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Extrinsic Bounds on λ_1 of Δ on a Compact Manifold

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1. Introduction

Although the Laplacian of a Riemannian manifold M is an intrinsic object (as well as the first nonzero eigenvalue λ_1), upper bounds on λ_1 may be computed in terms of extrinsic quantities (e.g. principal curvatures) of M relative to some isometric embedding of M into some euclidean space E^m. In order to convey some idea of the results we obtained, the following is a special case of Theorem I: For a compact orientable surface S immersed in E^3, λ_1 is bounded above by the average of the sum of the squares of the principal curvatures over S. Equality is achieved only in the case of a constant curvature sphere. Theorem I actually applies to manifolds with arbitrary dimension immersed with arbitrary codimension. A somewhat sharper result is found in Theorem II with the additional assumption that the mean curvature vector is parallel.

The primary means of obtaining these results is the minimum principle [1, p. 186] for λ_1, namely $\lambda_1 = \inf\{\int_M |df|^2/\int_M f^2 : \int_M f = 0,\ f \in C^1(M)\}$, and the infimum is achieved only for f such that $\Delta f = \lambda_1 f$. To obtain interesting results, we choose f such that df contains some geometric information, $\int_M f = 0$, and then state $\lambda_1 \leq \int_M |df|^2/\int_M f^2$. Actually we consider a parameterized family of such f say f_a, $a \in A$, where A is a manifold endowed with a measure and integrate both sides of $\lambda_1 \int_M f_a^2 \leq \int_M |df_a|^2$ over A to eliminate the parameter.

2. Definitions and Statements of Main Results

In general we follow the notation of [3]. Let $x: M \to \bar{M}$ be a isometric immersion and let ∇ and $\bar{\nabla}$ be the covarient differentiations on M and $\bar{M}$ respectively. For vector fields X and Y tangent to M, we define the vector field $h(X, Y)$ by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{[equation of Gauss].}$$

The normal bundle valued (symmetric) 2-tensor h on M is called the second fundamental form. For a normal vector field N on M we set

$$\bar{\nabla}_X N = -A_N(X) + D_X N \quad \text{[equation of Weingarten]},$$

$-A_N(X)$ and $D_X N$ being the tangential and normal parts of $\bar{\nabla}_X N$. We have

$$\langle A_N(X), Y\rangle = \langle h(X, Y), N\rangle.$$

The mean curvature vector field η is defined by $\eta(p) = 1/n$ trace $h = 1/n\sum_{i=1}^{n} h(e_i, e_i) = 1/n\sum_{\alpha=n+1}^{m} (\text{trace } A_\alpha)e_\alpha$ where $A_\alpha \equiv A_{e_\alpha}$ and (e_i) is an o.n. frame of T_pM and (e_α) is an o.n. frame of $T_pM^\perp$. We shall adopt the convention that latin indices run from 1 to n and greek from $n+1$ to m. Of course, we set $|A_\alpha|^2 = \sum_i \langle A_\alpha(e_i), A_\alpha(e_i)\rangle$ and $|A|^2 \equiv \sum_\alpha |A_\alpha|^2$.

THEOREM I. *Let M be a compact orientable n-manifold isometrically immersed in euclidean space E^m. Then $\lambda_1 \leq [\text{vol}(M)]^{-1} \int_M |A|^2$ with equality only in the case where M is a constant curvature sphere isometrically embedded in an $n+1$ dimensional subspace of E^m.*

THEOREM II. *With M as above and assuming η is parallel in the normal bundle, $\lambda_1 \leq [\text{vol}(M)]^{-1} \int_M |A_e|^2$, where $e = \eta/H$ and $H \equiv |\eta|$.*

3. Proof of Inequality in Theorem I

Let $x: M \to E^m$ be an isometric immersion. Now $dx \equiv (dx_1)\bar{e}_1 + \cdots + (dx_m)\bar{e}_m$ where $(\bar{e}_1, \ldots, \bar{e}_m)$ form a standard o.n. basis of E^m and $x(p) = x_1(p)\bar{e}_1 + \cdots + x_m(p)\bar{e}_m$, $p \in M$. Thus the dx_i's are 1-forms on M. We let $(dx_i)\bar{e}_i \wedge (dx_j)\bar{e}_j = (dx_i \wedge dx_j)(\bar{e}_i \wedge \bar{e}_j)$ and extend this product in the natural way to $\Lambda^*(M) \otimes \Lambda^*(E^m)$ (i.e. by linearity and $(\alpha \otimes e) \wedge (\beta \otimes f) = (\alpha \wedge \beta) \otimes (e \wedge f)$). In this context $1/n!\, dx \wedge \cdots \wedge dx$ (n times) $= (dV)f$ for some $f \in \Lambda^0(M) \otimes \Lambda^n(E^m)$ where dV is the volume element of M. For an oriented n-plane π of E^m with o.n. basis $(a_1, \ldots, a_n)$ we define the function f_π on M by $f_\pi(p) = (a_1 \wedge \cdots \wedge a_n) \cdot f(p)$ or perhaps more succinctly by $f_\pi\, dV = (a_1 \wedge \cdots \wedge a_n) \cdot 1/n!\, dx \wedge \cdots \wedge dx$ (n times). We note that $f_\pi\, dV = d[(a_1 \wedge \cdots \wedge a_n) \cdot 1/n!\, x \wedge dx \wedge \cdots \wedge dx]$ is exact, whence $\int_M f_\pi = 0$.

To derive a formula for f_π more amenable to calculating df_π, we note that we may write (locally) $dx = \sum w_i e_i$ where e_i is an o.n. frame field of M defined on a neighborhood of some point of M (here we identify vectors tangent to M with their images under x_*) and the w_i are the one-forms on M dual to the e_i. This is because dx, considered as an E^m-valued one-form on M, is the identity map under the above mentioned identification. Thus $1/n!\, dx \wedge \cdots \wedge dx = (w_1 \wedge \cdots \wedge w_n)(e_1 \wedge \cdots \wedge e_n) = dV(e_1 \wedge \cdots \wedge e_n)$. Thus $f = e_1 \wedge \cdots \wedge e_n$ and $f_\pi = (a_1 \wedge \cdots \wedge a_n) \cdot (e_1 \wedge \cdots \wedge e_n) = (a_{n+1} \wedge \cdots \wedge a_{n+p}) \cdot (e_{n+1} \wedge \cdots \wedge e_{n+p})$ where $n + p = m$ and $(a_1, \ldots, a_m)$ and $(e_1, \ldots, e_m)$ are oriented o.n. bases of E^m. To avoid writing $a_{n+1} \wedge \cdots \wedge a_{n+p}$ repeatedly we replace this expression by $\mathcal{A}$. Now $df_\pi = d(\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e_{n+p})$

$$= \sum_\alpha \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge de_\alpha \wedge \cdots \wedge e_{n+p}$$

$$= \sum_\alpha \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge \sum_i w_{\alpha i} e_i \wedge \cdots \wedge e_{n+p}$$

$$= -\sum_{\alpha,i} h^\alpha_{ij} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\alpha_i \wedge \cdots \wedge e_{n+p}) w_j$$

where $[(de_\alpha)(X)]^\top = -A_{e_\alpha}(X) = \sum_i w_{\alpha i}(X) e_i = -\sum_i h^\alpha_{ij} w_j(X) e_i$ for X tangent to M. Note $de_\alpha(X) \perp e_\alpha$. Here the super α indicates the α-th slot in the product.

$$|df_\pi|^2 = \sum_j \left(\sum_{\alpha,i} - h^\alpha_{ij} \mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\alpha_i \wedge \cdots \wedge e_{n+p} \right)^2$$

$$= \sum_{j,i,l,\alpha,\beta} h^\alpha_{ij} h^\beta_{lj} (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\alpha_i \wedge \cdots \wedge e_{n+p})$$

$$\times (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\beta_l \wedge \cdots \wedge e_{n+p}).$$

By the minimum principle for λ_1 we have $\lambda_1 \int_M f_\pi^2 \leq \int_M |df_\pi|^2$. However, the unwieldy expression for $|df_\pi|^2$ leaves much to be desired. Hence we integrate both sides of the inequality with respect to $\pi \in G(p, n) =$ oriented p-planes in $n+p$-space. To this end we apply Fubini's Theorem obtaining $\lambda_1 \int_M \int_G f_\pi^2 \, d\pi \, dV \leq \int_M \int_G |df_\pi|^2 \, d\pi \, dV$. In integrating $|df_\pi|^2$ with respect to π we consider h^α_{ij}, h^β_{lj} and the e's to be fixed and $\mathcal{A} = a_{n+1} \wedge \cdots \wedge a_{n+p}$ to vary. We need only consider

$$\int_G (\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\alpha_i \wedge \cdots \wedge e_{n+p})(\mathcal{A} \cdot e_{n+1} \wedge \cdots \wedge e^\beta_l \wedge \cdots \wedge e_{n+p}) \, d\pi.$$

Upon reflection (pardon pun) we see that the above is 0 unless $i = l$ and $\alpha = \beta$, for otherwise $\{e_{n+1}, \ldots, e^\alpha_i, \ldots, e_{n+p}\} \neq \{e_{n+1}, \ldots, e^\beta_l, \ldots, e_{n+p}\}$ and reflection of E^{n+p}, in a hyperplane perpendicular to a vector in one set but not in the other, induces an isometry of G under which the integrand changes sign. Thus the above integral is

$$\delta_{il}\,\delta_{\alpha\beta}\int_G (\mathcal{A}\cdot e_{n+1}\wedge\cdots\wedge e_i^\alpha\wedge\cdots\wedge e_{n+p})^2\,d\pi=\delta_{il}\,\delta_{\alpha\beta}\int_G f_\pi^2\,d\pi$$

and $\int_G |df_\pi|^2\,d\pi=\sum_{\alpha,i,j}(h_{ij}^\alpha)^2\int_G f_\pi^2\,d\pi$. Hence

$$\begin{aligned}\lambda_1\int_G f_\pi^2\,d\pi\int_M dV=\lambda_1\int_M\int_G f_\pi^2\,d\pi\,dV&\le\int_M\int_G |df_\pi|^2\,d\pi\,dV\\&=\int_G f_\pi^2\,d\pi\int_M\sum_{\alpha,i,j}(h_{ij}^\alpha)^2\,dV.\end{aligned}$$

Dividing by $\int_G f_\pi^2\,d\pi$ gives us the desired result

$$\lambda_1\operatorname{vol}(M)\le\int_M\sum_{\alpha,i,j}(h_{ij}^\alpha)^2\,dV=\int_M |A|^2.$$

4. Equality in Theorem I

In this section we show equality holds in Theorem I only in the case where M is a constant curvature sphere isometrically embedded in an $n+1$ dimensional subspace, E^{n+1}, of E^{n+p}.

If we assume equality holds in Theorem I, then for almost all $\pi\in G(n,p)$

$$\lambda_1=\frac{\int_M |d(\mathcal{A}\cdot\mathcal{E})|^2\,dV}{\int_M |\mathcal{A}\cdot\mathcal{E}|^2\,dV}$$

where $\mathcal{A}=\pi^\perp$ and $\mathcal{E}=e_{n+1}\wedge\cdots\wedge e_{n+p}$. Thus, for almost all $\pi\in G(n,p)$, the minimum principle implies that $\mathcal{A}\cdot\Delta\mathcal{E}=\Delta(\mathcal{A}\cdot\mathcal{E})=\lambda_1(\mathcal{A}\cdot\mathcal{E})=\mathcal{A}\cdot(\lambda_1\mathcal{E})$; hence $\Delta\mathcal{E}=\lambda_1\mathcal{E}$. We suppose $\lambda_1=n$, taking a homothetic transformation of E^{n+p} if necessary.

We now compute $\Delta\mathcal{E}$ at a point $m\in M$. Let $e_1, e_2,\ldots,e_{n+p}$ be a frame field in a neighborhood of $x(m)$ such that $e_1,\ldots,e_n$ are tangent to M and hence $e_{n+1},\ldots,e_{n+p}$ are normal to M. Suppose that $e_1,\ldots,e_n$ are parallel in TM at m, and $e_{n+1},\ldots,e_{n+p}$ are parallel in $TM^\perp$ at m. In fact, we may assume $D_{e_i}D_{e_i}e_\alpha=0$,

$1 \le i \le n$, $n+1 \le \alpha \le n+p$, by supposing that e_i and e_α are obtained from $e_i(m)$ and $e_\alpha(m)$ by parallel translation along geodesics through m in TM and $TM^\perp$, respectively.

Now, $\Delta \mathscr{E} = -\sum_{i=1}^n e_i e_i \mathscr{E}$. In a neighborhood of m, we have

$$\begin{aligned} e_i \mathscr{E} &= \sum_{\alpha=n+1}^{n+p} e_{n+1} \wedge \cdots \wedge de_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &= -\sum_\alpha e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} + \sum_\alpha e_{n+1} \wedge \cdots \wedge D_{e_i}(e_\alpha) \wedge \cdots \wedge e_{n+p} \end{aligned}$$

using Weingarten's equation; we have set $A_{e_\alpha} = A_\alpha$. At m,

$$\begin{aligned} e_i e_i \mathscr{E} = &\sum_{\alpha \ne \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha \ne \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge D_{e_i} e_\beta \wedge \cdots \wedge e_{n+p} \\ &- \sum_\alpha e_{n+1} \wedge \cdots \wedge \nabla_{e_i} A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &- \sum_\alpha e_{n+1} \wedge \cdots \wedge h(e_i, A_\alpha(e_i)) \wedge \cdots \wedge e_{n+p} \\ &- \sum_{\alpha \ne \beta} e_{n+1} \wedge \cdots \wedge D_{e_i} e_\alpha \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_{\alpha \ne \beta} e_{n+1} \wedge \cdots \wedge D_{e_i} e_\alpha \wedge \cdots \wedge D_{e_i} e_\beta \wedge \cdots \wedge e_{n+p} \\ &- \sum_\alpha e_{n+1} \wedge \cdots \wedge A_{D_{e_i} e_\alpha}(e_i) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_\alpha e_{n+1} \wedge \cdots \wedge D_{e_i} D_{e_i} e_\alpha \wedge \cdots \wedge e_{n+p}. \end{aligned}$$

In the preceding equation, the third and fourth terms arise from Gauss' equation. Also, note that the second, fifth, sixth, seventh and eighth terms vanish since $D_{e_i} e_\alpha = 0$ and $D_{e_i} D_{e_i} e_\alpha = 0$ at m. Thus

$$\begin{aligned} \Delta \mathscr{E} = &-\sum_i \sum_{\alpha \ne \beta} e_{n+1} \wedge \cdots \wedge A_\alpha(e_i) \wedge \cdots \wedge A_\beta(e_i) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_i \sum_\alpha e_{n+1} \wedge \cdots \wedge \nabla_{e_i} A_\alpha(e_i) \wedge \cdots \wedge e_{n+p} \\ &+ \sum_i \sum_\alpha e_{n+1} \wedge \cdots \wedge h(e_i, A_\alpha(e_i)) \wedge \cdots \wedge e_{n+p}. \end{aligned}$$

We now consider each term on the right-hand side of the preceding equation separately.

$$-\sum_i \sum_{\alpha\neq\beta} e_{n+1}\wedge\cdots\wedge A_\alpha(e_i)\wedge\cdots\wedge A_\beta(e_i)\wedge\cdots\wedge e_{n+p}$$

$$=-\sum_{\alpha\neq\beta}\sum_{i,j,k} e_{n+1}\wedge\cdots\wedge h^\alpha_{ij}e_j\wedge\cdots\wedge h^\beta_{ik}e_k\wedge\cdots\wedge e_{n+p}$$

$$=-\sum_{\alpha\neq\beta}\sum_{j<k}\left[\sum_i h^\alpha_{ij}h^\beta_{ik}-h^\alpha_{ik}h^\beta_{ij}\right]e_{n+1}\wedge\cdots\wedge e^\alpha_j\wedge\cdots\wedge e^\beta_k\wedge\cdots\wedge e_{n+p}$$

$$=-\sum_{\alpha\neq\beta}\sum_{j<k}\langle R^\perp(e_j,e_k)e_\alpha,e_\beta\rangle e_{n+1}\wedge\cdots\wedge e^\alpha_j\wedge\cdots\wedge e^\beta_k\wedge\cdots\wedge e_{n+p}$$

where $R^\perp$ is the curvature tensor of $TM^\perp$.

$$\sum_i\sum_\alpha e_{n+1}\wedge\cdots\wedge \nabla_{e_i}A_\alpha(e_i)\wedge\cdots\wedge e_{n+p}$$

$$=\sum_\alpha\sum_{i,j} e_{n+1}\wedge\cdots\wedge\langle\nabla_{e_i}A_\alpha(e_j),e_i\rangle e_j\wedge\cdots\wedge e_{n+p}$$

$$\overset{(*)}{=}\sum_\alpha\sum_{i,j} e_{n+1}\wedge\cdots\wedge\langle\nabla_{e_j}A_\alpha(e_i),e_i\rangle e_j\wedge\cdots\wedge e_{n+p}$$

$$=\sum_\alpha\sum_j e_{n+1}\wedge\cdots\wedge n\,dH_\alpha(e_j)e_j\wedge\cdots\wedge e_{n+p}$$

$$=n\sum_\alpha e_{n+1}\wedge\cdots\wedge\nabla H_\alpha\wedge\cdots\wedge e_{n+p},$$

where (*) follows from the Equation of Codazzi, which holds since e_α is parallel in $TM^\perp$ at m. Also H_α and ∇H_α denote the mean curvature and the gradient of the mean curvature in the direction of e_α.

$$\sum_i\sum_\alpha e_{n+1}\wedge\cdots\wedge h(e_i,A_\alpha(e_i))\wedge\cdots\wedge e_{n+p}$$

$$=\sum_i\sum_{\alpha,\beta} e_{n+1}\wedge\cdots\wedge\langle h(e_i,A_\alpha(e_i)),e_\beta\rangle e_\beta\wedge\cdots\wedge e_{n+p}$$

$$=\sum_i\sum_\alpha\langle A_\alpha(e_i),A_\alpha(e_i)\rangle e_{n+1}\wedge\cdots\wedge e_{n+p}=|A|^2\mathscr{E}.$$

Thus,

$$\Delta\mathscr{E}=|A|^2\mathscr{E}+n\sum_\alpha e_{n+1}\wedge\cdots\wedge\nabla H_\alpha\wedge\cdots\wedge e_{n+p}$$
$$-\sum_{\alpha\neq\beta}\sum_{j<k}\langle R^\perp(e_j,e_k)e_\alpha,e_\beta\rangle e_{n+1}\wedge\cdots\wedge e^\alpha_j\wedge\cdots\wedge e^\beta_k\wedge\cdots\wedge e_{n+p}.$$

Thus $\Delta\mathscr{C} = n\mathscr{C}$ if and only if the following hold:
1) $|A|^2 = n$.
2) $\nabla H_\alpha = 0$, for all α; this is equivalent to η being parallel in $TM^\perp$.
3) $\langle R^\perp(e_j, e_k)e_\alpha, e_\beta\rangle = 0$, for all j, k, α, β; that is, $TM^\perp$ is flat.

We will show that 1) and 2) imply $x(M) = S^n \subset E^{n+1} \subset E^{n+p}$. We set $e_{n+1} = -\eta/H$. Then $\Delta(x/nH) = e_{n+1}$ since $\Delta x = -n\eta$. Thus $\int_M (a \cdot e_{n+1})\, dV = 0$ for all unit vectors a in E^{n+p}. Also $de_{n+1} = -A_{n+1}$ since e_{n+1} is parallel in $TM^\perp$. Using the minimum principle for λ_1 again, we have

$$n \leqq \int_M |d(a \cdot e_{n+1})|^2 \Big/ \int_M (a \cdot e_{n+1})^2 = \int_M |a \cdot A_{n+1}|^2 \Big/ \int_M (a \cdot e_{n+1})^2$$

or

$$n\int_M (a \cdot e_{n+1})^2 \leqq \int_M |a \cdot A_{n+1}|^2.$$

We integrate this inequality over all $a \in S^{n+p-1}$, use Fubini's Theorem, and obtain

$$nV \leqq \int_M |A_{n+1}|^2.$$

But $|A_{n+1}|^2 \leq |A|^2 = n$. Thus $|A_{n+1}|^2 = |A|^2 = n$. Let $e_{n+2}, \ldots, e_{n+p}$ be an orthonormal basis of the complement of e_{n+1} in $TM^\perp$. Then $|A_{n+2}| = |A_{n+3}| = \cdots = |A_{n+p}| = 0$. Thus $A_{n+2} = \cdots = A_{n+p} = 0$. Let $\mathscr{F}$ be the $p-1$ plane in $\mathscr{C}$ orthogonal to e_{n+1}. It is easy to show that $d\mathscr{F} = 0$ since $\mathscr{F} = e_{n+2} \wedge \cdots \wedge e_{n+p}$ and $A_\alpha = 0$ for $\alpha = n+2, \ldots, n+p$. Hence $\mathscr{F} = \text{const}$. Thus $x : M \to E^{n+1}$ where E^{n+1} is a $(n+1)$-plane orthogonal to $\mathscr{F}$.

We must now prove that if $x : M^n \to E^{n+1}$ is an immersion for which $|A|^2 = n$ and $H = \text{constant}$ then x embeds M as a standard sphere in E^{n+1}. Consider $y = x + \eta : M \to E^{n+1}$. Then $\Delta y = \Delta x + \Delta\eta = -n\eta + n\eta = 0$ since $\Delta\eta = |A|^2\, \eta = n\eta$. By Hopf's Theorem $y = \text{constant}$. Thus the image of x is in a sphere with center y and radius $|\eta| = H$. Clearly then $H = 1$. Since M is compact $x(M) = S^n$. For $n \geq 2$, this is enough to imply x is an embedding. For $n = 1$ we use the fact that M and its image have the same length, 2π, to prove x is an embedding.

It is natural now to ask what can be said about an isometric immersion $x : M \to E^{n+p}$ of a compact orientable n-dimensional manifold M into E^{n+p} when $\Delta\mathscr{C} = \lambda\mathscr{C}$, where λ is a constant, not necessarily λ_1. We will characterize such x when $n = 1$, 2, or $p = 1$, or M has nonnegative sectional curvature.

PROPOSITION. *Let M be a compact orientable n-dimensional manifold. Suppose $x: M \to E^{n+p}$ is an isometric immersion with $\Delta\mathscr{C} = \lambda\mathscr{C}$, where λ is a constant. Then the following hold:*

i) *if $n = 1$, x is a covering map onto a circle in a 2-plane of E^{1+p};*

ii) *if $n > 1$, and $p = 1$, then x embeds M as standard sphere in an $(n+1)$-plane of E^{n+p} and $\lambda = \lambda_1$;*

iii) *if $n = 2$, then M has constant nonnegative Gaussian curvature, and is characterized in* iv);

iv) *if M has nonnegative sectional curvature, then, identifying M with its image, M is a product submanifold, $M^{n_1} \times \cdots \times M^{n_k}$, where M^{n_i} is an n_i-sphere of radius r_i in an $n_i + 1$-plane of E^{n+p}; moreover, $\lambda = \sum_{i=1}^{k} n_i/r_i^2$.*

Proof. As above, $\Delta\mathscr{C} = \lambda\mathscr{C}$ if and only if 1) $|A|^2 = \lambda$, 2) η is parallel in $TM^\perp$, and 3) $TM^\perp$ is flat.

One can easily show that i) follows from 2).

The argument in the last paragraph of the proof of Theorem I proves ii).

The scalar curvature $r = n^2H^2 - |A|^2$. Hence for $n = 2$, M has constant Gaussian curvature. Since 2) and 3) hold we may conclude from Lemma 2.5 [2, p. 108] and Theorem 2.1 [2, p. 106] of Chen that either x maps M into a 3-plane, E^3, of E^{2+p} or x maps M into a 3-sphere, S^3, in a 4-plane of E^{2+p}. Moreover, M has constant mean curvature relative to S^3. If x maps M into E^3 then clearly M is a standard 2-sphere. If x maps M into S^3 as a minimal surface, then M is a standard sphere or is flat by a theorem of Lawson [4]. If x maps M into S^3 with constant nonzero mean curvature then M is a standard sphere or is flat by a theorem of Klotz and Osserman [2, p. 118].

Finally, iv) follows from 2) and 3) and the fact that M has nonnegative sectional curvature by a theorem of Erbacher, Yano and Ishihara [2, p. 139]. One may easily compute $\lambda = |A|^2 = \sum_{i=1}^{k} n_i/r_i^2$.

5. Proof of Theorem II

In proving Theorem II we use the fact that for the immersion $x: M \to E^m$, we have $\Delta x \equiv (\Delta x_1, \ldots, \Delta x_m) = -n\eta$ where Δ is the laplacian on M, $n = \dim M$, and $\eta =$ mean curvature vector of $M \subset E^m$. Thus for any unit vector $a \in E^m$, $\Delta(x \cdot a) = n\eta \cdot a$ and since $\int_M \Delta f = 0$ for any $f \in C^\infty(M)$, we have $\int_M \eta \cdot a = 0$. By the minimum principle for λ_1 we have $\lambda_1 \int_M (\eta \cdot a)^2 \leq \int_M |d(\eta \cdot a)|^2$. Now $d(\eta \cdot a)(X) = X(\eta \cdot a) = (-A_\eta(X) + D_X\eta) \cdot a = -A_\eta(X) \cdot a = -HA_e(X) \cdot a$ since $D_X\eta = 0$. Thus $|d(\eta \cdot a)|^2 = \sum_i H^2(A_e(e_i) \cdot a)^2$ and $\int_{S^{m-1}} |d(\eta \cdot a)|^2 \, da = H^2 \sum_i |A_e(e_i)|^2 \int_{S^{m-1}} (v \cdot a)^2 \, da$ where v is a fixed unit vector. Thus integrating both sides of the above inequality with respect to "a" and applying Fubini, we obtain

$$\lambda_1 H^2 \int_{S^{m-1}} (v \cdot a)^2 \, da \int_M dV = \lambda_1 \int_{S^{m-1}} \int_M (\eta \cdot a)^2 \, dV \, da$$

$$\leq \int_{S^{m-1}} \int_M |d(\eta \cdot a)|^2 \, dV \, da = H^2 \int_{S^{m-1}} (v \cdot a)^2 \, da \int_M \sum_i |A_e(e_i)|^2$$

and dividing by $H^2 \int_{S^{m-1}} (v \cdot a)^2 \, da$ gives us $\lambda_1 \operatorname{vol}(M) \leq \int_M |A_e|^2$.

Remark. There are indications that in certain cases the upper bound of Theorem I is not so good, especially when M is close to being "extrinsically creased." For example, if one considers a family of oblate spheroids in E^3 with fixed intrinsic diameter but with minor axes approaching 0, the upper bounds of Theorem I approach infinity. However, by a result of Cheeger, λ_1 is bounded by $k[\operatorname{diam} M]^{-2}$ for some large constant k depending only on the dimension of M [1, p. 189].

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