [4, p. 184] to conclude that  $S$  is a generating set for  $D$ .

Suppose that  $S$  is an infinite set. By removing a finite number of elements from  $S$  we may assume that for each  $\phi \in S$ ,  $E(p, \phi p)$  is not a bounding side of  $R_p$ . By proposition 2.11  $\phi(R_p) \cap R_p = R_p \cap E(p, \phi p)$  is a single point  $q$  for  $\phi \in S$ . Since  $q \in \partial R_p$  it follows that  $q$  lies in some bounding side of  $R_p$  by proposition 2.9. Hence  $q$  must be a vertex of  $R_p$  by proposition 2.14. The proper discontinuity of  $D$  implies that only finitely many  $\phi \in S$  determine the same vertex  $\phi(R_p) \cap R_p$ , and therefore  $R_p$  has infinitely many vertices. However,  $R_p$  has only finitely many vertices since any two distinct bounding sides intersect at most once by proposition 2.8. This contradiction shows that  $S$  is a finite set and completes the proof. One may also show that if  $\phi_1, \dots, \phi_k$  are those elements of  $D$  such that  $E(p, \phi_i p)$ ,  $1 \leq i \leq k$ , are the bounding sides of  $R_p$ , then the elements  $\phi_1, \dots, \phi_k$  are a set of generators for  $D$ .

We now prove that 1)  $\rightarrow$  3). Let  $\tilde{p} \in H$  be given and suppose  $R_{\tilde{p}}$  has an infinite number of bounding sides  $E(\tilde{p}, \phi_n \tilde{p})$ ,  $n = 1, 2, \dots$ . Choose a point  $\tilde{p}_n \in E(\tilde{p}, \phi_n \tilde{p}) \cap R_{\tilde{p}}$ , which is possible by proposition 2.11, and let  $p_n = \pi(\tilde{p}_n)$ . By the choice of  $\tilde{p}_n$  the geodesic segments  $\sigma_n = \pi \circ \gamma_{\phi_n \tilde{p}, \tilde{p}_n}$  and  $\gamma_n = \pi \circ \gamma_{\tilde{p}_n, \tilde{p}_n}$  are distinct shortest geodesics in  $M$  from  $p = \pi \tilde{p}$  to  $p_n$ . By elementary covering space facts no two loops  $\alpha_n = \sigma_n^{-1} \gamma_n$  and  $\alpha_m = \sigma_m^{-1} \gamma_m$  are homotopic if  $m \neq n$  since  $\phi_n \neq \phi_m$ . If we pass to a subsequence the points  $p_n$  converge to some end  $A$  of  $M$  since the sequence  $p_n$  is divergent. Since  $\pi_1(M)$  is finitely generated it is known that  $M$  is homeomorphic to a compact surface with a finite number of points removed. For a surface of this type each end  $A$  has a neighborhood  $U$  homeomorphic to a punctured disk or equivalently to  $S^1 \times (0, \infty)$ . Applying the lemma above we obtain a contradiction.

We now prove theorem B. Let  $p \in M$  be given, and let  $\tilde{p} \in \pi^{-1}(p)$ ,  $\pi: H \rightarrow M$ , be arbitrarily chosen. By definition  $G_2(p)$  equals the cut locus of  $p$ ,  $C(p)$ , minus the vertices of  $C(p)$ . By the discussion following definition 2.3',  $G_2(p)$  is the image under  $\pi$  of the boundary of  $R_{\tilde{p}}$  minus the vertices of  $R_{\tilde{p}}$ . By proposition 2.9 and 2.11 the set  $\partial R_{\tilde{p}}$  minus vertices of  $R_{\tilde{p}}$  is the union of the interiors of the differentiable arcs  $E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}$ , where  $E(\tilde{p}, \phi \tilde{p})$  is a bounding side of  $R_{\tilde{p}}$ . To show that  $G_2(p)$  is a disjoint union of open differentiable arcs it suffices to show that if  $E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}$  and  $E(\tilde{p}, \psi \tilde{p}) \cap R_{\tilde{p}}$  are distinct bounding arcs of  $R_{\tilde{p}}$ , then the images under  $\pi$  of their interiors are either disjoint or identical. Suppose that these images intersect for some  $\phi, \psi \in D$ . Then there exists a point  $\tilde{q}$  in the interior of  $E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}$  and an element  $\xi \neq 1$  in  $D$  such that  $\xi \tilde{q}$  lies in the interior of  $E(\tilde{p}, \psi \tilde{p}) \cap R_{\tilde{p}}$ . Define geodesics segments

$\gamma_1 = \pi \circ \gamma_{\tilde{p}, \xi \tilde{q}} = \pi \circ \gamma_{\tilde{p}, \psi^{-1} \xi \tilde{q}}$ ,  $\gamma_2 = \pi \circ \gamma_{\tilde{p} \tilde{q}}$ ,  $\gamma_3 = \pi \circ \gamma_{\tilde{p}, \xi \tilde{q}}$  and  $\gamma_4 = \pi \circ \gamma_{\phi \tilde{p}, \tilde{q}} = \pi \circ \gamma_{\tilde{p}, \phi^{-1} \tilde{q}}$ . By the choice of  $\phi$ ,  $\psi$  and  $\xi$  it follows that these are all shortest geodesics in  $M$  from  $p$  to  $q = \pi(\tilde{q})$ . The point  $\tilde{q}$  is not a vertex of  $R_{\tilde{p}}$  since it lies in the interior of  $E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}$ , and therefore  $q$  is not a vertex of  $C(p)$ . It follows that at most two of the geodesics  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  are distinct. By inspection  $\gamma_2 \neq \gamma_3$  and  $\gamma_2 \neq \gamma_4$ . Hence  $\gamma_3 = \gamma_4$ , which implies that  $\xi = \phi^{-1}$ . Similarly  $\gamma_1 \neq \gamma_3$  and since  $\gamma_2 \neq \gamma_3$  it follows that  $\gamma_1 = \gamma_2$ , implying that  $\psi^{-1} \xi = 1$ . Therefore  $\psi = \phi^{-1}$  and  $E(\tilde{p}, \psi \tilde{p}) \cap R_{\tilde{p}} = \phi^{-1} \{E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}\}$  by proposition 2.12. Therefore the images under  $\pi$  of the interiors of  $E(\tilde{p}, \phi \tilde{p}) \cap R_{\tilde{p}}$  and  $E(\tilde{p}, \psi \tilde{p}) \cap R_{\tilde{p}}$  are identical.

Suppose now that  $M$  is a complete surface without conjugate points and that  $\pi_1(M)$  is finitely generated. Let  $p \in M$  be given, and let  $\tilde{p} \in H$ ,  $\tilde{p} \in \pi^{-1}(p)$ , be chosen arbitrarily. By theorem A  $R_{\tilde{p}}$  has a finite number of bounding sides, and by the discussion above it follows that  $G_2(p)$  has a finite number of connected components. Thus 1)  $\rightarrow$  3) in theorem B. Clearly 3)  $\rightarrow$  2). The discussion above also shows that each connected component of  $G_2(p)$  is the image under  $\pi$  of exactly two bounding arcs of  $R_{\tilde{p}}$ . If  $G_2(p)$  has a finite number  $k$  of connected components for some  $p \in M$ , then  $R_{\tilde{p}}$  has  $2k$  bounding sides for any  $\tilde{p} \in \pi^{-1}(p)$ . By theorem A  $\pi_1(M)$  is finitely generated and we have proved that 2)  $\rightarrow$  1).

We now prove theorem C. Let  $M$  be a complete surface without conjugate points and with finitely generated fundamental group, and let  $p \in M$  be given. If  $q \in M$  is a point for which there are at least 3 shortest geodesics from  $p$  to  $q$ , then by definition  $q$  lies in the cut locus,  $C(p)$ , of  $p$  and is a vertex of  $C(p)$ . Let  $\tilde{p} \in \pi^{-1}(p)$  be arbitrary. By corollary 2.15 and the discussion in section 2,  $q = \pi(\tilde{q})$ , where  $\tilde{q}$  is a vertex of  $R_{\tilde{p}}$ . By theorem A  $R_{\tilde{p}}$  has only a finite number of bounding sides. It follows that  $R_{\tilde{p}}$  has at most a finite number of vertices since a vertex lies in the intersection of two bounding sides, which must be a single point by proposition 2.8. Therefore  $C(p)$  has a finite number of vertices, which completes the proof of theorem C.

We do not know if the converse to theorem C is true although we suspect that it is. In principle it might be possible to have a deckgroup  $D$  of isometries of  $H$  for which each fundamental domain  $R_p$  has an infinite number of bounding sides, only finitely many of which intersect. The quotient surface  $H/D$  would then have an infinitely generated fundamental group, but each cut locus  $C(p)$  would have only a finite number of vertices.

*Remark.* Theorem A can be derived from theorem 2 of Marden [11] in the case that  $M$  is a noncompact, nonsimply connected orientable surface. Since  $M$  is orientable it has the structure of a Riemann surface, and  $M$  is therefore diffeomorphic to a quotient  $\Delta/G$ , where  $\Delta$  is the open unit disk in the complex plane and  $G$  is a freely acting, properly discontinuous group of fractional linear transformations preserving  $\Delta$ . If  $g$  is a metric without conjugate points in  $M$ , then the covering map  $\pi: \Delta \rightarrow M$  induces a metric  $\pi^*g$  without conjugate points in  $\Delta$ , and the elements of  $G$  are iso-

metries of the metric  $\pi^*g$ . The results of section 2 of this paper show that the canonical fundamental domains in  $\Delta$  determined by  $G$  are fundamental regions in the sense of Marden. Therefore if  $G \cong \pi_1(M)$  is finitely generated, then each canonical fundamental domain has finitely many bounding sides by Marden's theorem 2. This shows that 1) implies 3) in theorem A and the other assertions follow as above.

#### §4. Appendix

In this section we give the proofs of the results stated in section 2.

*Proof of proposition 2.4.* We shall need

**LEMMA 2.4.** *Let  $p$  and  $q$  be distinct points of  $H$ , and let  $t_0 = d(p, q)$ . Then there exists a unique point  $z$  in  $E(p, q)$  such that  $d(p, z) = d(q, z) = t_0/2$ , and moreover,  $d(y, p) \geq d(z, p)$  for any  $y \in E(p, q)$ . If  $t^* > t_0/2$  is any number, then there are exactly two points  $z_1, z_2$  in  $E(p, q)$  such that  $d(p, z_1) = d(p, z_2) = t^*$ .*

*Proof.* Let  $\gamma$  be the maximal geodesic  $\gamma_{pq}$ . If  $z$  is the midpoint of the geodesic segment  $\gamma_{pq}$  between  $p$  and  $q$ , then  $d(p, z) = d(q, z) = t_0/2$ . The uniqueness of  $z$  and the fact that  $d(y, p) \geq d(z, p)$  for every  $y \in E(p, q)$  follow immediately from the triangle inequality. Let  $H$  be given a fixed orientation. Given  $t^* > t_0/2$ , let  $\beta: [0, A] \rightarrow H$  be a unit speed parametrization of the circle of center  $p$  and radius  $t^*$  such that  $\beta(0)$  and  $\beta(A^*)$ ,  $0 < A^* < A$ , are the two points of this circle that lie on  $\gamma$  and  $\beta: (0, A^*) \rightarrow H$  parametrizes the semicircle lying to the left of  $\gamma$ . Let  $J(t) = d(q, \beta t)$ . Since  $\beta(t) \neq q$  for  $0 < t < A^*$  it follows from lemma 2.3 of [5] that  $J'(t) = -\langle \beta'(t), V(\beta t, q) \rangle$ . Since  $\beta'(t)$  is orthogonal to the vector  $V(\beta t, p)$  for every  $t$  and since  $V(\beta t, p)$  and  $V(\beta t, q)$  are not collinear for  $0 < t < A^*$  it follows that  $J'(t) \neq 0$  in this interval. Since  $J(0) = t_0 + t^*$  and  $J(A^*) = |t_0 - t^*| < t^*$  by a suitable choice of  $\beta$ , it follows that there exists a unique number  $s$  with  $0 < s < A^*$  such that  $d(p, \beta s) = d(q, \beta s) = t^*$ . Similarly there is a unique number  $s'$  with  $A^* < s' < A$  such that  $d(p, \beta s) = d(q, \beta s') = t^*$ . The points  $\beta s$  and  $\beta s'$  are on opposite sides of  $\gamma$ .

We now complete the proof of proposition 2.4. Given distinct points  $p$  and  $q$  in  $H$ , let  $\alpha(0)$  be the midpoint of the segment  $\gamma_{pq}$ . Let  $t > 0$  be given. Relative to a fixed orientation of  $H$ , let  $\alpha(t)$  be the unique point to the right of  $\gamma_{pq}$  such that  $\alpha(t)$  lies in  $E(p, q)$  and  $d(p, \alpha t) = t + t_0/2$ , where  $t_0 = d(p, q)$ . Let  $\alpha(-t)$  denote the unique point in  $E(p, q)$  such that  $\alpha(-t)$  lies to the left of  $\gamma_{pq}$  and  $d(p, \alpha(-t)) = t + t_0/2$ . This defines a map  $\alpha: \mathbf{R} \rightarrow E(p, q)$  which is a homeomorphism. The proof is complete.

*Proof of proposition 2.5.* The set  $E(p, q)$  is a  $C^\infty$  one dimensional manifold since it is the zero level set of the function  $r \rightarrow d(p, r) - d(q, r)$ , which is  $C^\infty$  on  $H - \{p \cup q\}$  and whose gradient is never zero on  $E(p, q)$ . Given a number  $t > 0$  let  $\beta: (-\varepsilon, \varepsilon) \rightarrow E(p, q)$  be a nonsingular  $C^\infty$  map such that  $\beta(0) = \alpha(t)$ . The  $C^\infty$  function  $\phi(u) = d(p, \beta u) - d(q, \beta u)$  is nonsingular at  $u = 0$ . If this were not the case, then the vectors



$V(\beta 0, p)$  and  $V(\beta 0, q)$  would both be perpendicular to  $\beta'(0)$ , which would imply that  $p, \beta 0$  and  $q$  are collinear, contradicting the fact that  $\beta(0) = \alpha(t)$ ,  $t > 0$ . If  $h(s)$  is the inverse function of  $\phi$ , then  $h$  is a  $C^\infty$  diffeomorphism of some neighborhood  $J$  of  $t + t_0/2$  onto some neighborhood  $I$  of 0,  $I \subseteq (-\varepsilon, \varepsilon)$ . It follows that  $d(p, \beta(hs)) = s$  for all  $s$  in  $J$ , and therefore  $\alpha(s - t_0/2) = \beta(h(s))$  for all  $s$  in  $J$ . Hence  $\alpha$  is  $C^\infty$  at every number  $t > 0$ .

*Proof of proposition 2.6.* Let  $g: H \rightarrow \mathbf{R}$  be given by  $g(r) = d(p, r) - d(q, r)$ . It is clear that  $E(p, q) = g^{-1}(0)$  and that the two components of  $H - E(p, q)$  are the sets  $A_1 = \{r: g(r) > 0\}$  and  $A_2 = \{r: g(r) < 0\}$ . The set  $A_1$  is starshaped relative to  $q$  since  $g$  is nonincreasing on every geodesic starting at  $q$ , and  $A_2$  is starshaped relative to  $p$  since  $g$  is nondecreasing on every geodesic starting at  $p$ .

Suppose now that  $\gamma$  is a unit speed geodesic with  $\gamma(0) = p$  such that  $\gamma$  meets  $E(p, q)$  twice at times  $s \neq 0$ ,  $t \neq 0$ . There are two cases: 1)  $q \in \gamma$  and 2)  $q \notin \gamma$ . In the first case  $\gamma = \gamma_{pq}$ . Now  $d(p, \gamma t) - d(q, \gamma t)$  vanishes for only one value of  $t$  since  $p, q$  and  $\gamma(t)$  are always collinear. Suppose now that  $q \notin \gamma$ . Then  $q, \gamma s$  and  $\gamma t$  are not collinear. If  $s$  and  $t$  both have the same sign with  $|t| > |s|$ , then  $d(p, \gamma t) = d(q, \gamma t) < d(q, \gamma s) + d(\gamma s, \gamma t) = d(p, \gamma s) + d(\gamma s, \gamma t) = d(p, \gamma t)$ , a contradiction. Suppose that  $s$  and  $t$  have opposite signs. Then  $d(\gamma s, \gamma t) < d(\gamma s, q) + d(q, \gamma t) = d(\gamma s, p) + d(p, \gamma t) = d(\gamma s, \gamma t)$ , another contradiction. Similarly no geodesic containing  $q$  can meet  $E(p, q)$  twice.

*Proof of proposition 2.7.* We prove the assertions only for the point  $p$ . The curve  $f(t) = V(p, \alpha t)$  is a continuous map of  $\mathbf{R}$  into  $S^1$ , the unit vectors in  $T_p(H)$ . Since any maximal geodesic through  $p$  meets  $E(p, q)$  at most once it follows that  $f$  is one-one and  $f(\mathbf{R})$  contains no pair of antipodal points. Therefore  $f(\mathbf{R})$  is an open arc in  $S^1$  with distinct endpoints  $v_1 = \lim_{t \rightarrow \infty} V(p, \alpha t)$  and  $v_2 = \lim_{t \rightarrow -\infty} V(p, \alpha t)$ . Suppose now that the maximal geodesic  $\gamma_1$  intersects  $E(p, q)$  at  $\gamma_1(s) = \alpha(t)$  for some numbers  $s$  and  $t$ . If  $s > 0$  then  $v_1 = \gamma_1'(0) = V(p, \alpha t)$  is an interior point of  $f(\mathbf{R})$ , which is impossible. If  $s < 0$  then  $-v_1 = V(p, \alpha t)$  is an interior point of  $f(\mathbf{R})$ , which implies that  $f(\mathbf{R})$  contains a pair of antipodal points near  $\{v_1, -v_1\}$ , a contradiction. Therefore  $\gamma_1$  does not meet  $E(p, q)$ . Similarly  $\gamma_2$  does not meet  $E(p, q)$ .

*Proof of proposition 2.8.* We shall need some preliminary results.

LEMMA 2.8a. Let  $p, q, r$  be distinct points in  $H$ . Let  $h: H \rightarrow \mathbf{R}$  be the function given by  $a \mapsto d(r, a) - d(q, a)$ , and let  $\alpha$  be the canonical parametrization of  $E(p, q)$ . Then  $h \circ \alpha$  has at most one relative maximum or minimum point. If  $h \circ \alpha$  has a relative maximum or minimum point at  $t_0 \in \mathbf{R}$  then either

- 1)  $r = \alpha(t_0)$
- 2)  $r \notin \alpha$ ;  $p, r$  and  $\alpha(t_0)$  are collinear with  $p$  and  $r$  on the same side of  $\alpha$  or
- 3)  $r \notin \alpha$ ;  $q, r$  and  $\alpha(t_0)$  are collinear with  $q$  and  $r$  on the same side of  $\alpha$ .

*Proof.* If  $r$  lies in  $\alpha$ , say  $r = \alpha(t^*)$ , then the triangle inequality implies that  $h \circ \alpha$  has a strict global minimum at  $t^*$ . Assuming that the latter part of the lemma has been proved it follows that in this case  $h \circ \alpha$  has no relative maximum or minimum at  $t_0 \neq t^*$ .

Suppose now that  $h \circ \alpha$  has a relative maximum or minimum at  $t_0$  and that  $r \neq \alpha(t_0)$ . Then  $h$  is  $C^\infty$  in a neighborhood of  $\alpha(t_0)$ . If  $t_0 = 0$  then  $\alpha$  is not  $C^\infty$  at  $t_0$ , but in any case we can find a  $C^\infty$  diffeomorphism  $\sigma: (-\varepsilon, \varepsilon) \rightarrow E(p, q)$  such that  $\sigma(0) = \alpha(t_0)$  since  $E(p, q)$  is a  $C^\infty$  submanifold. Therefore  $0 = (h \circ \sigma)'(0) = \langle \sigma'(0), \text{grad } h(\alpha t_0) \rangle$  since  $h \circ \alpha$  has a relative maximum or minimum at  $t_0$ . Define  $g: H \rightarrow \mathbf{R}$  by  $g(a) = d(p, a) - d(q, a)$ . Since  $E(p, q) = g^{-1}(0)$ ,  $g \circ \alpha \equiv 0$  and thus  $0 = (g \circ \sigma)'(0) = \langle \sigma'(0), \text{grad } g(\alpha t_0) \rangle$ . Since  $\sigma'(0)$  is nonzero it follows that either

- i)  $(\text{grad } g)(\alpha t_0) = 0$ ,
- ii)  $(\text{grad } h)(\alpha t_0) = 0$ , or
- iii)  $(\text{grad } g)(\alpha t_0)$  and  $(\text{grad } h)(\alpha t_0)$  are both nonzero and collinear.

If  $a \neq p$  and  $a \neq q$ , then  $(\text{grad } g)(a)$  exists and equals  $-V(a, p) + V(a, q)$ . In particular  $\text{grad } g$  is nonzero at all points of  $E(p, q)$  so case i) does not occur. If  $a \neq r$  and  $a \neq q$ , then  $(\text{grad } h)(a)$  exists and equals  $-V(a, r) + V(a, q)$ . The point  $\alpha(t_0)$  is neither  $p$  nor  $q$ , nor  $r$  by assumption so that both  $\text{grad } g$  and  $\text{grad } h$  exist at  $\alpha(t_0)$ . If  $(\text{grad } h)(\alpha t_0) = 0$  as in case ii), then  $V(\alpha t_0, r) = V(\alpha t_0, q)$ , which implies that  $q, r$  and  $\alpha(t_0)$  are collinear. Moreover  $r$  does not lie on  $\alpha$  since no geodesic through  $q$  intersects  $E(p, q)$  twice. Hence  $q$  and  $r$  lie on the same side of  $\alpha$ . Finally suppose that  $(\text{grad } g)(\alpha t_0)$  and  $(\text{grad } h)(\alpha t_0)$  are both nonzero and collinear. If the unit vectors  $V(\alpha t_0, r)$ ,  $V(\alpha t_0, p)$  and  $V(\alpha t_0, q)$  are all distinct, then it is easy to see from the expressions above that  $(\text{grad } g)(\alpha t_0)$  and  $(\text{grad } h)(\alpha t_0)$  are not collinear, a contradiction. Hence  $V(\alpha t_0, r) = V(\alpha t_0, p)$  (implying that  $\text{grad } h = \text{grad } g$  at  $\alpha(t_0)$ ). Thus  $p, r$  and  $\alpha(t_0)$  are collinear with  $p$  and  $r$  lying on the same side of  $\alpha$ . The point  $r$  cannot lie on  $\alpha$  since no geodesic from  $p$  meets  $E(p, q)$  twice. This property of geodesics through  $p$  or  $q$  now implies that  $h \circ \alpha$  has at most one relative maximum or minimum upon inspection of the possibilities 1), 2), 3) of the lemma.

**LEMMA 2.8b.** *Let  $p, q, r, h$  and  $\alpha$  be as in the previous lemma. Then one of the following must occur:*

- 1)  $h \circ \alpha$  has a unique global minimum at some number  $t_0$ , and  $h \circ \alpha$  is strictly monotone on  $(-\infty, t_0)$  and  $(t_0, \infty)$
- 2)  $h \circ \alpha$  has a unique global maximum at some number  $t_0$ , and  $h \circ \alpha$  is strictly monotone on  $(-\infty, t_0)$  and  $(t_0, \infty)$
- 3)  $h \circ \alpha$  is strictly monotone on  $\mathbf{R}$ .

Moreover,

- 1) occurs if  $r = \alpha(t_0)$  or if  $q, r, \alpha(t_0)$  are collinear with  $q$  and  $r$  on the same side of  $\alpha$ , and  $r$  between  $q$  and  $\alpha(t_0)$  or if  $p, r, \alpha(t_0)$  are collinear with  $p$  and  $r$  on the same side of  $\alpha$ , and  $r$  between  $p$  and  $\alpha(t_0)$ ,
- 2) occurs if  $q, r, \alpha(t_0)$  are collinear with  $q$  and  $r$  on the same side of  $\alpha$ , and  $q$  between  $r$  and  $\alpha(t_0)$  or if  $p, r, \alpha(t_0)$  are collinear with  $p$  and  $r$  on the same side of  $\alpha$ , and  $p$  between  $r$  and  $\alpha(t_0)$ ,

3) occurs if  $r$  and  $q$  lie on the same side of  $\alpha$  and  $\gamma_{qr}$  does not meet  $\alpha$  or if  $r$  and  $p$  lie on the same side of  $\alpha$  and  $\gamma_{pr}$  does not meet  $\alpha$

*Proof.* If  $h \circ \alpha$  has no relative maximum or minimum on  $\mathbf{R}$ , then it is one-one and hence strictly monotone on  $\mathbf{R}$ . If  $h \circ \alpha$  has a relative maximum or minimum at a number  $t_0$ , then by the previous lemma it is one-one hence strictly monotone on the intervals  $(-\infty, t_0)$  and  $(t_0, \infty)$ . Therefore  $t_0$  is a global maximum or minimum.

We now consider the various cases in which these possibilities occur. We have already observed that 1) occurs if  $r = \alpha(t_0)$  for some  $t_0$ . Suppose now that  $r$  does not lie on  $\alpha$ . If  $q$  and  $r$  lie on the same side of  $\alpha$  and  $\gamma_{qr}$  does not meet  $\alpha$ , then  $h \circ \alpha$  has no relative maximum or minimum by the previous lemma and hence case 3) occurs. Suppose now that  $q$  and  $r$  lie on the same side of  $\alpha$  and  $\gamma_{qr}$  meets  $\alpha$  at  $\alpha(t_0)$  (only one intersection is possible). If  $r$  lies between  $q$  and  $\alpha(t_0)$ , then for any  $s \in \mathbf{R}$   $(h \circ \alpha)(s) - (h \circ \alpha)(t_0) = d(r, \alpha s) - d(q, \alpha s) - d(r, \alpha t_0) + d(q, \alpha t_0) = d(r, \alpha s) - d(q, \alpha s) + d(q, r) \geq 0$ . Hence case 1) occurs. If  $q$  lies between  $r$  and  $\alpha(t_0)$ , then  $(h \circ \alpha)(s) - (h \circ \alpha)(t_0) = d(r, \alpha s) - d(q, \alpha s) - d(q, r) \leq 0$ . Hence case 2) occurs. The various cases where  $p$  and  $r$  lie on the same side of  $\alpha$  are handled in a similar fashion. Note that  $d(q, \alpha s) = d(p, \alpha s)$  for all  $s$ .

We begin the proof of proposition 2.8. We show first that  $E(p, q) \cap E(q, r)$  contains at most two points. This is equivalent to showing that  $h \circ \alpha$  is zero at most twice. If  $h \circ \alpha$  had at least three zeros, however, then it would have at least two relative maxima or minima, which is impossible by lemma 2.8a.

Let  $\beta$  denote the canonical parametrization of  $E(r, q)$ . Suppose that  $E(p, q) \cap E(r, q)$  contains two points  $\alpha(t_0) = \beta(\tilde{t}_0)$  and  $\alpha(t_1) = \beta(\tilde{t}_1)$ . By replacing  $\alpha$  if necessary by the other canonical parametrization of  $E(p, q)$ ,  $t \rightarrow \alpha(-t)$ , we may assume that  $t_0 < t_1$  and  $\tilde{t}_0 < \tilde{t}_1$ . Now let  $S^1$  denote the unit vectors in  $T_q(H)$ . Define continuous curves  $\gamma_1: [t_0, t_1] \rightarrow S^1$  and  $\gamma_2: [\tilde{t}_0, \tilde{t}_1] \rightarrow S^1$  by setting  $\gamma_1(t) = V(q, \alpha t)$  and  $\gamma_2(t) = V(q, \beta t)$ . Let  $z_1 = \gamma_1(t_0) = \gamma_2(\tilde{t}_0)$  and  $z_2 = \gamma_1(t_1) = \gamma_2(\tilde{t}_1)$ . Then  $\gamma_1$  and  $\gamma_2$  are both one-one arcs in  $S^1$  joining  $z_1$  to  $z_2$ , since each geodesic from  $q$  meets  $\alpha$  or  $\beta$  at most once. Therefore either

$$1) \gamma_1 \cup \gamma_2 = S^1 \text{ or}$$

$$2) \gamma_1 = \gamma_2.$$

Suppose that case 1) holds. Then any geodesic  $\gamma$  starting at  $q$  intersects  $\alpha[t_0, t_1] \cup \beta[\tilde{t}_0, \tilde{t}_1] \subseteq E(p, q) \cup E(q, r)$  at least twice, including one intersection point of the form  $\gamma(t)$ ,  $t > 0$  and one point of the form  $\gamma(t')$ ,  $t' < 0$ . Consider the geodesic  $\gamma$  such that  $\gamma'(0) = \lim_{t \rightarrow \infty} V(q, \gamma t)$ . By proposition 2.7  $\gamma$  never intersects  $\alpha = E(p, q)$ , so that  $\gamma$  must intersect  $\beta = E(q, r)$  at least twice by the preceding remark. This contradicts the fact that any maximal geodesic through  $q$  meets  $E(q, r)$  at most once, and hence case 1) is eliminated.

Suppose that case 2) holds. Choose any number  $t \in (t_0, t_1)$ . By hypothesis the geodesic  $\gamma_{qat}$  meets  $\beta(\tilde{t}_0, \tilde{t}_1)$  in some point  $\beta(t^*)$ , and  $\beta(t^*) \neq \alpha(t)$  since  $E(p, q) \cap E(q, r)$  consists of the points  $\alpha(t_0), \alpha(t_1)$ . If  $\beta(t^*)$  lies between  $q$  and  $\alpha(t)$ , then  $q$  and  $\alpha(t)$  lie

on opposite sides of  $E(q, r)$ . Hence  $\alpha(t_0, t_1)$  and  $q$  lie on opposite sides of  $E(q, r)$ . If  $\alpha(t)$  lies between  $q$  and  $\beta(t^*)$ , then it follows similarly that  $\beta(\tilde{t}_0, \tilde{t}_1)$  and  $q$  lie on opposite sides of  $E(p, q)$ . We shall only obtain a contradiction to the first possibility since the second reduces to the first by interchanging the roles of  $p$  and  $r$ .

We are given that  $\alpha(t_0, t_1)$  and  $q$  lie on opposite sides of  $E(q, r)$ . Define a continuous map  $\varrho: [t_0, t_1] \rightarrow E(q, r)$  by setting  $\varrho(t)$  equal to the unique point of intersection of  $\gamma_{q\alpha t}$  with  $E(q, r)$ . The fact that  $\gamma_1 = \gamma_2$ , the hypothesis of case 2), implies immediately that the point sets  $\varrho[t_0, t_1]$  and  $\beta[\tilde{t}_0, \tilde{t}_1]$  are equal. Let  $g: H \rightarrow \mathbf{R}$  be given by  $g(a) = d(p, a) - d(q, a)$ . Then  $g(\varrho t) > 0$  for any number  $t \in (t_0, t_1)$  since  $\varrho(t)$  lies between  $q$  and  $\alpha(t)$  and  $g$  is strictly monotone decreasing on the segment  $\gamma_{q\alpha t}$ . Therefore  $(g \circ \beta) > 0$  on  $(\tilde{t}_0, \tilde{t}_1)$ , which implies that  $g \circ \beta$  has a global maximum in  $(\tilde{t}_0, \tilde{t}_1)$  by lemma 2.8b and the fact that  $g \circ \beta$  vanishes at  $\tilde{t}_0$  and  $\tilde{t}_1$ . Lemma 2.8b implies further that either

- i)  $\gamma_{pq}$  meets  $E(q, r)$  in a point  $z$  and  $q$  lies between  $p$  and  $z$  or
- ii)  $\gamma_{pr}$  meets  $E(q, r)$  in a point  $z$ , and  $r$  lies between  $p$  and  $z$ .

We treat these cases separately.

If i) holds then  $g(z) = d(p, z) - d(q, z) = d(p, q) > 0$ . Thus  $z = \beta(t)$  for some  $t$  in  $(\tilde{t}_0, \tilde{t}_1)$  since  $(g \circ \beta) \leq 0$  on  $(-\infty, \tilde{t}_0]$  and  $[\tilde{t}_1, \infty)$ . If  $t^* \in (t_0, t_1)$  is that number such that  $\beta(t) = \varrho(t^*)$ , then  $\gamma_{pq}$  meets  $E(p, q)$  twice, once between  $p$  and  $q$  and once at  $\alpha(t^*)$ , which is beyond  $z = \beta(t) = \varrho(t^*)$ . This contradicts the fact that any geodesic from  $p$  meets  $E(p, q)$  at most once.

Suppose that ii) holds. Then  $\gamma_{pr}$  meets  $E(p, r)$  at a point  $y$  between  $p$  and  $r$  and meets  $E(r, q)$  at a point  $z$  as assumed. The point  $r$  is thus an interior point of the segment  $\gamma_{yz}$ . Now if  $u$  and  $v$  are the points of intersection of  $E(p, q)$  and  $E(r, q)$  then they are equidistant from  $p, q$  and  $r$  and hence also lie in  $E(p, r)$ . Let  $\delta$  be the canonical parametrization of  $E(p, r)$  with  $u = \delta(t_0^*)$ ,  $v = \delta(t_1^*)$  and  $t_0^* < t_1^*$ . As in case 1) earlier we let  $S^1$  denote the unit vectors in  $T_r(H)$ . Define continuous curves  $\delta_1: [t_0^*, t_1^*] \rightarrow S^1$  and  $\delta_2: [\tilde{t}_0, \tilde{t}_1] \rightarrow S^1$  by  $\delta_1(t) = V(r, \delta t)$  and  $\delta_2(t) = V(r, \beta t)$ . Then  $\delta_1$  and  $\delta_2$  are both one-one arcs in  $S^1$  joining  $V(r, u)$  to  $V(r, v)$ . Either

- 1)  $\delta_1 \cup \delta_2 = S^1$  or
- 2)  $\delta_1 = \delta_2$ .

The case 1) is impossible by the same argument used earlier in the proof. Suppose that  $\delta_1 = \delta_2$ . We show first that  $y \in \delta[t_0^*, t_1^*]$  and  $z \in \beta[\tilde{t}_0, \tilde{t}_1]$ . By the definition of  $z$ ,  $g \circ \beta$  has a global maximum at  $t^*$ , where  $\beta(t^*) = z$ , and  $t^* \in (\tilde{t}_0, \tilde{t}_1)$ . Moreover,  $V(r, z)$  is an interior point of  $\delta_2$ . Now  $\delta_2$  does not contain any pair of antipodal points of  $S^1$ ; if  $V(r, a)$  and  $V(r, b) = -V(r, a)$  both lay in  $\delta_2$  for points  $a, b$  in  $\beta[\tilde{t}_0, \tilde{t}_1]$ , then the geodesic  $\gamma_{ab} = \gamma_{ra} = \gamma_{rb}$  would intersect  $E(q, r)$  at  $a$  and  $b$ , contradicting the fact that any geodesic containing  $r$  meets  $E(q, r)$  at most once. Therefore  $\delta_2$  is an arc in  $S^1$  of length  $< \pi$ . The fact that  $V(r, z)$  is an interior point of  $\delta_2$ , whose endpoints are  $V(r, u)$  and  $V(r, v)$ , now implies that  $u$  and  $v$  lie on opposite sides of the maximal geodesic  $\gamma_{rz} = \gamma_{pr}$ . Now  $\delta[t_0^*, t_1^*]$  is a curve joining  $u$  to  $v$  so  $\delta$  must intersect  $\gamma_{pr}$  in a point  $y^*$ .

Since  $\gamma_{pr}$  meets  $E(p, r)$  in the points  $y$  and  $y^*$  it follows that  $y = y^*$ . Thus  $y \in \delta[t_0^*, t_1^*]$ . The previous work has shown that  $V(r, y) \in \delta_1$  and  $V(r, z) = -V(r, y) \in \delta_2$ . Since  $\delta_1 = \delta_2$  by hypothesis, the geodesic  $\gamma_{yz} = \gamma_{rz} = \gamma_{ry}$  meets  $E(p, r)$  (and  $E(q, r)$ ) twice, on opposite sides of the point  $r$ . This contradiction completes the proof of the proposition.

*Proof of proposition 2.9.* We have already observed that  $\partial R_p$  is contained in the union of the sets  $E(p, \phi p)$ ,  $\phi \in D$ . Let a point  $q \in \partial R_p$  be given. The proper discontinuity of  $D$  implies that  $q$  is contained in only finitely many equidistant sets  $E(p, \phi_i p)$ ,  $1 \leq i \leq n$ . Since only finitely many of the sets  $E(p, \phi p)$  meet any compact neighborhood of  $q$ , it follows that for a sufficiently small open set  $0$  containing  $q$  and any  $\phi \in D - \{\phi_1, \dots, \phi_n\}$ ,  $0 \subseteq \text{Int } E^+(p, \phi p) = \{r \in H : d(p, r) < d(\phi p, r)\}$ . Choose  $x \in 0 - R_p$ ; this can be done since  $q \in \partial R_p$ . Moreover, let  $p, q$  and  $x$  be noncollinear. The geodesic segment  $\gamma_{px}$  meets  $\partial R_p$ , and in fact if  $\gamma_{px}$  intersects  $E(p, \phi p)$  then  $\phi \in \{\phi_1, \dots, \phi_n\}$  by the way in which  $0$  was chosen. Now  $\gamma_{px}$  meets each of the sets  $E(p, \phi_i p)$  at most once so there are a finite number  $k \leq n$  of intersections of  $\gamma_{px}$  with  $\bigcup_{i=1}^n E(p, \phi_i p)$ . Let  $\gamma_{px}(t_i)$ ,  $1 \leq i \leq k$ , be these intersections, where  $t_i < t_{i+1}$ , and let  $r$  be that integer such that  $\gamma_{px}(t_1) \in E(p, \phi_r p)$ . Note that  $\gamma_{px}(t_1)$  lies in exactly one of the sets  $E(p, \phi_i p)$  since the unique point of intersection of any two sets  $E(p, \phi_i p)$ ,  $E(p, \phi_j p)$  is  $q$ .

We assert that  $E(p, \phi_r p)$  is a bounding side. Let  $t^* \in (t_1, t_2)$  be arbitrary. If  $z = \gamma_{px}(t^*)$  then the geodesic segment  $\gamma_{pz}$  intersects  $E(p, \phi_r p)$  but not  $E(p, \phi p)$  if  $\phi \neq \phi_r$ ,  $\phi \neq 1$ . Therefore  $d(p, z) < d(\phi p, z)$  if  $\phi \neq \phi_r$ ,  $\phi \neq 1$  and  $z \in \bigcap_{\substack{\phi \neq \phi_r \\ \phi \neq 1}} E^+(p, \phi p)$ . However  $z \notin R_p$  since  $\gamma_{pz}$  intersects  $E(p, \phi_r p)$ , implying that  $d(p, z) < d(\phi_r p, z)$ . This proves that  $E(p, \phi_r p)$  is a bounding side.

*Proof of corollary 2.10.* If  $A$  denotes the intersection of these sets it is clear from the definition of  $R_p$  that  $R_p \subseteq A$ . If  $R_p$  were a proper subset of  $A$ , then for any point  $q$  in  $A - R_p$  the geodesic segment  $\gamma_{pq}$  would meet  $\partial R_p$  in a point  $q^*$  in the interior of  $\gamma_{pq}$ . By the preceding result  $q^*$  lies in some bounding side  $E(p, \phi p)$ , which implies that points on  $\gamma_{pq}$  beyond  $q^*$ , in particular  $q$ , lie in  $H - E^+(p, \phi p)$ . This contradicts the hypothesis that  $q \in A \subseteq E^+(p, \phi p)$ .

*Proof of proposition 2.11.* We first establish the following

**LEMMA 2.11.** *If  $R_p \cap E(p, \phi p)$  is nonempty, then  $R_p \cap E(p, \phi p)$  is an arc connected subset of  $E(p, \phi p)$ .*

*Proof.* We may assume that  $R_p \cap E(p, \phi p)$  contains at least two points, for otherwise the result is vacuously true. Let  $q_1$  and  $q_2$  be two points in  $E(p, \phi p) \cap R_p$ . Giving  $E(p, \phi p)$  the canonical parametrization  $\alpha$ , we know that  $q_1 = \alpha(s)$  and  $q_2 = \alpha(t)$  for some numbers  $s$  and  $t$ . We may assume that  $s < t$ . If  $\alpha(u) \in H - R_p$  for some number  $u$  with  $s < u < t$ , then  $\alpha(u) \in H - E^+(p, \psi p)$  for some nonidentity element  $\psi \neq \phi$ . Let  $f: H \rightarrow \mathbb{R}$  be the function given by  $f(r) = d(p, r) - d(\psi p, r)$ . Now  $f(\alpha s) \leq 0$  and  $f(\alpha t) \leq 0$  since  $\alpha(s)$  and  $\alpha(t)$  lie in  $R_p$ . On the other hand  $f(\alpha u) > 0$  by hypothesis.



Hence  $f \circ \alpha$  equals zero at some points  $s^*$  and  $t^*$  with  $s \leq s^* < u < t^* \leq t$ . Therefore  $E(p, \phi p) \cap E(p, \psi p)$  contains the distinct points  $\alpha(s^*)$  and  $\alpha(t^*)$ , contradicting proposition 2.8. Therefore  $\alpha[s, t] \subseteq R_p \cap E(p, \phi p)$ .

We begin the proof of proposition 2.11. Suppose that  $E(p, \phi p) \cap R_p$  is nonempty but  $E(p, \phi p)$  is not a bounding side of  $R_p$ . Assuming that  $E(p, \phi p) \cap R_p$  contains more than one point we see by the previous lemma that  $E(p, \phi p) \cap R_p$  consists of an entire subarc of  $E(p, \phi p)$ . Let  $q$  be an interior point of  $E(p, \phi p) \cap R_p$ , and choose a number  $\varepsilon > 0$  such that the set  $A = \overline{B_\varepsilon(q)} \cap E(p, \phi p)$  is a compact subarc of  $E(p, \phi p)$  that is contained in  $R_p$ . In particular  $A \subseteq \partial R_p$ . ( $\overline{B_\varepsilon(q)}$  denotes the closed ball of radius  $\varepsilon$  and center  $q$  in  $H$ ). Let  $\psi_1, \dots, \psi_k$  be those elements of  $D$  such that  $E(p, \psi_i p)$ ,  $1 \leq i \leq k$ , are the only bounding sides of  $R_p$  that intersect  $A$ . The element  $\phi$  is not equal to  $\psi_i$  for any  $i$  since  $E(p, \phi p)$  is not a bounding side of  $R_p$ . Hence  $E(p, \phi p) \cap E(p, \psi_i p)$  is at most one point for each  $i$  by proposition 2.8. Therefore the set  $A - \bigcup_{i=1}^k E(p, \psi_i p)$  is an infinite set. Let  $r$  be an arbitrary point of point of  $A - \bigcup_{i=1}^k E(p, \psi_i p)$ . Proposition 2.9 implies that  $r$  lies in some bounding side of  $R_p$  since  $r \in \partial R_p$ . However, the only bounding sides of  $R_p$  that meet  $A$  are  $E(p, \psi_i p)$ ,  $1 \leq i \leq k$ , contradicting the choice of  $r$ . Therefore  $E(p, \phi p) \cap R_p$  is a single point.

Next let  $E(p, \phi p)$  be a bounding side of  $R_p$ . We show first that  $E(p, \phi p) \cap R_p$  is nonempty. By definition there exists a point  $q$  in  $\bigcap_{\substack{\psi \neq 1 \\ \psi \neq \phi}} E^+(p, \psi p) - R_p$ . Let  $q^*$  be the unique point of intersection of the geodesic segment  $\gamma_{pq}$  with  $E(p, \phi p)$ . We claim that  $q^* \in E(p, \phi p) \cap R_p$ . For every  $\psi$  in  $D$  the set  $E^+(p, \psi p)$  is starshaped relative to  $p$ , and any geodesic from  $p$  that meets  $E(p, \psi p)$  leaves  $E^+(p, \psi p)$  after intersecting  $E(p, \psi p)$ . If  $\psi \neq \phi$  it follows that  $q^* \in E^+(p, \psi p)$  but not in  $E(p, \psi p)$  since  $q \in E^+(p, \psi p)$ . Since  $q^* \in E(p, \phi p) \subseteq E^+(p, \phi p)$  by the choice of  $q^*$  it follows that  $q^* \in R_p \cap E(p, \phi p)$ .

We show that  $R_p$  contains a set  $U$  such that  $q^* \in U \subseteq E(p, \phi p)$  and  $U$  is open in  $E(p, \phi p)$ . If this were not the case, then we could find a sequence of points  $q_n^* \subseteq (H - R_p) \cap E(p, \phi p)$  that converges to  $q^*$ . Therefore we could find a sequence  $\phi_n \subseteq D$  such that  $\phi_n \neq \phi$  and  $d(\phi_n p, q_n^*) < d(p, q_n^*)$  for every  $n$ . There are only finitely many distinct elements  $\phi_n$  by the proper discontinuity of  $D$  since the points  $\phi_n(p)$  are a bounded sequence in  $H$ . Passing to a subsequence we may assume that  $\phi_n = \psi \neq \phi$  for every  $n$ . Since  $E(p, \psi p)$  is closed  $q^* \in E(p, \psi p)$ , which contradicts the fact proved above that  $q^* \notin E(p, \psi p)$  if  $\psi \neq \phi$ . Since  $E(p, \phi p) \cap R_p$  contains more than one point it consists of an entire subarc of  $R_p$  by the previous lemma.

Finally we show that no interior point of  $E(p, \phi p) \cap R_p$  can be a vertex of  $R_p$ . Suppose that  $q$  is a vertex of  $R_p$  and also an interior point of  $E(p, \phi p) \cap R_p$  for some bounding side  $E(p, \phi p)$ . By the definition of vertex there exists an element  $\psi \neq \phi$  in  $D$  such that  $E(p, \psi p)$  is a bounding side for  $R_p$  and  $q \in E(p, \psi p)$ . Relative to canonical parametrizations  $\alpha, \beta$  for  $E(p, \phi p), E(p, \psi p)$  we can write  $q = \alpha(t_0) = \beta(s_0)$ . There exists by hypothesis an  $\varepsilon > 0$  such that  $\alpha t \in E(p, \phi p) \cap R_p$  for  $|t - t_0| < \varepsilon$ . Since

$E(p, \psi p) \cap R_p$  is an arc there exists some  $\delta > 0$  such that  $\beta(s) \in R_p$  for all  $s \in [s_0 - \delta, s_0]$  or all  $s \in [s_0, s_0 + \delta]$ . Hence there exists  $t \neq t_0$  such that  $|t - t_0| < \varepsilon$  and  $\gamma_{p\alpha t}$  intersects  $E(p, \psi p) \cap R_p$  in a point  $q^*$ . However  $\alpha t \in \partial R_p$  also and since any geodesic from  $p$  meets  $\partial R_p$  at most once it follows that  $\alpha t = q^*$ . This implies that both  $q$  and  $q^*$  lie in  $E(p, \phi p) \cap E(p, \psi p)$ , contradicting proposition 2.8.

*Proof of proposition 2.12.* We show first that if  $q \in E(p, \phi p) \cap R_p$  then  $\phi^{-1}q \in R_p \cap E(p, \phi^{-1}p)$ . Let such a point  $q$  be given. Clearly  $\phi^{-1}q \in E(p, \phi^{-1}p)$  so it suffices to show that  $\phi^{-1}q \in R_p$ . For any nonidentity element  $\psi$  in  $D$  we know that  $d(\psi p, \phi^{-1}q) = d(\phi\psi p, q) \geq d(p, q) = d(\phi p, q) = d(p, \phi^{-1}q)$ . Therefore  $\phi^{-1}q \in R_p$ , and this implies that  $\phi^{-1}\{E(p, \phi p) \cap R_p\} \subseteq E(p, \phi^{-1}p) \cap R_p$ . Reversing the roles of  $\phi$  and  $\phi^{-1}$  we see that  $\phi\{E(p, \phi^{-1}p) \cap R_p\} \subseteq E(p, \phi p) \cap R_p$ , which implies that  $E(p, \phi^{-1}p) \cap R_p \subseteq \phi^{-1}\{E(p, \phi p) \cap R_p\}$  and proves that  $E(p, \phi^{-1}p) \cap R_p = \phi^{-1}\{E(p, \phi p) \cap R_p\}$ . The set  $E(p, \phi p) \cap R_p$  is an arc by the preceding result, hence  $E(p, \phi^{-1}p) \cap R_p$  is an arc. This implies that  $E(p, \phi^{-1}p)$  is a bounding side for  $R_p$ , again by the previous result.

*Proof of proposition 2.13.* Let  $q$  be a vertex of  $R_p$  and suppose that  $q$  lies on three distinct bounding sides  $L_1 = E(p, \phi_1 p)$ ,  $L_2 = E(p, \phi_2 p)$  and  $L_3 = E(p, \phi_3 p)$ . Fix an orientation of the tangent space  $T_p(H)$ , and set  $v = V(p, q)$ . For each positive number  $\varepsilon$  we let  $B_\varepsilon^+(v) = \{w \in T_p(H) : \|w\| = 1 \text{ and } 0 \leq \angle(v, w) < \varepsilon\}$  and  $B_\varepsilon^-(v) = \{w \in T_p(H) : \|w\| = 1 \text{ and } -\varepsilon < \angle(v, w) \leq 0\}$ . By  $\angle(v, w) > 0$  (respectively  $< 0$ ) we mean that the pair  $\{v, w\}$  is positively (respectively negatively) oriented relative to the given orientation of  $T_p(H)$ . Since  $L_i$  is a bounding side of  $R_p$  for each  $i$  the point  $q$  is an endpoint of some arc  $\beta_i$  contained in  $L_i \cap R_p$ . Therefore for each  $i = 1, 2, 3$  we can find a number  $\varepsilon_i > 0$  such that one of the following two possibilities occurs:

- i) For any vector  $w \in B_{\varepsilon_i}^+(v)$ ,  $\gamma_w$  intersects  $L_i \cap R_p$ .
- ii) For any vector  $w \in B_{\varepsilon_i}^-(v)$ ,  $\gamma_w$  intersects  $L_i \cap R_p$ .

Since we have three bounding sides  $L_1, L_2, L_3$  we can find an  $\varepsilon > 0$  such that one of the half neighborhoods of  $v$ , say  $B_\varepsilon^+(v)$ , corresponds to two of the bounding sides. Denote these sides by  $L$  and  $L'$ . Now  $L \cap L' = \{q\}$  by proposition 2.8 so that if  $w \in B_\varepsilon^+(v)$  is not equal to  $v$ , then  $\gamma_w$  meets  $L \cap R_p$  and  $L' \cap R_p$  in distinct points  $r$  and  $r'$ . The points  $r$  and  $r'$  both lie in  $\partial R_p$ . However, any geodesic  $\gamma$  from  $p$  intersects  $\partial R_p$  in at most one point  $q$ , for if  $q \in E(p, \phi p)$  for some  $\phi \neq 1$  in  $D$ , then all points on  $\gamma_{pq}$  beyond  $q$  lie in  $H - E^+(p, \phi p)$ . We have obtained a contradiction to our assumption that  $q$  lies in three bounding sides of  $R_p$ .

*Proof of proposition 2.14.* Since  $q \in \partial R_p$ ,  $q$  lies in some bounding side  $E(p, \xi p)$ ,  $\xi \in D$ , by proposition 2.9. One of the elements  $\{\phi, \psi\}$ , say  $\phi$ , is not equal to  $\xi$ . If  $E(p, \phi p)$  is a bounding side of  $R_p$ , then we are done so we may suppose that  $E(p, \phi p)$  is not a bounding side of  $R_p$ . By proposition 2.11  $E(p, \phi p) \cap R_p$  is the single point  $q$ . Let  $\alpha$  be the canonical parametrization of  $E(p, \phi p)$ , and let  $f: H \rightarrow \mathbb{R}$  be the function  $r \rightarrow d(\xi p, r) - d(p, r)$ . Now  $q = \alpha(t_0)$  for some number  $t_0$ , and hence  $(f \circ \alpha) = 0$ . Since  $\xi p \neq \phi p$  lemma 2.8a implies that  $f \circ \alpha$  is nonzero at any relative maximum or minimum

point. Therefore  $(f \circ \alpha)$  has no relative maximum or minimum at  $t_0$ , and lemma 2.8b implies that  $f \circ \alpha$  is strictly monotone in some neighborhood  $U$  of  $t_0$ . Therefore either

i)  $(f \circ \alpha)(t) > 0$  for  $t > t_0, t \in U$  or

ii)  $(f \circ \alpha)(t) > 0$  for  $t < t_0, t \in U$ .

Without loss of generality we may assume that i) occurs. Let  $t_n$  be any sequence of numbers such that  $t_n > t_0$  for each  $n$  and  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$ . If  $q_n = \alpha(t_n)$ , then  $q_n \in H - R_p$  since  $E(p, \phi p) \cap R_p = q = \alpha(t_0)$ . By assumption  $(f \circ \alpha)(t_n) = d(\xi p, q_n) - d(p, q_n) > 0$  for large  $n$ . The geodesic segment  $\gamma_{pq_n}$  meets  $\partial R_p$  in a point  $r_n$ , and by the triangle inequality  $d(p, r_n) \leq d(p, r_n) + d(r_n, q_n) + d(r_n, \xi p) - d(q_n, \xi p) = d(p, q_n) - d(\xi p, q_n) + d(r_n, \xi p) < d(\xi p, r_n)$ . Hence  $r_n$  does not lie in  $E(p, \xi p)$  for large  $n$ . Since  $r_n$  is a bounded sequence in the boundary of  $R_p$ , proposition 2.9 and the proper discontinuity of  $D$  imply that by passing to a subsequence we can find an element  $\psi \neq \xi$  in  $D$  such that  $r_n \in E(p, \psi p)$  for all  $n$  and  $E(p, \psi p)$  is a bounding side of  $R_p$ . Passing to a further subsequence we may assume that  $r_n$  converges to a point  $r$  in  $E(p, \psi p)$ . Since  $r_n$  lies on the geodesic  $\gamma_{pq_n}$  for every  $n$  it follows that  $r$  lies on the geodesic  $\gamma_{pq}$ . Hence  $\gamma_{pq}$  meets  $\partial R_p$  at both  $q$  and  $r$ , and this implies that  $r = q$  since a geodesic starting at  $p$  can meet  $\partial R_p$  at most once. Therefore  $q$  lies on the distinct bounding sides  $E(p, \psi p)$  and  $E(p, \xi p)$ , which by definition means that  $q$  is a vertex of  $R_p$ .

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