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Extrinsic Bounds on λ_1 of Δ on a Compact Manifold

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1. Introduction

Although the Laplacian of a Riemannian manifold M is an intrinsic object (as well as the first nonzero eigenvalue λ_1), upper bounds on λ_1 may be computed in terms of extrinsic quantities (e.g. principal curvatures) of M relative to some isometric embedding of M into some euclidean space E^m . In order to convey some idea of the results we obtained, the following is a special case of Theorem I: For a compact orientable surface S immersed in E^3 , λ_1 is bounded above by the average of the sum of the squares of the principal curvatures over S . Equality is achieved only in the case of a constant curvature sphere. Theorem I actually applies to manifolds with arbitrary dimension immersed with arbitrary codimension. A somewhat sharper result is found in Theorem II with the additional assumption that the mean curvature vector is parallel.

The primary means of obtaining these results is the minimum principle [1, p. 186] for λ_1 , namely $\lambda_1 = \inf \{ \int_M |df|^2 / \int_M f^2 : \int_M f = 0, f \in C^1(M) \}$, and the infimum is achieved only for f such that $\Delta f = \lambda_1 f$. To obtain interesting results, we choose f such that df contains some geometric information, $\int_M f = 0$, and then state $\lambda_1 \leq \int_M |df|^2 / \int_M f^2$. Actually we consider a parameterized family of such f say f_a , $a \in A$, where A is a manifold endowed with a measure and integrate both sides of $\lambda_1 \int_M f_a^2 \leq \int_M |df_a|^2$ over A to eliminate the parameter.

2. Definitions and Statements of Main Results

In general we follow the notation of [3]. Let $x: M \rightarrow \bar{M}$ be a isometric immersion and let ∇ and $\bar{\nabla}$ be the covariant differentiations on M and \bar{M} respectively. For vector fields X and Y tangent to M , we define the vector field $h(X, Y)$ by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad [\text{equation of Gauss}].$$

The normal bundle valued (symmetric) 2-tensor h on M is called the second fundamental form. For a normal vector field N on M we set

$$\bar{\nabla}_X N = -A_N(X) + D_X N \quad [\text{equation of Weingarten}],$$

$-A_N(X)$ and $D_X N$ being the tangential and normal parts of $\bar{\nabla}_X N$. We have

$$\langle A_N(X), Y \rangle = \langle h(X, Y), N \rangle.$$

The mean curvature vector field η is defined by $\eta(p) = 1/n$ trace $h = 1/n \sum_{i=1}^n h(e_i, e_i) = 1/n \sum_{\alpha=n+1}^m (\text{trace } A_\alpha) e_\alpha$ where $A_\alpha \equiv A_{e_\alpha}$ and (e_i) is an o.n. frame of $T_p M$ and (e_α) is an o.n. frame of $T_p M^\perp$. We shall adopt the convention that latin indices run from 1 to n and greek from $n+1$ to m . Of course, we set $|A_\alpha|^2 = \sum_i \langle A_\alpha(e_i), A_\alpha(e_i) \rangle$ and $|A|^2 \equiv \sum_\alpha |A_\alpha|^2$.

THEOREM I. *Let M be a compact orientable n -manifold isometrically immersed in euclidean space E^m . Then $\lambda_1 \leq [\text{vol}(M)]^{-1} \int_M |A|^2$ with equality only in the case where M is a constant curvature sphere isometrically embedded in an $n+1$ dimensional subspace of E^m .*

THEOREM II. *With M as above and assuming η is parallel in the normal bundle, $\lambda_1 \leq [\text{vol}(M)]^{-1} \int_M |A_e|^2$, where $e = \eta/H$ and $H \equiv |\eta|$.*

3. Proof of Inequality in Theorem I

Let $x: M \rightarrow E^m$ be an isometric immersion. Now $dx \equiv (dx_1)\bar{e}_1 + \cdots + (dx_m)\bar{e}_m$ where $(\bar{e}_1, \dots, \bar{e}_m)$ form a standard o.n. basis of E^m and $x(p) = x_1(p)\bar{e}_1 + \cdots + x_m(p)\bar{e}_m$, $p \in M$. Thus the dx_i 's are 1-forms on M . We let $(dx_i)\bar{e}_i \wedge (dx_j)\bar{e}_j = (dx_i \wedge dx_j)(\bar{e}_i \wedge \bar{e}_j)$ and extend this product in the natural way to $\Lambda^*(M) \otimes \Lambda^*(E^m)$ (i.e. by linearity and $(\alpha \otimes e) \wedge (\beta \otimes f) = (\alpha \wedge \beta) \otimes (e \wedge f)$). In this context $1/n! dx \wedge \cdots \wedge dx$ (n times) $= (dV)f$ for some $f \in \Lambda^0(M) \otimes \Lambda^n(E^m)$ where dV is the volume element of M . For an oriented n -plane π of E^m with o.n. basis (a_1, \dots, a_n) we define the function f_π on M by $f_\pi(p) = (a_1 \wedge \cdots \wedge a_n) \cdot f(p)$ or perhaps more succinctly by $f_\pi dV = (a_1 \wedge \cdots \wedge a_n) \cdot 1/n! dx \wedge \cdots \wedge dx$ (n times). We note that $f_\pi dV = d[(a_1 \wedge \cdots \wedge a_n) \cdot 1/n! x \wedge dx \wedge \cdots \wedge dx]$ is exact, whence $\int_M f_\pi = 0$.

To derive a formula for f_π more amenable to calculating df_π , we note that we may write (locally) $dx = \sum w_i e_i$ where e_i is an o.n. frame field of M defined on a neighborhood of some point of M (here we identify vectors tangent to M with

their images under x_*) and the w_i are the one-forms on M dual to the e_i . This is because dx , considered as an E^m -valued one-form on M , is the identity map under the above mentioned identification. Thus $1/n! dx \wedge \cdots \wedge dx = (w_1 \wedge \cdots \wedge w_n)(e_1 \wedge \cdots \wedge e_n) = dV(e_1 \wedge \cdots \wedge e_n)$. Thus $f = e_1 \wedge \cdots \wedge e_n$ and $f_\pi = (a_1 \wedge \cdots \wedge a_n) \cdot (e_1 \wedge \cdots \wedge e_n) = (a_{n+1} \wedge \cdots \wedge a_{n+p}) \cdot (e_{n+1} \wedge \cdots \wedge e_{n+p})$ where $n + p = m$ and (a_1, \dots, a_m) and (e_1, \dots, e_m) are oriented o.n. bases of E^m . To avoid writing $a_{n+1} \wedge \cdots \wedge a_{n+p}$ repeatedly we replace this expression by \mathcal{A} . Now $df_\pi = d(\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{n+p})$

$$\begin{aligned} &= \sum_{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge de_{\alpha} \wedge \cdots \wedge e_{n+p} \\ &= \sum_{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge \sum_i w_{\alpha i} e_i \wedge \cdots \wedge e_{n+p} \\ &= - \sum_{\alpha, i} h_{ij}^{\alpha} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_i^{\alpha} \wedge \cdots \wedge e_{n+p}) w_j \end{aligned}$$

where $[(de_{\alpha})(X)]^T = -A_{e_{\alpha}}(X) = \sum_i w_{\alpha i}(X) e_i = -\sum_i h_{ij}^{\alpha} w_j(X) e_i$ for X tangent to M . Note $de_{\alpha}(X) \perp e_{\alpha}$. Here the super α indicates the α -th slot in the product.

$$\begin{aligned} |df_{\pi}|^2 &= \sum_j \left(\sum_{\alpha, i} -h_{ij}^{\alpha} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_i^{\alpha} \wedge \cdots \wedge e_{n+p} \right)^2 \\ &= \sum_{j, i, l, \alpha, \beta} h_{ij}^{\alpha} h_{lj}^{\beta} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_i^{\alpha} \wedge \cdots \wedge e_{n+p}) \\ &\quad \times (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_l^{\beta} \wedge \cdots \wedge e_{n+p}). \end{aligned}$$

By the minimum principle for λ_1 we have $\lambda_1 \int_M f_{\pi}^2 \leq \int_M |df_{\pi}|^2$. However, the unwieldy expression for $|df_{\pi}|^2$ leaves much to be desired. Hence we integrate both sides of the inequality with respect to $\pi \in G(p, n) =$ oriented p -planes in $n + p$ -space. To this end we apply Fubini's Theorem obtaining $\lambda_1 \int_M \int_G f_{\pi}^2 d\pi dV \leq \int_M \int_G |df_{\pi}|^2 d\pi dV$. In integrating $|df_{\pi}|^2$ with respect to π we consider h_{ij}^{α} , h_{lj}^{β} and the e 's to be fixed and $\mathcal{A} = a_{n+1} \wedge \cdots \wedge a_{n+p}$ to vary. We need only consider

$$\int_G (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_i^{\alpha} \wedge \cdots \wedge e_{n+p}) (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_l^{\beta} \wedge \cdots \wedge e_{n+p}) d\pi.$$

Upon reflection (pardon pun) we see that the above is 0 unless $i = l$ and $\alpha = \beta$, for otherwise $\{e_{n+1}, \dots, e_i^{\alpha}, \dots, e_{n+p}\} \neq \{e_{n+1}, \dots, e_l^{\beta}, \dots, e_{n+p}\}$ and reflection of E^{n+p} , in a hyperplane perpendicular to a vector in one set but not in the other,

induces an isometry of G under which the integrand changes sign. Thus the above integral is

$$\delta_{il} \delta_{\alpha\beta} \int_G (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_i^\alpha \wedge \cdots \wedge e_{n+p})^2 d\pi = \delta_{il} \delta_{\alpha\beta} \int_G f_\pi^2 d\pi$$

and $\int_G |df_\pi|^2 d\pi = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 \int_G f_\pi^2 d\pi$. Hence

$$\begin{aligned} \lambda_1 \int_G f_\pi^2 d\pi \int_M dV &= \lambda_1 \int_M \int_G f_\pi^2 d\pi dV \leq \int_M \int_G |df_\pi|^2 d\pi dV \\ &= \int_G f_\pi^2 d\pi \int_M \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 dV. \end{aligned}$$

Dividing by $\int_G f_\pi^2 d\pi$ gives us the desired result

$$\lambda_1 \text{vol}(M) \leq \int_M \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 dV = \int_M |A|^2.$$

4. Equality in Theorem I

In this section we show equality holds in Theorem I only in the case where M is a constant curvature sphere isometrically embedded in an $n+1$ dimensional subspace, E^{n+1} , of E^{n+p} .

If we assume equality holds in Theorem I, then for almost all $\pi \in G(n, p)$

$$\lambda_1 = \frac{\int_M |d(\mathcal{A} \cdot \mathcal{E})|^2 dV}{\int_M |\mathcal{A} \cdot \mathcal{E}|^2 dV}$$

where $\mathcal{A} = \pi^\perp$ and $\mathcal{E} = e_{n+1} \wedge \cdots \wedge e_{n+p}$. Thus, for almost all $\pi \in G(n, p)$, the minimum principle implies that $\mathcal{A} \cdot \Delta \mathcal{E} = \Delta(\mathcal{A} \cdot \mathcal{E}) = \lambda_1(\mathcal{A} \cdot \mathcal{E}) = \mathcal{A} \cdot (\lambda_1 \mathcal{E})$; hence $\Delta \mathcal{E} = \lambda_1 \mathcal{E}$. We suppose $\lambda_1 = n$, taking a homothetic transformation of E^{n+p} if necessary.

We now compute $\Delta \mathcal{E}$ at a point $m \in M$. Let e_1, e_2, \dots, e_{n+p} be a frame field in a neighborhood of $x(m)$ such that e_1, \dots, e_n are tangent to M and hence e_{n+1}, \dots, e_{n+p} are normal to M . Suppose that e_1, \dots, e_n are parallel in TM at m , and e_{n+1}, \dots, e_{n+p} are parallel in TM^\perp at m . In fact, we may assume $D_{e_i} D_{e_i} e_\alpha = 0$,

$1 \leq i \leq n$, $n+1 \leq \alpha \leq n+p$, by supposing that e_i and e_α are obtained from $e_i(m)$ and $e_\alpha(m)$ by parallel translation along geodesics through m in TM and TM^\perp , respectively.

Now, $\Delta \mathcal{E} = -\sum_{i=1}^n e_i e_i \mathcal{E}$. In a neighborhood of m , we have

$$\begin{aligned} e_i \mathcal{E} &= \sum_{\alpha=n+1}^{n+p} e_{n+1} \wedge \cdots \wedge de_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &= -\sum_{\alpha} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} + \sum_{\alpha} e_{n+1} \wedge \cdots \wedge D_{e_i}(e_\alpha) \wedge \cdots \wedge e_{n+p} \end{aligned}$$

using Weingarten's equation; we have set $A_{e_\alpha} = A_\alpha$. At m ,

$$\begin{aligned} e_i e_i \mathcal{E} &= \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad - \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge D_{e_i} e_\beta \wedge \cdots \wedge e_{n+p} \\ &\quad - \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla_{e_i} A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad - \sum_{\alpha} e_{n+1} \wedge \cdots \wedge h(e_i, A_\alpha(e_i)) \wedge \cdots \wedge e_{n+p} \\ &\quad - \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge D_{e_i} e_\alpha \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad + \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge D_{e_i} e_\alpha \wedge \cdots \wedge D_{e_i} e_\beta \wedge \cdots \wedge e_{n+p} \\ &\quad - \sum_{\alpha} e_{n+1} \wedge \cdots \wedge A_{D_{e_i} e_\alpha}(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad + \sum_{\alpha} e_{n+1} \wedge \cdots \wedge D_{e_i} D_{e_i} e_\alpha \wedge \cdots \wedge e_{n+p}. \end{aligned}$$

In the preceding equation, the third and fourth terms arise from Gauss' equation. Also, note that the second, fifth, sixth, seventh and eighth terms vanish since $D_{e_i} e_\alpha = 0$ and $D_{e_i} D_{e_i} e_\alpha = 0$ at m . Thus

$$\begin{aligned} \Delta \mathcal{E} &= -\sum_i \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad + \sum_i \sum_{\alpha} e_{n+1} \wedge \cdots \wedge \nabla_{e_i} A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &\quad + \sum_i \sum_{\alpha} e_{n+1} \wedge \cdots \wedge h(e_i, A_\alpha(e_i)) \wedge \cdots \wedge e_{n+p}. \end{aligned}$$

We now consider each term on the right-hand side of the preceding equation separately.

$$\begin{aligned}
& - \sum_i \sum_{\alpha \neq \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\
& = - \sum_{\alpha \neq \beta} \sum_{i,j,k} e_{n+1} \wedge \cdots \wedge h_{ij}^\alpha e_j \wedge \cdots \wedge h_{ik}^\beta e_k \wedge \cdots \wedge e_{n+p} \\
& = - \sum_{\alpha \neq \beta} \sum_{j < k} \left[\sum_i h_{ij}^\alpha h_{ik}^\beta - h_{ik}^\alpha h_{ij}^\beta \right] e_{n+1} \wedge \cdots \wedge e_j^\alpha \wedge \cdots \wedge e_k^\beta \wedge \cdots \wedge e_{n+p} \\
& = - \sum_{\alpha \neq \beta} \sum_{j < k} \langle R^\perp(e_j, e_k) e_\alpha, e_\beta \rangle e_{n+1} \wedge \cdots \wedge e_j^\alpha \wedge \cdots \wedge e_k^\beta \wedge \cdots \wedge e_{n+p}
\end{aligned}$$

where R^\perp is the curvature tensor of TM^\perp .

$$\begin{aligned}
& \sum_i \sum_\alpha e_{n+1} \wedge \cdots \wedge \nabla_{e_i} A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\
& = \sum_\alpha \sum_{i,j} e_{n+1} \wedge \cdots \wedge \langle \nabla_{e_i} A_\alpha(e_j), e_i \rangle e_j \wedge \cdots \wedge e_{n+p} \\
& \stackrel{(*)}{=} \sum_\alpha \sum_{i,j} e_{n+1} \wedge \cdots \wedge \langle \nabla_{e_j} A_\alpha(e_i), e_i \rangle e_j \wedge \cdots \wedge e_{n+p} \\
& = \sum_\alpha \sum_j e_{n+1} \wedge \cdots \wedge n dH_\alpha(e_j) e_j \wedge \cdots \wedge e_{n+p} \\
& = n \sum_\alpha e_{n+1} \wedge \cdots \wedge \nabla H_\alpha \wedge \cdots \wedge e_{n+p},
\end{aligned}$$

where (*) follows from the Equation of Codazzi, which holds since e_α is parallel in TM^\perp at m . Also H_α and ∇H_α denote the mean curvature and the gradient of the mean curvature in the direction of e_α .

$$\begin{aligned}
& \sum_i \sum_\alpha e_{n+1} \wedge \cdots \wedge h(e_i, A_\alpha(e_i)) \wedge \cdots \wedge e_{n+p} \\
& = \sum_i \sum_{\alpha, \beta} e_{n+1} \wedge \cdots \wedge \langle h(e_i, A_\alpha(e_i)), e_\beta \rangle e_\beta \wedge \cdots \wedge e_{n+p} \\
& = \sum_i \sum_\alpha \langle A_\alpha(e_i), A_\alpha(e_i) \rangle e_{n+1} \wedge \cdots \wedge e_{n+p} = |A|^2 \mathcal{E}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta \mathcal{E} &= |A|^2 \mathcal{E} + n \sum_\alpha e_{n+1} \wedge \cdots \wedge \nabla H_\alpha \wedge \cdots \wedge e_{n+p} \\
&\quad - \sum_{\alpha \neq \beta} \sum_{j < k} \langle R^\perp(e_j, e_k) e_\alpha, e_\beta \rangle e_{n+1} \wedge \cdots \wedge e_j^\alpha \wedge \cdots \wedge e_k^\beta \wedge \cdots \wedge e_{n+p}.
\end{aligned}$$

Thus $\Delta \mathcal{E} = n\mathcal{E}$ if and only if the following hold:

- 1) $|A|^2 = n$.
- 2) $\nabla H_\alpha = 0$, for all α ; this is equivalent to η being parallel in TM^\perp .
- 3) $\langle R^\perp(e_j, e_k)e_\alpha, e_\beta \rangle = 0$, for all j, k, α, β ; that is, TM^\perp is flat.

We will show that 1) and 2) imply $x(M) = S^n \subset E^{n+1} \subset E^{n+p}$. We set $e_{n+1} = -\eta/H$. Then $\Delta(x/nH) = e_{n+1}$ since $\Delta x = -n\eta$. Thus $\int_M (a \cdot e_{n+1}) dV = 0$ for all unit vectors a in E^{n+p} . Also $de_{n+1} = -A_{n+1}$ since e_{n+1} is parallel in TM^\perp . Using the minimum principle for λ_1 again, we have

$$n \leq \int_M |d(a \cdot e_{n+1})|^2 / \int_M (a \cdot e_{n+1})^2 = \int_M |a \cdot A_{n+1}|^2 / \int_M (a \cdot e_{n+1})^2$$

or

$$n \int_M (a \cdot e_{n+1})^2 \leq \int_M |a \cdot A_{n+1}|^2.$$

We integrate this inequality over all $a \in S^{n+p-1}$, use Fubini's Theorem, and obtain

$$nV \leq \int_M |A_{n+1}|^2.$$

But $|A_{n+1}|^2 \leq |A|^2 = n$. Thus $|A_{n+1}|^2 = |A|^2 = n$. Let e_{n+2}, \dots, e_{n+p} be an orthonormal basis of the complement of e_{n+1} in TM^\perp . Then $|A_{n+2}| = |A_{n+3}| = \dots = |A_{n+p}| = 0$. Thus $A_{n+2} = \dots = A_{n+p} = 0$. Let \mathcal{F} be the $p-1$ plane in \mathcal{E} orthogonal to e_{n+1} . It is easy to show that $d\mathcal{F} = 0$ since $\mathcal{F} = e_{n+2} \wedge \dots \wedge e_{n+p}$ and $A_\alpha = 0$ for $\alpha = n+2, \dots, n+p$. Hence $\mathcal{F} = \text{const}$. Thus $x: M \rightarrow E^{n+1}$ where E^{n+1} is a $(n+1)$ -plane orthogonal to \mathcal{F} .

We must now prove that if $x: M^n \rightarrow E^{n+1}$ is an immersion for which $|A|^2 = n$ and $H = \text{constant}$ then x embeds M as a standard sphere in E^{n+1} . Consider $y = x + \eta: M \rightarrow E^{n+1}$. Then $\Delta y = \Delta x + \Delta \eta = -n\eta + n\eta = 0$ since $\Delta \eta = |A|^2 \eta = n\eta$. By Hopf's Theorem $y = \text{constant}$. Thus the image of x is in a sphere with center y and radius $|\eta| = H$. Clearly then $H = 1$. Since M is compact $x(M) = S^n$. For $n \geq 2$, this is enough to imply x is an embedding. For $n = 1$ we use the fact that M and its image have the same length, 2π , to prove x is an embedding.

It is natural now to ask what can be said about an isometric immersion $x: M \rightarrow E^{n+p}$ of a compact orientable n -dimensional manifold M into E^{n+p} when $\Delta \mathcal{E} = \lambda \mathcal{E}$, where λ is a constant, not necessarily λ_1 . We will characterize such x when $n = 1, 2$, or $p = 1$, or M has nonnegative sectional curvature.

PROPOSITION. *Let M be a compact orientable n -dimensional manifold. Suppose $x:M \rightarrow E^{n+p}$ is an isometric immersion with $\Delta \mathcal{E} = \lambda \mathcal{E}$, where λ is a constant. Then the following hold:*

- i) *if $n = 1$, x is a covering map onto a circle in a 2-plane of E^{1+p} ;*
- ii) *if $n > 1$, and $p = 1$, then x embeds M as standard sphere in an $(n+1)$ -plane of E^{n+p} and $\lambda = \lambda_1$;*
- iii) *if $n = 2$, then M has constant nonnegative Gaussian curvature, and is characterized in iv);*
- iv) *if M has nonnegative sectional curvature, then, identifying M with its image, M is a product submanifold, $M^{n_1} \times \cdots \times M^{n_k}$, where M^{n_i} is an n_i -sphere of radius r_i in an $n_i + 1$ -plane of E^{n+p} ; moreover, $\lambda = \sum_{i=1}^k n_i/r_i^2$.*

Proof. As above, $\Delta \mathcal{E} = \lambda \mathcal{E}$ if and only if 1) $|A|^2 = \lambda$, 2) η is parallel in TM^\perp , and 3) TM^\perp is flat.

One can easily show that i) follows from 2).

The argument in the last paragraph of the proof of Theorem I proves ii).

The scalar curvature $r = n^2 H^2 - |A|^2$. Hence for $n = 2$, M has constant Gaussian curvature. Since 2) and 3) hold we may conclude from Lemma 2.5 [2, p. 108] and Theorem 2.1 [2, p. 106] of Chen that either x maps M into a 3-plane, E^3 , of E^{2+p} or x maps M into a 3-sphere, S^3 , in a 4-plane of E^{2+p} . Moreover, M has constant mean curvature relative to S^3 . If x maps M into E^3 then clearly M is a standard 2-sphere. If x maps M into S^3 as a minimal surface, then M is a standard sphere or is flat by a theorem of Lawson [4]. If x maps M into S^3 with constant nonzero mean curvature then M is a standard sphere or is flat by a theorem of Klotz and Osserman [2, p. 118].

Finally, iv) follows from 2) and 3) and the fact that M has nonnegative sectional curvature by a theorem of Erbacher, Yano and Ishihara [2, p. 139]. One may easily compute $\lambda = |A|^2 = \sum_{i=1}^k n_i/r_i^2$.

5. Proof of Theorem II

In proving Theorem II we use the fact that for the immersion $x:M \rightarrow E^m$, we have $\Delta x \equiv (\Delta x_1, \dots, \Delta x_m) = -n\eta$ where Δ is the laplacian on M , $n = \dim M$, and η = mean curvature vector of $M \subset E^m$. Thus for any unit vector $a \in E^m$, $\Delta(x \cdot a) = n\eta \cdot a$ and since $\int_M \Delta f = 0$ for any $f \in C^\infty(M)$, we have $\int_M \eta \cdot a = 0$. By the minimum principle for λ_1 we have $\lambda_1 \int_M (\eta \cdot a)^2 \leq \int_M |d(\eta \cdot a)|^2$. Now $d(\eta \cdot a)(X) = X(\eta \cdot a) = (-A_\eta(X) + D_X \eta) \cdot a = -A_\eta(X) \cdot a = -HA_e(X) \cdot a$ since $D_X \eta = 0$. Thus $|d(\eta \cdot a)|^2 = \sum_i H^2(A_e(e_i) \cdot a)^2$ and $\int_{S^{m-1}} |d(\eta \cdot a)|^2 da = H^2 \sum_i |A_e(e_i)|^2 \int_{S^{m-1}} (v \cdot a)^2 da$ where v is a fixed unit vector. Thus integrating

both sides of the above inequality with respect to “ a ” and applying Fubini, we obtain

$$\begin{aligned} \lambda_1 H^2 \int_{S^{m-1}} (v \cdot a)^2 da \int_M dV &= \lambda_1 \int_{S^{m-1}} \int_M (\eta \cdot a)^2 dV da \\ &\leq \int_{S^{m-1}} \int_M |d(\eta \cdot a)|^2 dV da = H^2 \int_{S^{m-1}} (v \cdot a)^2 da \int_M \sum_i |A_e(e_i)|^2 \end{aligned}$$

and dividing by $H^2 \int_{S^{m-1}} (v \cdot a)^2 da$ gives us $\lambda_1 \text{vol}(M) \leq \int_M |A_e|^2$.

Remark. There are indications that in certain cases the upper bound of Theorem I is not so good, especially when M is close to being “extrinsically creased.” For example, if one considers a family of oblate spheroids in E^3 with fixed intrinsic diameter but with minor axes approaching 0, the upper bounds of Theorem I approach infinity. However, by a result of Cheeger, λ_1 is bounded by $k[\text{diam } M]^{-2}$ for some large constant k depending only on the dimension of M [1, p. 189].

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