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G Maps and the Projective Class Group

TED PETRIE¹

0. Introduction and Motivation

Let G be a compact Lie group and $f: X \rightarrow Y$ be a G normal map (see §1) between smooth closed G manifolds X and Y . We are interested in the relation between the homological dimension over $H_*(G, R)$ of $K_*(f, R) = \ker(H_*(X, R) \rightarrow H_*(Y, R))$ and Smith theory. The latter states that if f is a G map between two G spaces (not necessarily manifolds) which induces an isomorphism in mod p homology, then for each p subgroup K of G , the fixed point mapping f^K also induces an isomorphism in mod p homology.

To study this relationship we introduce an invariant $\chi(f) \in \tilde{K}_0(Z(G/G_0))$ (the reduced projective class group of the group ring of G/G_0) for a G map $f: X \rightarrow Y$ which satisfies the conclusions of Smith theory for each p subgroup K of G . Here X and Y need not be manifolds.

We expect that $\chi(f)$ will be a useful tool in other areas of G homotopy theory. Since our application is in G normal cobordism theory, we emphasize the relationship mentioned in the first paragraph.

In order to motivate the ideas, let X and Y be smooth closed oriented G manifolds. The singular set of X written sX is the set of points of X whose isotropy groups are not principle. If G acts freely on X , then ${}^sX = \emptyset$ and X/G is a manifold of dimension $m - g$ if $\dim X = m$ and $\dim G = g$.

The following results serve as a starting point for our study.

THEOREM 0.1. (Folklore) *If G is connected and acts freely on Y and $K_i(f) = 0$ for $i < \lambda = [(m - g)/2]$ and $m - g$ is even, then $K_*(f) = H_*(G) \otimes K_\lambda(f)$ as an $H_*(G)$ module and $K_\lambda(f)$ is free over Z .*

THEOREM 0.2 [5] and [12]. *If G is finite, so $H_*(G) = Z(G)$, and acts freely on Y with $K_i(f) = 0$ for $i < \lambda$ and m is even, then $K_i(f) = 0$ for $i \neq \lambda$ and $K_\lambda(f)$ is $Z(G)$ projective and zero in $\tilde{K}_0(Z(G))$. If m is odd and $K_i(f) = 0$ for $i < \lambda$ and $K_\lambda(f)$ is a Z torsion module, then $K_\lambda(f)$ has homological dimension ≤ 1 over $Z(G)$ and gives zero in $\tilde{K}_0(Z(G))$.*

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Observe that the condition that G act freely on Y implies ${}^sY = \phi$ a very restrictive condition; however, examples show that some restrictions on sX and sY are necessary for conclusions like those of (0.1) and (0.2). The conclusions of Smith theory are restrictions on sX and sY and together with the assumption that $K_i(f, Z) = 0$ for $i < \lambda$ are just the conditions necessary to establish the analog (6.1) of 0.1 and 0.2. Of course some condition on sX e.g. $\dim {}^sX/G < \frac{1}{2} \dim X/G$ is necessary to achieve $K_i(f, Z) = 0$ for $i < \lambda$. Not only do the singular sets appear implicitly in the definition of $\chi(f)$ (5.2), but also in its calculation (5.4) where $\chi(f) = \chi({}^s f)$.

The relation between $\chi(f)$, $K_*(f)$ and Smith theory is (6.1) which under the conditions there gives $\chi(f) = \pm[K_\lambda(f, Z)^*]$. One of the interesting consequences of this is that $\chi(f)$ (and so $K_\lambda(f, Z)$) depends not only on the p subgroups of G but on all subgroups (§9). This is certainly a new feature in G homotopy theory.

This paper is organized as follows: The first four sections are technical. In §5 we define $\chi(f)$. A key ingredient here is a paper of Rim [9]. In §6 we give the main result, the structure of $K_*(f, Z)$ as an $H_*(G)$ module. In §7 we give a very brief outline of the application of $\chi(f)$ to the G normal cobordism problem. In §8 we discuss the Swan homomorphism $\sigma_G: Z_n^* \rightarrow \tilde{K}_0(Z(G))$, relate it to $\chi(f)$ and prove geometrically a theorem of [11]. In §9 we give examples where $\chi(f) \neq 0$ and in §10 we give an application to equivariant homotopy groups of spheres.

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1. Notation

Throughout we consider only compact Lie Groups. Let G be such a group and g its dimension. Its connected component is denoted by G_0 , its maximal torus by T and N is the normalizer of T . If p is a prime, G_p is the inverse image in G of the Sylow p subgroup $(G/G_0)_p$ of G/G_0 .

$$\mathcal{H}(N_p) = \{H \subset N_p \mid H \neq 1, \quad H/H_0 \text{ is a } p \text{ group}\}. \quad (1.1)$$

The sets $\mathcal{H}(N_p)$ play a central role and have two important properties

- (i) If G is finite or abelian and H and K are in $\mathcal{H}(N_p)$, so is $H \cdot K$.

- (ii) If $L \subset N_p$, it has a finite normal subgroup $F \subset L \cap N_0$ with $F_p = 1$, $L/F \cong L_p$ and if $N(L)$ denotes the normalizer of L , $N(L) \subset N(L_p) \cdot L$ (by Sylow's theorem on the conjugacy of Sylow subgroups). Note $N(L)_p \subset N(L_p)_p$. These normalizers are taken in N_p .

If X and Y are G spaces and $f: X \rightarrow Y$ is a G map, M_f is the mapping cone of f . It is a G space with a canonical fixed point $q \in (M_f)^G = M_f^G$ corresponding to the point obtained by identifying X to a point. Here $f^G: X^G \rightarrow Y^G$ is the induced map of the fixed point sets. The equality

$$(M_f)^G = M_f^G \tag{1.3}$$

is important and includes the convention that $M_h = \text{point}$ if h is a map of the empty set. The isotropy group of a point $x \in X$ is denoted by G_x and the singular set of X denoted by sX is defined as

$${}^sX = \{x \in X \mid G_x \neq \text{principal isotropy group}\}. \tag{1.4}$$

Let ${}^s f: {}^sX \rightarrow {}^sY$ denote the restriction of f to sX . Then

$${}^s(M_f) = M_{{}^s f}. \tag{1.5}$$

Suppose that E is a contractible G space on which G acts freely so the orbit space E/G is the classifying space B_G of G . Let $C_*^G(X)$ denote the chain complex of $X \times_G E$. If M is a module over the group ring $\Lambda = Z(G/G_0)$, we write $H_*^G(X, M)$ and $H_G^*(X, M)$ for the homology of the chain complexes $C_*^G(X) \otimes_\Lambda M$ and $\text{Hom}_\Lambda(C_*^{G_0}(X), M)$. In particular

$$H_G^*(X, \Lambda) = H_{G_0}^*(X, Z), \quad H_G^*(X, Z) = H^*(X \times_G E, Z). \tag{1.6}$$

If A is an algebra over Λ and M is an A module, then $H_G^*(X, M)$ is an $H_G^*(X, A)$ module. Set $\tilde{H}_G(X, A) = \ker(H_G^*(X, A) \rightarrow H^*(X, A))$.

When G is a finite group, $\tilde{K}_0(\Lambda)$ is the reduced projective class group of Λ . That is the Grothendieck group of Λ modules of finite homological dimension modulo the subgroup generated by free modules. The involution of $\tilde{K}_0(\Lambda)$ defined by $M \rightarrow \text{Hom}_Z(M, Z) = M^*$ is denoted by $*$.

In what follows, all manifolds are smooth and *oriented* and all G spaces have only a finite number of conjugacy classes of isotropy subgroups. Let X and Y be smooth closed G manifolds of dimension m .

DEFINITION

A G normal map $f: X \rightarrow Y$ consists of a G map f whose degree is 1 together with a specific G bundle map $F: \nu_X \rightarrow \xi$ covering f from the stable G normal bundle ν_X of a G imbedding of X in a real G module to some G vector bundle ξ over Y . Briefly this is denoted by (X, f) . Note F defines an isomorphism $\nu_X \cong f^*\xi$.

The definition of a G normal cobordism between two G normal maps (X_i, f_i) $i = 0, 1$ (Y is fixed) is straightforward. This generalizes the definition of [4] where $G = 1$.

The G normal cobordism problem: Given a G normal map (X, f) to Y . When is (X, f) G normally cobordant to (X', f') with f' a homotopy equivalence?

Define $K_*(f, R) = \ker(H_*(X, R) \rightarrow H_*(Y, R))$, $K^*(f, R) = \text{coker}(H^*(Y, R) \rightarrow H^*(X, R))$. These groups satisfy duality $K^i(f, R) \cong K_{m-i}(f, R)$ and a universal coefficient theorem $K_i(f, R) = K_i(f, Z) \otimes_Z R \oplus \text{Tor}(K_{i-1}(f, Z), R)$ and similarly for $K^*(f, R)$. When R is Z , we abbreviate $K_*(f, Z)$ and $K^*(f, Z)$ by $K_*(f)$ and $K^*(f)$. For a G normal map (X, f) to Y we have

$$K^i(f, R) = H^{i+1}(M_f, q, R) \quad \text{and} \quad K_i(f, R) = H_{i+1}(M_f, q, R). \quad (1.9)$$

2. Behavior of $H_G(X, M)$ for Subgroups

LEMMA 2.1. *Suppose X is an N space and M is a $Z_p(N/N_0)$ module. Then $H_N^*(X, M) \rightarrow H_{N_p}^*(X, M)$ is a monomorphism.*

Proof. The composition of restriction $H_N^*(X, M) \rightarrow H_{N_p}^*(X, M)$ and transfer $H_{N_p}^*(X, M) \rightarrow H_N^*(X, M)$ is multiplication by the index of N_p in N .

LEMMA 2.2. *Let A be a Λ algebra on which G/G_0 acts as the identity. Then $H_G^*(X, A)$ is a subalgebra of $H_N^*(X, A)$.*

Proof. This follows from [2] applied to the fibration $G/N \rightarrow X \times_N E \rightarrow X \times_G E$. There is a homomorphism $t: H_N^*(X, A) \rightarrow H_G^*(X, A)$ with $\pi^*t(x) = \chi(G/N) \cdot x$ for $x \in H_G^*(X, A)$. Since the Euler number of G/N is 1, the result follows.

We need two results about finite generation over $H_G^*(q, Z_p)$.

LEMMA 2.3. *Suppose G is connected and X is a G space whose total Z_p cohomology is finite dimensional over Z_p . Then $H_G^*(X, Z_p)$ is a finitely generated $H_G^*(q, Z_p)$ module.*

Proof. $H_G^*(q, Z_p)$ is Noetherian and there is a spectral sequence of $H_G^*(q, Z_p)$ algebras $E_2 = H_G^*(q, Z_p) \otimes_{Z_p} H^*(X, Z_p) \Rightarrow H_G^*(X, Z_p)$.

Since E_2 is finitely generated, the result follows.

LEMMA 2.4. *Suppose G is a finite p group and M is a finitely generated $Z_p(G)$ module. Then $H_G^*(q, M)$ is a finitely generated $H_G^*(q, Z_p)$ module.*

Proof. Let I be the kernel of the augmentation $Z_p(G) \rightarrow Z_p$. Then I is nilpotent, say $I^n = 0$ [1]. Filter M as $M \supset IM \supset \dots \supset I^n M = 0$. We have an exact triangle

$$\begin{array}{ccc} H_G^*(q, I^{k+1}M) & \longrightarrow & H_G^*(q, I^kM) \\ & \swarrow & \searrow \\ & H_G^*(q, I^kM/I^{k+1}M) & \end{array}$$

and each $I^kM/I^{k+1}M$ is a Z_p vector space with trivial action of G . The result follows by induction.

LEMMA 2.5. *Suppose M is a (graded) finitely generated $H_G^*(q, Z_p)$ module and for each multiplicative subset $s \in \tilde{H}_G(q, Z_p)$, $s^{-1}M = 0$. Then M is zero for large i .*

Proof. Suppose $\Gamma = H_G^*(q, Z_p)$ has one algebra generator y of positive dimension and M has one generator m as a Γ module. Let s be the set of powers of y . Since $s^{-1}M = 0$, $y^k m = 0$ for some k . Then $M^i = 0$ for $i > k$ dimension $(y) \cdot$ dimension (m) . The general case is similar.

3. $K^*(f)$ as an $H^*(G)$ module- G connected

LEMMA 3.1. *Let W be a G space with $q \in W^G \neq \phi$ and $H^*(W^H, q, Z_p) = 0$ for all $H \in \mathcal{H}(N_p)$. Then $H^*({}_pW, q, Z_p) = 0$ where ${}_pW = \cup_{H \in \mathcal{H}(N_p)} W^H$.*

Proof. If N_p is finite or abelian, this follows from Meyer-Vietoris and induction by (1.2)(i). In general we show $H_G^*({}_pW, q) = 0$ implying $H^*({}_pW, q) = 0$. (Z_p coefficients understood.) We can suppose $G = N_p$ and choose $P \subset \mathcal{H}(G)$ with $W^P \neq W^G$ and contained in no other P' in $\mathcal{H}(G)$ with this property. Order the conjugacy classes of isotropy groups Q_i containing P so that $G = Q_0$ and if some

conjugate of Q_i contains Q_j then $i < j$. Note $Q_{ip} = P$ for $i \neq 0$. As a matter of notation, let r be the largest index and $Q_r = P$ (eventhough P may not be an isotropy subgroup). Define $W_0 = W^G$ and $W_{n+1} = GW^{Q_{n+1}} \cup W_n$. The W_i give a G filtration of GW^P and the W_i^P give an $N(P)$ filtration of W^P . These filtrations produce spectral sequences $E_r \Rightarrow H_G^*(GW^P, W^G)$ and $E'_r \Rightarrow H_{N(P)}^*(W^P, W^G)$ and the inclusion of spaces a map of spectral sequences $E_r \rightarrow E'_r$ which is an isomorphism of E_1 to E'_1 because $H_G(W_i, W_{i-1}) \rightarrow H_{N(P)}(W_i^P, W_{i-1}^P)$ is an isomorphism for all i . In fact this map is the composition of these isomorphisms:

$$\begin{aligned} H_G^*(W_i, W_{i-1}) &= H_G^*(Gx_{N(Q_i)}(W^{Q_i}, W_{i-1}^{Q_i})) \\ &\cong H_{N(Q_i)}^*(W^{Q_i}, W_{i-1}^{Q_i}) =_\alpha H_{N(P) \cap N(Q_i)}^*(W^{Q_i}, W_{i-1}^{Q_i}) \\ &\cong H_{N(P)}^*(N(P) \times_{N(P) \cap N(Q_i)}(W^{Q_i}, W_{i-1}^{Q_i})) \\ &\cong H_{N(P)}^*(N(P)W^{Q_i}, N(P)W_{i-1}^{Q_i}) =_\beta H_{N(P)}^*(W_i^P, W_{i-1}^P). \end{aligned}$$

Only steps α and β require comment. Since $Q_{ip} = P$, $N(Q_i)_p = (N(Q_i) \cap N(P))_p$ by (1.2)(ii). Since $H_L^*(A, B) = H_{L_p}^*(A, B)$ for L in N_p by (1.2)(ii), this shows α is true. For β the key facts are $(GW^{Q_i})^P = N(P)Q^{Q_i}$ and $N(P)(GW^{Q_i})^{Q_i} = N(P)W^{Q_i}$ if some conjugate of Q_j contains Q_i . For $gQ_jg^{-1} \supset Q_i \supset P$ implies $g \in N(P)Q_j$ by Sylow's theorem.

This argument shows the natural map $H_G^*(GW^P, W^G) \rightarrow H_{N(P)}^*(W^P, W^G)$ is an isomorphism, but the latter group is zero because $P \in \mathcal{H}(N_p)$. The proof now follows by induction considering ${}_pW/GW^P$.

LEMMA 3.2. *Let W satisfy the hypothesis of (3.1). Then for each multiplicative set $s \in \tilde{H}_{N_p}^*(q, Z_p)$ (the kernel of $H_{N_p}^*(q, Z_p) \rightarrow H^*(q, Z_p)$), $s^{-1}\tilde{H}_{N_p}^*(W, q, Z_p) = 0$. If $s \in \tilde{H}_{N_p/N_0}^*(q)$, then $s^{-1}H_{N_p}(W, q) = 0$.*

Proof. $s^{-1}H_{N_p}^*(W, q, Z_p) \rightarrow s^{-1}H_{N_p}^*({}_pW, q, Z_p)$ is an isomorphism. To see this note that each $x \in W - {}_pW$ has isotropy group $(N_p)_x$ which is finite or order prime to p by 1.2(ii). This means that s maps to zero in $H_{(N_p)_x}^*(q, Z_p)$; so $s^{-1}H_{(N_p)_x}^*(q, Z_p) = 0$. This implies $s^{-1}H_{N_p}^*(W, {}_pW, Z_p) = 0$. Since $H^*({}_pW, q, Z_p) = 0$ by (3.1), $H_{N_p}^*({}_pW, q, Z_p) = 0$. For the second statement, note that each $x \in W - {}_pW$ has isotropy group $(N_p)_x \in N_0$, $\tilde{H}_{N_p/N_0}^*(q) \rightarrow \tilde{H}_{N_0}^*(q)$ is zero and $H_{N_p}^*({}_pW, q) \rightarrow H_{N_0}^*({}_pW, q)$ is an isomorphism by (3.1).

COROLLARY 3.3. *Let G be connected and W satisfy the hypothesis of (3.1) and have its total mod p cohomology finite dimensional over Z_p . Then $H_G^i(W, q, Z_p) = 0$ for large i .*

Proof. By (2.1) and (2.2), $H_G^*(W, q, Z_p)$ is a subalgebra of $H_{N_p}^*(W, q, Z_p)$. Let

$s \in \tilde{H}_G(q, Z_p) \subset \tilde{H}_{N_p}(q, Z_p)$ be any multiplicative set. Then $s^{-1}H_{N_p}^*(W, q, Z_p) = 0$ (3.2); so $s^{-1}H_G^*(W, q, Z_p) = 0$. But $H_G^*(W, q, Z_p)$ is a finitely generated $H_G^*(q, Z_p)$ module by (2.3). The result follows from (2.5).

THEOREM 3.4. *Let G be a compact connected Lie group with $H_*(G)$ Z torsion free and W a G space with $q \in W^G \neq \emptyset$. Suppose that (i) for some integer m , $H^i(W, q, R) \cong H_{m-i+2}(W^r, q, R)$ for all i and every R , (ii) if $\lambda = [(m-g)/2] + 1$, $H_i(W, q) = 0$ for $i < \lambda$, (iii) $H_\lambda(W, q)$ is a Z torsion module if $m-g$ is odd and (iv) for each prime p and for each $K \in \mathcal{H}(N_p)$ $H^*(W^K, q, Z_p) = 0$. Then there is a filtration of $H_*(W, q)$ such that $E_0(H_*(W, q)) = H_*(G) \otimes H_*^G(W, q)$; moreover, $H_i^G(W, q) = 0$ for $i \neq \lambda$ and if $m-g$ is even $H_\lambda^G(W, q)$ is Z free and is Z torsion if $m-g$ is odd. In particular for $m-g$ even, $H_*(W, q)$ is a free $H_*(G)$ module and the hypothesis $H_*(G)$ is torsion free is superfluous.*

Proof. First note that $H_G^i(W, q, Z_p) = 0$ for large i (3.3). Let d be the largest i such that $H_G^i(W, q, Z_p) \neq 0$. The spectral sequence $H_G^*(W, q, Z_p) \otimes H^*(G, Z_p) \Rightarrow H^*(W, q, Z_p)$ has a non zero term in E_2 of bidegree (d, g) as $E_2^{d,g} = H_G^d(W, q, Z_p) \otimes H^g(G, Z_p)$. This term survives to E_∞ and shows $H^{g+d}(W, q, Z_p) \neq 0$. But then $H_{m-g-d+2}(W, q, Z_p) \neq 0$ so $m-g-d+2 \geq [(m-g)/2] + 1$ and $d \leq m-g - [(m-g)/2] + 1$. Also $H_G^i(W, q, Z_p) = 0$ for $i < [(m-g)/2] + 1$ since the same is true of $H^i(W, q, Z_p)$. Thus $H_G^i(W, q, Z_p) = 0$ for $i \neq \lambda$ if $m-g$ is even and for $i \neq \lambda, \lambda + 1$ if $m-g$ is odd. This shows that $H_G^i(W, q) = 0$ for $i \neq \lambda$ and $H_\lambda^G(W, q)$ is Z free if $m-g$ is even. If $m-g$ is odd $H_\lambda^{G+1}(W, q)$ is a Z torsion module and $H_G^i(W, q) = 0$ $i \neq \lambda + 1$. In either case the spectral sequence $H^*(G) \otimes H_G^*(W, q) \Rightarrow H^*(W, q)$ collapses implying the homology spectral sequences collapses giving $E_0(H_*(W, q)) = H_*(G) \otimes H_*^G(W, q)$ as an $H_*(G)$ module.

THEOREM 3.5. *Let G be connected and $H_*(G)$ be Z torsion free. Let $f: X \rightarrow Y$ a G normal map between oriented smooth closed G manifolds of dimension m . Suppose for each prime p for each $H \in \mathcal{H}(N_p)$, $K^*(f^H, Z_p) = 0$, $K_i(f) = 0$ for $i < [(m-g)/2] = \lambda$ and if $m-g$ is odd $K_\lambda(f)$ is a Z torsion module. Then there is a filtration of $K_*(f)$ such that $E_0K_*(f) = H_*(G) \otimes H_*^G(M_f, q)$; moreover, $H_i^G(M_f, q) = 0$ for $i \neq \lambda$ and if $m-g$ is even $K_\lambda(f) = H_{\lambda+1}^G(M_f, q)$ is Z torsion free and is Z torsion if $m-g$ is odd. In particular for $m-g$ even, $K_*(f)$ is a free $H_*(G)$ module and the hypothesis $H_*(G)$ is torsion free is superfluous.*

Proof. Since the degree of f^K (for each component of X^K) is a unit of Z_p [6], for each $H \in \mathcal{H}(N_p)$, $K^i(f^H, Z_p) = H^{i+1}(M_f^H, q, Z_p)$. Since $(M_f)^H = M_{f^H}$ (1.3), $H^*(M_f^H, q, Z_p) = 0$ for all p and all $H \in \mathcal{H}(N_p)$. Now apply (3.4) with $W = M_f$ noting $K^{m-i}(f) \cong K_i(f)$ and (1.9).

Remark 3.6. Certainly the hypothesis that $H_*(G)$ be torsion free can be removed from the hypothesis with only minor changes in the conclusion.

4. Localization in $H_{G_p}^*(q, Z_p)$ and homological dimension of $Z(G)$ modules

Throughout this section G is finite. Using [9], we show a relation between homological dimension of $Z(G)$ modules and localization in $H_{G_p}(q, Z_p)$. The first result is an easy consequence of the universal coefficient theorem and [9] (4.11):

THEOREM 4.1 [9]. *A finitely generated $Z(G)$ module M which is Z torsion free is projective iff for each prime p $M \otimes_Z Z_p$ is $Z_p(G)$ projective.*

This together with the results of [9] and a few elementary lemmas gives

THEOREM 4.2. *A finitely generated $Z(G)$ module M has homological dimension ≤ 1 if for each prime p , $H_{G_p}^i(q, M \otimes Z_p) = 0$ for large i . If in addition M is Z torsion free, then M is projective over $Z(G)$. (Moreover if $M \otimes Z_p$ is replaced by M , the condition is necessary and sufficient.)*

Using the fact that $H_{G_p}^*(q, M \otimes Z_p)$ is an $H_{G_p}^*(q, Z_p)$ module, we have this more convenient statement:

THEOREM 4.3. *Let M be a finitely generated $Z(G)$ module (which is Z free) then the homological dimension of M is ≤ 1 (≤ 0) if for each prime p and each multiplicative set $s \in \tilde{H}_{G_p}^*(q, Z_p)$, $s^{-1}H_{G_p}^*(q, M \otimes Z_p) = 0$. Moreover if Z_p is replaced by Z , the condition is necessary and sufficient for zero homological dimension.*

Proof. This is immediate from (4.2) and (2.5).

Our principle application occurs when M is a (graded) module arising from the cohomology of a G space. Say $M = H^*(W, q)$. The universal coefficient theorem $H^*(W, q, Z_p) = H^*(W, q) \otimes Z_p \oplus \text{Tor}(H^{*+1}(W, q), Z)$ clearly implies

COROLLARY 4.4. *Let W be a G space, with $q \in W^G$. If $H^i(W, q)$ is a finitely generated Λ module (with each $H^i(W, q)$ Z free) then the homological dimension of each $H^i(W, q)$ is ≤ 1 (≤ 0) if for each prime p and multiplicative set $s \in \tilde{H}_{G_p}^*(q, Z_p)$, $s^{-1}H_{G_p}^*(q, H^i(W, Z_p)) = 0$ or if for each $s \in \tilde{H}_{G_p}^*(q)$, $s^{-1}H_{G_p}^*(q, H^i(W, q)) = 0$.*

5. Defining $\chi(f)$

Throughout this section $f: X \rightarrow Y$ is a G map between G spaces whose total cohomology is finitely generated over Z . Then $H_{G_0}^i(M_f, q)$ is a finitely generated $Z(G/G_0)$ module for each i . We give conditions insuring that the definition

$$\chi(f) = \Sigma(-1)^i H_{G_0}^i(M_f, q) \in \tilde{K}_0(Z(G/G_0)) \tag{5.1}$$

makes sense. Clearly $\chi(f) = 0$ if f is a homotopy equivalence. It measures the deviation from being a homotopy equivalence.

THEOREM 5.2. *Suppose for each prime p and each $K \in \mathcal{H}(N_p)$ that $H^*(M_f^K, q, Z_p) = 0$. Then $H_{G_0}^i(M_f, q) = 0$ for i large. If also the spectral sequence $H_{G_p/G_0}^*(q, H_{G_0}^*(M_f, q, Z_p)) \Rightarrow H_{G_p}^*(M_f, q, Z_p)$ collapses for each p , then each $H_{G_0}^i(M_f, q)$ has homological dimension ≤ 1 (≤ 0 if $H_{G_0}^i(M_f, q)$ is Z free) over $Z(G/G_0)$ and $\chi(f)$ makes sense. Alternatively if the spectral sequence collapses with integral coefficients the same conclusion is valid.*

Proof. The total cohomology of M_f is a finitely generated Z module. Suppose $H^i(M_f, q) = 0$ for $i > N$. Then $H^i(M_f, q, Z_p) = 0$ for $i > N + 1$ for each prime p . By (3.3) $H_{G_0}^i(M_f, q, Z_p) = 0$ for i large. Examining the spectral sequence $H^*(G_0, Z_p) \otimes H_{G_0}^*(M_f, q, Z_p) \Rightarrow H^*(M_f, q, Z_p)$, we see that if d is the largest integer with $H_{G_0}^d(M_f, q, Z_p) \neq 0$ then $g + d \leq N + 1$. Since this holds for each p , $H_{G_0}^i(M_f, q) = 0$ for $i > N + 1 - g$.

Now suppose the spectral sequence in the statement of the theorem collapses. Then there is a filtration (of $Z(G_p/G_0)$ modules) of $H_{G_p}^*(M_f, q, Z_p)$ with $E_0 H_{G_p}^*(M_f, q, Z_p)$ equal to $H_{G_p/G_0}^*(q, H_{G_0}^*(M_f, q, Z_p))$. Let s be any multiplicative set in $\tilde{H}_{G_p/G_0}(q, Z_p)$. This gives rise to a multiplicative set again called s in $H_{G_p}^*(q, Z_p)$ under the obvious algebra homomorphism. By (3.2), $s^{-1} H_{G_p}^*(M_f, q, Z_p) = 0$. Since localization is exact, s^{-1} and E_0 commute; thus $s^{-1} H_{G_p/G_0}^*(q, H_{G_0}^*(M_f, q, Z_p)) = 0$. Apply (4.4) replacing G by G/G_0 and W by $M_f \times_{G_0} E/q \times_{G_0} E$. This shows $H_{G_0}^i(M_f, q)$ has homological dimension ≤ 1 over $Z(G/G_0)$.

Remark 5.3. The spectral sequence of 5.2 certainly collapses if $H_{G_0}^i(M_f, q, Z_p) = 0$ for all but one value of i . This is a frequent situation of application. See e.g. (3.5).

THEOREM 5.4. *Suppose G is a finite group and there is a point $y \in Y$ with $G_y = 1$. Then $\chi(f) = \chi({}^s f)$ provided both are defined.*

Proof. G operates freely on $M_f - {}^sM_f$ which is $M_f - M_{s_f}$ by (1.5). Thus the cellular cochain complex $C^*(M_f, M_{s_f}) = C^*$ is a free Λ module. Clearly $\chi(C^*) = \Sigma(-1)^i C^i$ is zero in $\tilde{K}_0(\Lambda)$.

The exact sequence of cochain complexes $0 \rightarrow C^*(M_f, M_{s_f}) \rightarrow C^*(M_f, q) \rightarrow C^*(M_{s_f}, q) \rightarrow 0$ gives rise to an exact triangle

$$\begin{array}{ccc} H^*(M_f, M_{s_f}) & \longrightarrow & H^*(M_f, q) \\ & \swarrow & \searrow \\ & H^*(M_{s_f}, q) & \end{array}$$

which implies that $H^i(M_f, M_{s_f})$ has finite homological dimension over $Z(G)$ so $\chi(f, {}^s f) = \Sigma(-1)^i H^i(M_f, M_{s_f}) = \chi(C^*) = 0$. But $\chi(f) = \chi({}^s f) + \chi(f, {}^s f)$.

LEMMA 5.5. *If G is a p group and the conditions of 5.2 are satisfied, $\chi({}^s f)$ is defined.*

Proof. Apply (3.1) with $W = M_f$. Then ${}_p W = M_{s_f}$ and $H^*(M_{s_f}, q, Z_p) = 0$; so $H^*(M_{s_f}, q)$ is a Z torsion module with no p torsion and for each i , $H^i(M_{s_f}, q)$ has homological dimension ≤ 1 over $Z(G)$ by (4.3).

6. $K_*(f)$ as an $H_*(G)$ module

We are now prepared to discuss the structure of $K_*(f)$ as an $H_*(G)$ module. The homology algebra $H_*(G)$ is the “twisted” tensor product $H_*(G_0) \otimes_t Z(G/G_0)$. In fact $H_*(G_0)$ is a $Z(G/G_0)$ module. $x^g = \bar{g}^{-1} x \bar{g}$ for $x \in H_*(G_0)$, $g \in G/G_0$ and $\bar{g} \in G$ representing g . The multiplication in the twisted tensor product is given by $x \otimes w \cdot x' \otimes w' = x \cdot x' \otimes ww'$ for $x, x' \in H_*(G_0)$.

THEOREM 6.1. *Let $H_*(G_0)$ be Z torsion free and $f: X \rightarrow Y$ be a G normal map between smooth closed oriented G manifolds of dimension m . Suppose for each prime p and for each $H \in \mathcal{H}(N_p)$ that $K^*(f^H, Z_p) = 0$, $K_i(f) = 0$ for $i < [(m - g)/2] = \lambda$ and if $m - g$ is odd $K_\lambda(f)$ is a Z torsion module. Then there is a filtration of $K_*(f)$ by $H_*(G_0)$ modules such that $E_0 K_*(f) = H_*(G_0) \otimes K_\lambda(f)$ and $K_\lambda(f)$ is a projective $Z(G/G_0)$ module if $m - g$ is even and has homological dimension ≤ 1 if $m - g$ is odd; moreover, when $m - g$ is even, the hypothesis on $H_*(G_0)$ is superfluous, $\chi(f) = \pm [K_*(f)^*]$ and $K_*(f)$ is a stably free $H_*(G)$ module iff $\chi(f) = 0$.*

Proof. The first conclusion is a restatement of (3.5) noting $H_\lambda^{G_0}(M_f, q) = K_\lambda(f)$. For the second, note that $H_{G_0}^i(M_f, q) = 0$ unless $i = \lambda$ when $m - g$ is even or

$i = \lambda + 1$ when $m - g$ is odd by the universal coefficient theorem. Thus the spectral sequence of (5.2) collapses and $H_{G_0}^i(M_f, q) = K^i(f)$ has homological dimension ≤ 1 for $i = \lambda$ ($m - g$ even) or $i = \lambda + 1$ ($m - g$ odd). In the first case $K^\lambda(f)$ is Z torsion free since $K_i(f) = 0$ for $i < \lambda$; so in this case $K^\lambda(f)$ is a projective $Z(G/G_0)$ module. When $m - g$ is odd, $K_\lambda(f) = \text{Ext}_Z^1(K^{\lambda+1}(f), Z)$; so it too has homological dimension ≤ 1 .

Since $H_{G_0}^i(M_f, q) = 0$ for $i \neq \lambda$ or $\lambda + 1$ depending on $m - g$, $\chi(f) = \pm[K^\lambda(f)]$ or $\pm[K^{\lambda+1}(f)]$. Moreover, in the first case $K_\lambda(f) = \text{Hom}_Z(K^\lambda(f), Z) = K^\lambda(f)^*$ by the universal coefficient theorem; so $K_\lambda(f)$ is also $Z(G/G_0)$ projective. If it is free over $Z(G/G_0)$, then $K_*(f)$ is free over $H_*(G)$.

7. Application to the G normal cobordism problem

Let $\gamma \in H_*(G_0)$ denote the orientation class and define a homomorphism $w_1: G/G_0 \rightarrow Z_2 = \{\pm 1\}$ by

$$\gamma^g = w_1(g)\gamma \quad \text{for } g \in G/G_0 \tag{7.1}$$

Let $[X] \in H_*(X)$ denote the orientation class for X and define $w_2: G/G_0 \rightarrow Z_2$ by

$$g[X] = w_2(g)[X] \tag{7.2}$$

When the hypothesis of (6.1) hold and $m - g$ is even, we can define an integral valued non singular bilinear form $\langle \rangle$ on $K_\lambda(f)$ using the intersection pairing \circ in $H_*(X)$;

$$\langle x, y \rangle = x \circ (\gamma \cdot y) \in Z; \quad x, y \in K_\lambda(f) \tag{7.3}$$

Then for $g \in G/G_0$, $\langle gx, gy \rangle = w(g)\langle x, y \rangle$ where $w(g) = w_1(g)w_2(g)$. This follows from the fact that $\gamma \cdot (gy) = (\gamma g) \cdot y = (g\gamma^g) \cdot y = g(\gamma^g \cdot y)$ and $g\alpha \circ g\beta = w_2(g)(\alpha \circ \beta)$. The fact that $\langle \rangle$ is non singular i.e. induces an isomorphism $K_\lambda(f) \cong \text{Hom}_Z(K_\lambda(f), Z)$ of $Z(G/G_0)$ modules follows from the fact that the intersection pairing $K_\lambda(f) \otimes K_{\lambda+g}(f) \rightarrow Z$ is non singular and the isomorphism of $H_*(G_0)$ modules of $H_*(G_0) \otimes K_\lambda(f)$ and $K_*(f)$ is defined by $\alpha \otimes \beta \rightarrow \alpha \cdot \beta$ i.e. by the structure of $K_*(f)$ as an $H_*(G_0)$ module. Thus we have

COROLLARY 7.4. *If the hypothesis of (6.1) hold, $K_\lambda(f)$ is a projective $Z(G/G_0)$ module supporting a Z valued non singular bilinear form $\langle \rangle$ satisfying $\langle gx, gy \rangle = w(g)\langle x, y \rangle$ for $g \in G/G_0$, $x, y \in K_\lambda(f)$ and $w(g) = w_1(g)w_2(g)$.*

Of course we can also view $\langle \rangle$ as a bilinear form (over Λ) on $K_\lambda(f)$ with values

in Λ by setting

$$(x, y) = \sum_{g \in G/G_0} \langle x, g^{-1}y \rangle g$$

This is to conform to the standard notation for this situation when $G = G/G_0$ acts freely on Y [13]. Under certain hypothesis on sX e.g. $\dim {}^sX/G < \frac{1}{2} \dim X/G$, it is possible to define a self intersection form $\mu : K_\lambda(f) \rightarrow \Lambda/I$ where I is the subgroup of Λ consisting of $\nu + (-1)^{\lambda-1}\bar{\nu}$ for $\nu \in \Lambda$ and $\nu \rightarrow \bar{\nu}$ the automorphism of Λ defined by $\sum \overline{a_g}g = \sum_{g \in G} w(g)a_gg^{-1}$.

When $\chi(f) = 0$, so $K_\lambda(f)$ is Λ free,

$$\sigma(f) = (K_\lambda(f), (,), \mu) \in L_{2\lambda}(G/G_0, w) \tag{7.5}$$

represents an element of the group $L_{2\lambda}(G/G_0, w)$ of Wall [13]. Under suitable hypothesis e.g. trivial principle isotropy group, $\pi_1(Y) = 0$ and $\dim {}^sX/G < \frac{1}{2} \dim X/G$, $\sigma(f)$ is the only obstruction to finding a G normal cobordism between (X, f) and (X', f') where $f' : X' \rightarrow Y$ is a homotopy equivalence. Thus $\chi(f)$ is a primary obstruction and $\sigma(f)$ a secondary obstruction to making f a homotopy equivalence. Of course this is all relative to the hypothesis of (6.1).

To achieve the full obstruction theory for the G normal cobordism problem (1.8), we first generalize $\chi(f)$ and $\sigma(f)$ slightly by introducing $\chi(f, Z_{(p)}) \in \tilde{K}_0(Z_{(p)}(G/G_0))$ and $\sigma(f, Z_{(p)}) \in L_{2\lambda}(Z_{(p)}(G/G_0), w)$ where $Z_{(p)}$ is Z localized at p . This is to be able to treat maps whose degree is a unit in $Z_{(p)}$. For each p , partially order the conjugacy classes of groups in $\mathcal{H}(N_p)$ by setting $K \leq H$ if K contains a conjugate of H . Roughly each conjugacy class K in $\mathcal{H}(N_p)$ contributes two obstructions $\chi_K(f) = \chi(f^K, Z_{(p)})$ and $\sigma_K(f) = \sigma(f^K, Z_{(p)})$ as K/K_0 is a p group. In fact $\chi_K(f)$ is defined only if $\chi_L(f) = 0$ and $\sigma_L(f) = 0$ for $L < K$ and corresponds to replacing G by $N(K)/K$ and f by $f^K : X^K \rightarrow Y^K$ in our preceding discussion. Here $N(K)$ is the normalizer of K and $\chi_L(f) \in \tilde{K}_0(Z_{(p)}(L''))$, $\sigma_L(f) \in L_\alpha(Z_{(p)}(L''), w_L)$ where L/L_0 is a p group, $L' = N(L)/L$ and $L'' = L'/L'_0$.

This very brief discussion illustrates the obstruction theory for dealing with the hypothesis of (6.1) and shows how the Smith theory conditions show up in a constructive manner for handling the G normal cobordism problem.

For a complete discussion of the application of the obstruction theory for $G = S^1$ see [6]. There all the obstructions $\chi_L(f)$ vanish because L'' is 1.

8. The homomorphism $\sigma_G : Z_n^* \rightarrow \tilde{K}_0(Z(G))$

As a consequence of (4.3), we see that if the order of G is n and q is prime to

n , Z_q viewed as a $Z(G)$ module has homological dimension ≤ 1 ; so represents an element $[Z_q] \in \tilde{K}_0(Z(G))$. Swan showed [11] that this gives rise to a homomorphism $\sigma_G : Z_n^* \rightarrow \tilde{K}_0(Z(G))$ from the multiplicative group of units of the ring Z_n to $\tilde{K}_0(Z(G))$. He proved the

THEOREM 8.1 [11]. σ_G is zero if G is cyclic.

Since this is important for our study, we give a very simple geometric proof.

Proof. Let $G' = S^1$ and $G = Z_p \subset S^1$ be the cyclic group of order p (not necessarily a prime). Let N and M be the complex two dimensional G' modules defined by

- (i) $N : t(z_0, z_1) = (t^p z_0, t^q z_1), \quad z = (z_0, z_1) \in N$
- (ii) $M : t(z_0, z_1) = (tz_0, t^{pq} z_1), \quad z = (z_0, z_1) \in M$

Here $t \in S^1 \subset C$ and q is an integer prime to p . Choose integers a and b so that $-ap + bq = 1$. Define a G' map $w : N \rightarrow M$ by

$$\omega(z_0, z_1) = (\bar{z}_0^a z_1^b, z_0^q + z_1^p) \tag{6.2}$$

This gives rise to a G' map from the unit sphere of N to the unit sphere of $M : f : S(N) \rightarrow S(M)$ by $f(z) = \omega(z) / \|\omega(z)\|$.

Restrict the action to G and set $X = S(N)$, $Y = S(M)$. Since the degree of f is 1 [8], [7], f is a homotopy equivalence so $\chi(f)$ is zero. Note that G acts semi-freely on X and Y with $X^G = \{(z_0, 0) \mid |z_0| = 1\}$ and $Y^G = \{(0, z_1) \mid |z_1| = 1\}$; moreover, $f^G(z_0, 0) = (0, z_0^q)$ is a map of degree q . Clearly $H^2(M_{f^G}) = Z_q$ and $H^i(M_f, q) = 0$ for $i \neq 2$. Since $M_{f^G} = (M_f)^G$, G acts trivially on Z_q . Since G acts semi-freely on X and Y , ${}^s f = f^G$. Thus $\chi({}^s f) = \chi(f^G) = [Z_q] = \sigma_G(q)$. Since $\chi(f) = \chi({}^s f)$ by (5.4), $0 = \chi(f) = \chi({}^s f) = \sigma_G(q)$.

COROLLARY 8.3. *Let G be an arbitrary finite group of order n acting semi-freely on X and Y and $f : X \rightarrow Y$ a G map. Suppose each $H^i(M_{f^G}, q, Z_n) = 0$. Then $\chi({}^s f)$ is defined. If $\chi(f)$ is also defined $\chi(f) \in \text{image } \sigma_G$.*

Proof. Each $H^i(M_{f^G}, q)$ is a Z torsion module of order prime to n and hence has homological dimension ≤ 1 over $Z(G)$. Since G acts trivially on $H^i(M_{f^G}, q)$, the class it represents in $\tilde{K}_0(Z(G))$ is in the image of σ_G . Since G acts semi-freely on X and Y , $f^G = {}^s f$; so $\chi(f) = \chi(f^G) \in \text{image } \sigma_G$.

COROLLARY 8.4. *Suppose G is Z_p with p prime. Suppose also the hypothesis of (5.2). Then $\chi(f) = 0$.*

Proof. The hypothesis of (5.2) guarantee $H^*(M_{f^Q}, q, Z_p) = 0$. The result now follows from (8.3) and (8.1).

9. An example with $\chi(f) \neq 0$

Let $G = Q$ be the quaternion group; so $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbf{H}$ where \mathbf{H} is the quaternion skew field. Viewing \mathbf{H} as a left complex vector space, it is a complex Q module with Q acting by right multiplication. Note that the function $h: \mathbf{H} \rightarrow C$ defined by $h(x + yj) = x^4 + y^4$ is Q invariant if Q acts trivially on C and x and y are the complex coordinates of $x + yj \in \mathbf{H}$. This shows that for each integer λ , the variety

$$V_\lambda = \{(z_0, z_1, z_2, x, y) \in C^3 \times \mathbf{H} \mid h_\lambda = 0\}$$

$$h_\lambda(z_0, z_1, z_2, x, y) = z_0^\lambda + z_1^2 + z_2^2 + x^4 + y^4$$

is Q invariant. Here Q acts on $C^3 \times \mathbf{H}$ by $(u, v)q = (u, vq)$ for $q \in Q$, $\mu \in C^3$ and $v \in \mathbf{H}$. Set

$$L_\lambda = V_\lambda \cap S(C^3 \times \mathbf{H})$$

where $S(C^3 \times \mathbf{H})$ is the unit sphere in $C^3 \times \mathbf{H}$. Clearly L_λ is Q invariant.

The subvariety $W_\lambda = \{(z_0, z_1, z_2, x, y) \in L_\lambda \mid x = y = 0\}$ is the fixed point set L_λ^Q and its homology is given by

$$H_1(W_\lambda) = Z_\lambda, \quad H_i(W_\lambda) = Z, \quad i = 0, 3$$

and $H_2(W_\lambda) = 0$. See [3], p. 275. The action of Q on L_λ is semi-free so the singular set ${}^sL_\lambda$ is $L_\lambda^Q = W_\lambda$.

Let λ be an odd integer and choose integers a and b such that $-2a + \lambda b = 1$. Define a Q map $f: L_\lambda \rightarrow S(C^2 \times \mathbf{H})$ by

$$f(z_0, z_1, z_2, x, y) = \frac{(\bar{z}_0^a \cdot z_1^b, z_2, x, y)}{\|(\bar{z}_0^a z_1^b, z_2, x, y)\|}$$

Then

- (i) Both f and f^Q have degree 1
- (ii) $f_*^Q: H_*(L_\lambda^Q, Z_2) \rightarrow H_*(S(C^2 \times \mathbf{H})^Q, Z_2)$ is an isomorphism
- (iii) $H^i(M_f, q) = 0$ for $i \neq 5$ and $H^5(M_f, q) \cong H^4(L_\lambda)$ is a Z torsion module of

odd order [3], p. 279.

(iv) $H^i(M_{f^\alpha}, q) = 0$ for $i \neq 3$ and $H^3(M_{f^\alpha}, q) = H^2(W_\lambda) = Z_\lambda$

These facts insure that both $\chi(f)$ and $\chi(f^Q)$ are defined and

THEOREM 9.1. $\chi(f) = \chi(f^Q) = \sigma_Q(\lambda)$. For $\lambda = 3$, $\chi(f) \neq 0$.

Proof. Since the actions are semi-free, the first equality follows from (5.4) while the second follows from (iv). The fact that $\sigma_Q(3) \neq 0$, is a result of Swan [11].

Remark 9.2. The map $f: L_\lambda \rightarrow S(C^2 \times \mathbf{H})$ is a Q normal map. The Q normal bundle of $L_\lambda \subset C^3 \times \mathbf{H}$ is $L_\lambda \times R^3$ with trivial Q action on R^3 .

One might suspect that the invariant $\chi(f)$ is completely determined by the Sylow subgroups, a phenomenon which occurs for example for the cohomology of a group. This is not the case. To see this let $J_\lambda \subset S(C^3 \times \mathbf{H})$ be the subvariety $z_0^\lambda + z_1^2 + z_2^2 + z_3^{12} + z_4^{12} = 0$. The group $G = Z_3 \times Q$ acts semi-freely on J_λ . The action is induced by the action of $Z_3 \times Q$ on \mathbf{H} defined by viewing Z_3 as the multiplicative subgroup of C of 3rd roots of unity and allowing Z_3 to act via left multiplication on \mathbf{H} and Q via right multiplication. The same map f as above gives a G normal map $f: J_\lambda \rightarrow S(C^2 \times \mathbf{H})$ and again $\chi(f) = \pm[Z_\lambda] = \sigma_G(\lambda) \in \tilde{K}_0(Z(G))$. The order of G is 24 and $\sigma_G(17) \neq 0$ but $\sigma_{Z_3}(17) = 0$ and $\sigma_Q(17) = 0$. See [11].

Remark 9.3. The Q variety L_λ has higher dimensional analogs generated by the functions $z_0^\lambda + z_1^2 + \dots + z_{2k}^2 + x_1^4 + \dots + x_{2l}^4$ as k and l vary.

Remark 9.4. The fact that $\chi(f) = \chi(f^Q) = \sigma_G(3)$ when $\lambda = 3$, shows that (L_λ, f) is never Q normally cobordant rel L_λ^Q to (X', f') with f' a homotopy equivalence even though $f_*^Q: H_*(L^Q, Z_2) \rightarrow H_*(S(C^2 \times \mathbf{H})^Q, Z_2)$ is an isomorphism.

10. Application to Equivariant Homotopy Groups of Spheres

If Σ_i $i = 0, 1$ are homotopy spheres supporting an action of G and $f: \Sigma_0 \rightarrow \Sigma_1$ is a G map of degree 1, then $f^H: \Sigma_0^H \rightarrow \Sigma_1^H$ is a map whose degree is non zero mod p for every p group H in G (Smith theory). In particular this means that if G acts semi-freely on Σ_i (i.e. the only isotropy groups are G and 1) then $\deg f^G$ is a unit in Z_n where $n = \text{order } G$. For cyclic groups, $\deg f^G$ can be an arbitrary element of Z_n^* . See e.g. the example of (8.1). In general there are additional restrictions, namely

PROPOSITION 10.1. Let $f: \Sigma_0 \rightarrow \Sigma_1$ be a degree 1 G map where G acts

semi-freely on Σ_i and suppose Σ_i^G is a homotopy sphere for $i=0, 1$. Then $\sigma_G(\deg f^G) = 0$ in $\tilde{K}_0(Z(G))$.

Proof. $\sigma_G(\deg f^G) = \chi(f^G) = \chi(f) = 0$ because f is a homotopy equivalence.

For example if $G = Q$ is the quaternion group of section 8, then $\deg f^G \neq \pm 3(8)$.

Proposition 10.1 is an example of the relation between the homological invariants of G manifolds and G maps. For another example, if Σ_i $i=0, 1$ are rational homotopy spheres supporting an S^1 action with $\Sigma_i^{S^1} = \phi$ and $f: \Sigma_0 \rightarrow \Sigma_1$ is an S^1 map, then $\deg f$ is uniquely determined by the S^1 manifolds Σ_i .

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