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Extrinsic Bounds on λ_1 of Δ on a Compact Manifold

DAVID D. BLEECKER and JOEL L. WEINER

1. Introduction

Although the Laplacian of a Riemannian manifold M is an intrinsic object (as well as the first nonzero eigenvalue λ_1), upper bounds on λ_1 may be computed in terms of extrinsic quantities (e.g. principal curvatures) of M relative to some isometric embedding of M into some euclidean space E^m . In order to convey some idea of the results we obtained, the following is a special case of Theorem I: For a compact orientable surface S immersed in E^3 , λ_1 is bounded above by the average of the sum of the squares of the principal curvatures over S. Equality is achieved only in the case of a constant curvature sphere. Theorem I actually applies to manifolds with arbitrary dimension immersed with arbitrary codimension. A somewhat sharper result is found in Theorem II with the additional assumption that the mean curvature vector is parallel.

The primary means of obtaining these results is the minimum principle [1, p. 186] for λ_1 , namely $\lambda_1 = \inf \{ \int_M |df|^2 / \int_M f^2 : \int_M f = 0, f \in C^1(M) \}$, and the infimum is achieved only for f such that $\Delta f = \lambda_1 f$. To obtain interesting results, we choose f such that df contains some geometric information, $\int_M f = 0$, and then state $\lambda_1 \leq \int_M |df|^2 / \int_M f^2$. Actually we consider a parameterized family of such f say f_a , $a \in A$, where A is a manifold endowed with a measure and integrate both sides of $\lambda_1 \int_M f_a^2 \leq \int_M |df_a|^2$ over A to eliminate the parameter.

2. Definitions and Statements of Main Results

In general we follow the notation of [3]. Let $x:M\to \overline{M}$ be a isometric immersion and let \overline{V} and \overline{V} be the covarient differentiations on M and \overline{M} respectively. For vector fields X and Y tangent to M, we define the vector field h(X,Y) by

 $\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$ [equation of Gauss].

The normal bundle valued (symmetric) 2-tensor h on M is called the second fundamental form. For a normal vector field N on M we set

$$\bar{\nabla}_X N = -A_N(X) + D_X N$$
 [equation of Weingarten],

 $-A_N(X)$ and D_XN being the tangential and normal parts of $\bar{\nabla}_XN$. We have

$$\langle A_N(X), Y \rangle = \langle h(X, Y), N \rangle.$$

The mean curvature vector field η is defined by $\eta(p) = 1/n$ trace $h = 1/n\sum_{i=1}^{n}h(e_i,e_i)=1/n\sum_{\alpha=n+1}^{m}(\operatorname{trace} A_{\alpha})e_{\alpha}$ where $A_{\alpha} \equiv A_{e_{\alpha}}$ and (e_i) is an o.n. frame of T_pM and (e_{α}) is an o.n. frame of T_pM^{\perp} . We shall adopt the convention that latin indices run from 1 to n and greek from n+1 to m. Of course, we set $|A_{\alpha}|^2 = \sum_i \langle A_{\alpha}(e_i), A_{\alpha}(e_i) \rangle$ and $|A|^2 \equiv \sum_{\alpha} |A_{\alpha}|^2$.

THEOREM I. Let M be a compact orientable n-manifold isometrically immersed in euclidean space E^m . Then $\lambda_1 \leq [\operatorname{vol}(M)]^{-1} \int_M |A|^2$ with equality only in the case where M is a constant curvature sphere isometrically embedded in an n+1 dimensional subspace of E^m .

THEOREM II. With M as above and assuming η is parallel in the normal bundle, $\lambda_1 \leq [\operatorname{vol}(M)]^{-1} \int_M |A_e|^2$, where $e = \eta/H$ and $H \equiv |\eta|$.

3. Proof of Inequality in Theorem I

Let $x: M \to E^m$ be an isometric immersion. Now $dx = (dx_1)\bar{e}_1 + \cdots + (dx_m)\bar{e}_m$ where $(\bar{e}_1, \ldots, \bar{e}_m)$ form a standard o.n. basis of E^m and $x(p) = x_1(p)\bar{e}_1 + \cdots + x_m(p)\bar{e}_m$, $p \in M$. Thus the dx_i 's are 1-forms on M. We let $(dx_i)\bar{e}_i \wedge (dx_j)\bar{e}_j = (dx_i \wedge dx_j)(\bar{e}_i \wedge \bar{e}_j)$ and extend this product in the natural way to $\Lambda^*(M) \otimes \Lambda^*(E^m)$ (i.e. by linearity and $(\alpha \otimes e) \wedge (\beta \otimes f) = (\alpha \wedge \beta) \otimes (e \wedge f)$). In this context $1/n! dx \wedge \cdots \wedge dx$ (n times) = (dV)f for some $f \in \Lambda^\circ(M) \otimes \Lambda^n(E^m)$ where dV is the volume element of M. For an oriented n-plane π of E^m with o.n. basis (a_1, \ldots, a_n) we define the function f_π on M by $f_\pi(p) = (a_1 \wedge \cdots \wedge a_n) \cdot f(p)$ or perhaps more succinctly by $f_\pi dV = (a_1 \wedge \cdots \wedge a_n) \cdot 1/n! dx \wedge \cdots \wedge dx$ (n times). We note that $f_\pi dV = d[(a_1 \wedge \cdots \wedge a_n) \cdot 1/n! x \wedge dx \wedge \cdots \wedge dx]$ is exact, whence $\int_M f_\pi = 0$.

To derive a formula for f_{π} more amenable to calculating df_{π} , we note that we may write (locally) $dx = \sum w_i e_i$ where e_i is an o.n. frame field of M defined on a neighborhood of some point of M (here we identify vectors tangent to M with

their images under x_*) and the w_i are the one-forms on M dual to the e_i . This is because dx, considered as an E^m -valued one-form on M, is the identity map under the above mentioned identification. Thus $1/n! dx \wedge \cdots \wedge dx = (w_1 \wedge \cdots \wedge w_n)(e_1 \wedge \cdots \wedge e_n) = dV(e_1 \wedge \cdots \wedge e_n)$. Thus $f = e_1 \wedge \cdots \wedge e_n$ and $f_{\pi} = (a_1 \wedge \cdots \wedge a_n) \cdot (e_1 \wedge \cdots \wedge e_n) = (a_{n+1} \wedge \cdots \wedge a_{n+p}) \cdot (e_{n+1} \wedge \cdots \wedge e_{n+p})$ where n + p = m and (a_1, \ldots, a_m) and (e_1, \ldots, e_m) are oriented o.n. bases of E^m . To avoid writing $a_{n+1} \wedge \cdots \wedge a_{n+p}$ repeatedly we replace this expression by \mathcal{A} . Now $df_{\pi} = d(\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{n+p})$

$$= \sum_{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge de_{\alpha} \wedge \cdots \wedge e_{n+p}$$

$$= \sum_{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge \sum_{i} w_{\alpha i} e_{i} \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha,i} h_{ij}^{\alpha} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\alpha} \wedge \cdots \wedge e_{n+p}) w_{j}$$

where $[(de_{\alpha})(X)]^{\top} = -A_{e_{\alpha}}(X) = \sum_{i} w_{\alpha i}(X)e_{i} = -\sum_{i} h_{ij}^{\alpha}w_{j}(X)e_{i}$ for X tangent to M. Note $de_{\alpha}(X) \perp e_{\alpha}$. Here the super α indicates the α -th slot in the product.

$$|df_{\pi}|^{2} = \sum_{j} \left(\sum_{\alpha,i} -h_{ij}^{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\alpha} \wedge \cdots \wedge e_{n+p} \right)^{2}$$

$$= \sum_{j,i,l,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\alpha} \wedge \cdots \wedge e_{n+p})$$

$$\times (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\beta} \wedge \cdots \wedge e_{n+p}).$$

By the minimum principle for λ_1 we have $\lambda_1 \int_M f_\pi^2 \le \int_M |df_\pi|^2$. However, the unwieldy expression for $|df_\pi|^2$ leaves much to be desired. Hence we integrate both sides of the inequality with respect to $\pi \in G(p, n) =$ oriented p-planes in n + p-space. To this end we apply Fubini's Theorem obtaining $\lambda_1 \int_M \int_G f_\pi^2 d\pi dV \le \int_M \int_G |df_\pi|^2 d\pi dV$. In integrating $|df_\pi|^2$ with respect to π we consider h_{ij}^α , h_{ij}^β and the e's to be fixed and $\mathcal{A} = a_{n+1} \wedge \cdots \wedge a_{n+p}$ to vary. We need only consider

$$\int_{G} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\alpha} \wedge \cdots \wedge e_{n+p}) (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\beta} \wedge \cdots \wedge e_{n+p}) d\pi.$$

Upon reflection (pardon pun) we see that the above is 0 unless i = l and $\alpha = \beta$, for otherwise $\{e_{n+1}, \ldots, e_i^{\alpha}, \ldots, e_{n+p}\} \neq \{e_{n+1}, \ldots, e_l^{\beta}, \ldots, e_{n+p}\}$ and reflection of E^{n+p} , in a hyperplane perpendicular to a vector in one set but not in the other,

induces an isometry of G under which the integrand changes sign. Thus the above integral is

$$\delta_{il} \, \delta_{\alpha\beta} \int_{G} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{i}^{\alpha} \wedge \cdots \wedge e_{n+p})^{2} \, d\pi = \delta_{il} \, \delta_{\alpha\beta} \int_{G} f_{\pi}^{2} \, d\pi$$

and $\int_G |df_{\pi}|^2 d\pi = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2 \int_G f_{\pi}^2 d\pi$. Hence

$$\lambda_{1} \int_{G} f_{\pi}^{2} d\pi \int_{M} dV = \lambda_{1} \int_{M} \int_{G} f_{\pi}^{2} d\pi dV \le \int_{M} \int_{G} |df_{\pi}|^{2} d\pi dV$$
$$= \int_{G} f_{\pi}^{2} d\pi \int_{M} \sum_{\alpha, i, j} (h_{ij}^{\alpha})^{2} dV.$$

Dividing by $\int_G f_{\pi}^2 d\pi$ gives us the desired result

$$\lambda_1 \operatorname{vol}(M) \le \int_M \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2 dV = \int_M |A|^2.$$

4. Equality in Theorem I

In this section we show equality holds in Theorem I only in the case where M is a constant curvature sphere isometrically embedded in an n+1 dimensional subspace, E^{n+1} , of E^{n+p} .

If we assume equality holds in Theorem I, then for almost all $\pi \in G(n, p)$

$$\lambda_1 = \frac{\int_M |d(\mathcal{A} \cdot \mathcal{E})|^2 dV}{\int_M |\mathcal{A} \cdot \mathcal{E}|^2 dV}$$

where $\mathcal{A} = \pi^{\perp}$ and $\mathcal{C} = e_{n+1} \wedge \cdots \wedge e_{n+p}$. Thus, for almost all $\pi \in G(n, p)$, the minimum principle implies that $\mathcal{A} \cdot \Delta \mathcal{C} = \Delta(\mathcal{A} \cdot \mathcal{C}) = \lambda_1(\mathcal{A} \cdot \mathcal{C}) = \mathcal{A} \cdot (\lambda_1 \mathcal{C})$; hence $\Delta \mathcal{C} = \lambda_1 \mathcal{C}$. We suppose $\lambda_1 = n$, taking a homothetic transformation of E^{n+p} if necessary.

We now compute $\Delta \mathcal{E}$ at a point $m \in M$. Let $e_1, e_2, \ldots, e_{n+p}$ be a frame field in a neighborhood of x(m) such that e_1, \ldots, e_n are tangent to M and hence e_{n+1}, \ldots, e_{n+p} are normal to M. Suppose that e_1, \ldots, e_n are parallel in TM at m, and e_{n+1}, \ldots, e_{n+p} are parallel in TM^{\perp} at m. In fact, we may assume $D_{e_i}D_{e_i}e_{\alpha}=0$,

 $1 \le i \le n$, $n+1 \le \alpha \le n+p$, by supposing that e_i and e_{α} are obtained from $e_i(m)$ and $e_{\alpha}(m)$ by parallel translation along geodesics through m in TM and TM^{\perp} , respectively.

Now, $\Delta \mathscr{E} = -\sum_{i=1}^{n} e_i e_i \mathscr{E}$. In a neighborhood of m, we have

$$e_{i} \mathscr{E} = \sum_{\alpha=n+1}^{n+p} e_{n+1} \wedge \cdots \wedge de_{\alpha}(e_{i}) \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha} e_{n+1} \wedge \cdots \wedge A_{\alpha}(e_{i}) \wedge \cdots \wedge e_{n+p} + \sum_{\alpha} e_{n+1} \wedge \cdots \wedge D_{e_{i}}(e_{\alpha}) \wedge \cdots \wedge e_{n+p}$$

using Weingarten's equation; we have set $A_{e_{\alpha}} = A_{\alpha}$. At m,

$$\begin{split} e_{i}e_{i}\mathscr{E} &= \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_{\alpha}(e_{i}) \wedge \cdots \wedge A_{\beta}(e_{i}) \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_{\alpha}(e_{i}) \wedge \cdots \wedge D_{e_{i}}e_{\beta} \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla_{e_{i}}A_{\alpha}(e_{i}) \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha} e_{n+1} \wedge \cdots \wedge h(e_{i}, A_{\alpha}(e_{i})) \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge D_{e_{i}}e_{\alpha} \wedge \cdots \wedge A_{\beta}(e_{i}) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge D_{e_{i}}e_{\alpha} \wedge \cdots \wedge D_{e_{i}}e_{\beta} \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha} e_{n+1} \wedge \cdots \wedge A_{D_{e_{i}}e_{\alpha}}(e_{i}) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_{\alpha} e_{n+1} \wedge \cdots \wedge D_{e_{i}}D_{e_{i}}e_{\alpha} \wedge \cdots \wedge e_{n+p}. \end{split}$$

In the preceding equation, the third and fourth terms arise from Gauss' equation. Also, note that the second, fifth, sixth, seventh and eighth terms vanish since $D_{e_i}e_{\alpha} = 0$ and $D_{e_i}D_{e_i}e_{\alpha} = 0$ at m. Thus

$$\Delta \mathscr{E} = -\sum_{i} \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_{\alpha}(e_{i}) \wedge \cdots \wedge A_{\beta}(e_{i}) \wedge \cdots \wedge e_{n+p}$$

$$+ \sum_{i} \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla_{e_{i}} A_{\alpha}(e_{i}) \wedge \cdots \wedge e_{n+p}$$

$$+ \sum_{i} \sum_{\alpha} e_{n+1} \wedge \cdots \wedge h(e_{i}, A_{\alpha}(e_{i})) \wedge \cdots \wedge e_{n+p}.$$

We now consider each term on the right-hand side of the preceding equation separately.

$$-\sum_{i} \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_{\alpha}(e_{i}) \wedge \cdots \wedge A_{\beta}(e_{i}) \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha \neq \beta} \sum_{i,j,k} e_{n+1} \wedge \cdots \wedge h_{ij}^{\alpha} e_{j} \wedge \cdots \wedge h_{ik}^{\beta} e_{k} \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha \neq \beta} \sum_{j < k} \left[\sum_{i} h_{ij}^{\alpha} h_{ik}^{\beta} - h_{ik}^{\alpha} h_{ij}^{\beta} \right] e_{n+1} \wedge \cdots \wedge e_{j}^{\alpha} \wedge \cdots \wedge e_{k}^{\beta} \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha \neq \beta} \sum_{j < k} \langle R^{\perp}(e_{j}, e_{k}) e_{\alpha}, e_{\beta} \rangle e_{n+1} \wedge \cdots \wedge e_{j}^{\alpha} \wedge \cdots \wedge e_{k}^{\beta} \wedge \cdots \wedge e_{n+p}$$

where R^{\perp} is the curvature tensor of TM^{\perp} .

$$\sum_{i} \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla_{e_{i}} A_{\alpha}(e_{i}) \wedge \cdots \wedge e_{n+p}$$

$$= \sum_{\alpha} \sum_{i,j} e_{n+1} \wedge \cdots \wedge \langle \nabla_{e_{i}} A_{\alpha}(e_{j}), e_{i} \rangle e_{j} \wedge \cdots \wedge e_{n+p}$$

$$\stackrel{(*)}{=} \sum_{\alpha} \sum_{i,j} e_{n+1} \wedge \cdots \wedge \langle \nabla_{e_{i}} A_{\alpha}(e_{i}), e_{i} \rangle e_{j} \wedge \cdots \wedge e_{n+p}$$

$$= \sum_{\alpha} \sum_{i,j} e_{n+1} \wedge \cdots \wedge n \ dH_{\alpha}(e_{j}) e_{j} \wedge \cdots \wedge e_{n+p}$$

$$= n \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla H_{\alpha} \wedge \cdots \wedge e_{n+p},$$

where (*) follows from the Equation of Codazzi, which holds since e_{α} is parallel in TM^{\perp} at m. Also H_{α} and ∇H_{α} denote the mean curvature and the gradient of the mean curvature in the direction of e_{α} .

$$\sum_{i} \sum_{\alpha} e_{n+1} \wedge \cdots \wedge h(e_{i}, A_{\alpha}(e_{i})) \wedge \cdots \wedge e_{n+p}$$

$$= \sum_{i} \sum_{\alpha,\beta} e_{n+1} \wedge \cdots \wedge \langle h(e_{i}, A_{\alpha}(e_{i})), e_{\beta} \rangle e_{\beta} \wedge \cdots \wedge e_{n+p}$$

$$= \sum_{i} \sum_{\alpha} \langle A_{\alpha}(e_{i}), A_{\alpha}(e_{i}) \rangle e_{n+1} \wedge \cdots \wedge e_{n+p} = |A|^{2} \%.$$

Thus,

$$\Delta \mathscr{E} = |A|^2 \mathscr{E} + n \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla H_{\alpha} \wedge \cdots \wedge e_{n+p}$$

$$- \sum_{\alpha \neq \beta} \sum_{j < k} \langle R^{\perp}(e_j, e_k) e_{\alpha}, e_{\beta} \rangle e_{n+1} \wedge \cdots \wedge e_{j}^{\alpha} \wedge \cdots \wedge e_{k}^{\beta} \wedge \cdots \wedge e_{n+p}.$$

Thus $\Delta \mathcal{E} = n\mathcal{E}$ if and only if the following hold:

- 1) $|A|^2 = n$.
- 2) $\nabla H_{\alpha} = 0$, for all α ; this is equivalent to η being parallel in TM^{\perp} .
- 3) $\langle R^{\perp}(e_j, e_k)e_{\alpha}, e_{\beta}\rangle = 0$, for all j, k, α , β ; that is, TM^{\perp} is flat.

We will show that 1) and 2) imply $x(M) = S^n \subset E^{n+1} \subset E^{n+p}$. We set $e_{n+1} = -\eta/H$. Then $\Delta(x/nH) = e_{n+1}$ since $\Delta x = -n\eta$. Thus $\int_M (a \cdot e_{n+1}) dV = 0$ for all unit vectors a in E^{n+p} . Also $de_{n+1} = -A_{n+1}$ since e_{n+1} is parallel in TM^{\perp} . Using the minimum principle for λ_1 again, we have

$$n \leq \int_{M} |d(a \cdot e_{n+1})|^{2} / \int_{M} (a \cdot e_{n+1})^{2} = \int_{M} |a \cdot A_{n+1}|^{2} / \int_{M} (a \cdot e_{n+1})^{2}$$

or

$$n \int_{M} (a \cdot e_{n+1})^2 \leq \int_{M} |a \cdot A_{n+1}|^2.$$

We integrate this inequality over all $a \in S^{n+p-1}$, use Fubini's Theorem, and obtain

$$nV \leq \int_{M} |A_{n+1}|^2.$$

But $|A_{n+1}|^2 \le |A|^2 = n$. Thus $|A_{n+1}|^2 = |A|^2 = n$. Let e_{n+2}, \ldots, e_{n+p} be an orthonormal basis of the complement of e_{n+1} in TM^{\perp} . Then $|A_{n+2}| = |A_{n+3}| = \cdots = |A_{n+p}| = 0$. Thus $A_{n+2} = \cdots = A_{n+p} = 0$. Let \mathscr{F} be the p-1 plane in \mathscr{E} orthogonal to e_{n+1} . It is easy to show that $d\mathscr{F} = 0$ since $\mathscr{F} = e_{n+2} \wedge \cdots \wedge e_{n+p}$ and $A_{\alpha} = 0$ for $\alpha = n+2, \ldots, n+p$. Hence $\mathscr{F} = \text{const.}$ Thus $x: M \to E^{n+1}$ where E^{n+1} is a (n+1)-plane orthogonal to \mathscr{F} .

We must now prove that if $x:M^n \to E^{n+1}$ is an immersion for which $|A|^2 = n$ and H = constant then x embeds M as a standard sphere in E^{n+1} . Consider $y = x + \eta : M \to E^{n+1}$. Then $\Delta y = \Delta x + \Delta \eta = -n\eta + n\eta = 0$ since $\Delta \eta = |A|^2 \eta = n\eta$. By Hopf's Theorem y = constant. Thus the image of x is in a sphere with center y and radius $|\eta| = H$. Clearly then H = 1. Since M is compact $x(M) = S^n$. For $n \ge 2$, this is enough to imply x is an embedding. For n = 1 we use the fact that M and its image have the same length, 2π , to prove x is an embedding.

It is natural now to ask what can be said about an isometric immersion $x: M \to E^{n+p}$ of a compact orientable *n*-dimensional manifold M into E^{n+p} when $\Delta \mathcal{E} = \lambda \mathcal{E}$, where λ is a constant, not necessarily λ_1 . We will characterize such x when n = 1, 2, or p = 1, or M has nonnegative sectional curvature.

PROPOSITION. Let M be a compact orientable n-dimensional manifold. Suppose $x: M \to E^{n+p}$ is an isometric immersion with $\Delta \mathscr{E} = \lambda \mathscr{E}$, where λ is a constant. Then the following hold:

- i) if n = 1, x is a covering map onto a circle in a 2-plane of E^{1+p} ;
- ii) if n > 1, and p = 1, then x embeds M as standard sphere in an (n + 1)-plane of E^{n+p} and $\lambda = \lambda_1$;
- iii) if n = 2, then M has constant nonnegative Gaussian curvature, and is characterized in iv);
- iv) if M has nonnegative sectional curvature, then, identifying M with its image, M is a product submanifold, $M^{n_1} \times \cdots \times M^{n_k}$, where M^{n_i} is an n_i -sphere of radius r_i in an $n_i + 1$ -plane of E^{n+p} ; moreover, $\lambda = \sum_{i=1}^k n_i/r_i^2$.

Proof. As above, $\Delta \mathcal{C} = \lambda \mathcal{C}$ if and only if 1) $|A|^2 = \lambda$, 2) η is parallel in TM^{\perp} , and 3) TM^{\perp} is flat.

One can easily show that i) follows from 2).

The argument in the last paragraph of the proof of Theorem I proves ii).

The scalar curvature $r = n^2H^2 - |A|^2$. Hence for n = 2, M has constant Gaussian curvature. Since 2) and 3) hold we may conclude from Lemma 2.5 [2, p. 108] and Theorem 2.1 [2, p. 106] of Chen that either x maps M into a 3-plane, E^3 , of E^{2+p} or x maps M into a 3-sphere, S^3 , in a 4-plane of E^{2+p} . Moreover, M has constant mean curvature relative to S^3 . If x maps M into E^3 then clearly M is a standard 2-sphere. If x maps M into S^3 as a minimal surface, then M is a standard sphere or is flat by a theorem of Lawson [4]. If x maps M into S^3 with constant nonzero mean curvature then M is a standard sphere or is flat by a theorem of Klotz and Osserman [2, p. 118].

Finally, iv) follows from 2) and 3) and the fact that M has nonnegative sectional curvature by a theorem of Erbacher, Yano and Ishihara [2, p. 139]. One may easily compute $\lambda = |A|^2 = \sum_{i=1}^k n_i/r_i^2$.

5. Proof of Theorem II

In proving Theorem II we use the fact that for the immersion $x: M \to E^m$, we have $\Delta x \equiv (\Delta x_1, \ldots, \Delta x_m) = -n\eta$ where Δ is the laplacian on M, $n = \dim M$, and $\eta = \text{mean curvature vector of } M \subset E^m$. Thus for any unit vector $a \in E^m$, $\Delta(x \cdot a) = n\eta \cdot a$ and since $\int_M \Delta f = 0$ for any $f \in C^\infty(M)$, we have $\int_M \eta \cdot a = 0$. By the minimum principle for λ_1 we have $\lambda_1 \int_M (\eta \cdot a)^2 \leq \int_M |d(\eta \cdot a)|^2$. Now $d(\eta \cdot a)(X) = X(\eta \cdot a) = (-A_{\eta}(X) + D_X \eta) \cdot a = -A_{\eta}(X) \cdot a = -HA_{\varepsilon}(X) \cdot a$ since $D_X \eta = 0$. Thus $|d(\eta \cdot a)|^2 = \sum_i H^2(A_{\varepsilon}(e_i) \cdot a)^2$ and $\int_{S^{m-1}} |d(\eta \cdot a)|^2 da = H^2 \sum_i |A_{\varepsilon}(e_i)|^2 \int_{S^{m-1}} (v \cdot a)^2 da$ where v is a fixed unit vector. Thus integrating

both sides of the above inequality with respect to "a" and applying Fubini, we obtain

$$\lambda_1 H^2 \int_{S^{m-1}} (v \cdot a)^2 da \int_M dV = \lambda_1 \int_{S^{m-1}} \int_M (\eta \cdot a)^2 dV da$$

$$\leq \int_{S^{m-1}} \int_M |d(\eta \cdot a)|^2 dV da = H^2 \int_{S^{m-1}} (v \cdot a)^2 da \int_M \sum_i |A_e(e_i)|^2$$

and dividing by $H^2 \int_{S^{m-1}} (v \cdot a)^2 da$ gives us $\lambda_1 \operatorname{vol}(M) \leq \int_M |A_e|^2$.

Remark. There are indications that in certain cases the upper bound of Theorem I is not so good, especially when M is close to being "extrinsically creased." For example, if one considers a family of oblate spheroids in E^3 with fixed intrinsic diameter but with minor axes approaching 0, the upper bounds of Theorem I approach infinity. However, by a result of Cheeger, λ_1 is bounded by $k[\text{diam } M]^{-2}$ for some large constant k depending only on the dimension of M [1, p. 189].

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