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**Autor:** Fait, Maria / Krzyz, J.G. / Zygmunt, Jadwiga  
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# Explicit Quasiconformal Extensions for some Classes of Univalent Functions

MARIA FAIT, JAN G. KRZYŻ AND JADWIGA ZYGMUNT

## 1. Introduction. Notations

Let  $S$  be the class of functions analytic and univalent in  $\Delta = \{z : |z| < 1\}$  for which  $f(0) = f'(0) - 1 = 0$ .

We say that  $f \in S_k$ ,  $0 \leq k < 1$ , if  $f \in S$  and  $f$  has a quasiconformal extension on the whole plane  $\mathbb{C}$  with complex dilatation  $\mu_f = f_{\bar{z}}/f_z$  that satisfies  $|\mu_f(z)| \leq k$  almost everywhere in  $\mathbb{C}$ . The symbols  $f_z, f_{\bar{z}}$  denote formal derivatives of  $f$ .

Let  $S^*(\alpha)$ ,  $0 \leq \alpha < 1$ , denote the subclass of  $S$  consisting of strongly starlike functions of order  $\alpha$ , cf. [1], [5], i.e. of functions  $f$  that satisfy:

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \alpha \frac{\pi}{2}, \quad z \in \Delta. \tag{1}$$

As shown in [1],  $f(\Delta)$  is a Jordan domain for any  $f \in S^*(\alpha)$ .

In this paper we find an explicit quasiconformal extension for an arbitrary function  $f \in S^*(\alpha)$ . We show that  $S^*(\alpha) \subset S_k$ , where  $k \leq \sin \alpha\pi/2$ .

We construct this extension by means of an auxiliary mapping which may be called a reflection with respect to a starlike Jordan curve (Lemma 1). In what follows we call a  $k$ -circle a Jordan curve that is a homeomorphic image of the unit circumference under a quasiconformal mapping  $F$  of the extended plane  $\mathbb{C}$  onto itself whose complex dilatation  $\mu_F$  satisfies  $|\mu_F(z)| \leq k < 1$  a.e.

We obtain explicit quasiconformal extensions for bounded convex functions and for functions with bounded boundary rotation (Theorems 3,4). In particular we show that any convex Jordan curve contained in an annulus  $\{w : r \leq |w| \leq R\}$  is a  $k$ -circle with  $k \leq \sqrt{1 - (r/R)^2}$ .

Similarly, any strongly starlike curve of order  $\alpha$  is a  $k$ -circle with  $k \leq \sin \alpha\pi/2$ .

## 2. Quasiconformal Extension for the Class $S^*(\alpha)$

In this section we shall prove

**THEOREM 1.** *If  $f \in S^*(\alpha)$ ,  $0 \leq \alpha < 1$ , then the mapping  $F$  defined by the formula*

$$F(z) = \begin{cases} f(z) & \text{for } |z| \leq 1 \\ |f(\xi)|^2 / f\left(\frac{1}{\bar{z}}\right), & \text{for } |z| \geq 1, \end{cases} \quad (2)$$

where  $\xi$  satisfies the conditions:  $|\xi| = 1$ ,  $\arg f(\xi) = \arg f(1/\bar{z})$ , belongs to the class  $S_k$  and  $|\mu_F(z)| \leq k = \sin \alpha\pi/2$  a.e.

We first prove

**LEMMA 1.** *Suppose that  $G$  is a domain bounded by a Jordan curve  $\Gamma$  starlike with respect to the origin. Suppose, moreover, that*

$$w = R(\varphi)e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi, \quad (3)$$

is the parametric equation of  $\Gamma$ , where  $R(\varphi)$  is absolutely continuous and positive,  $R(0) = R(2\pi)$  and

$$|R'(\varphi)|[R^2(\varphi) + R'^2(\varphi)]^{-1/2} \leq k < 1 \quad (4)$$

almost everywhere in  $[0, 2\pi]$ . Then the mapping

$$\phi(w) = R^2(\varphi)/\bar{w}, \quad \varphi = \arg w, \quad (5)$$

is an antiquasiconformal mapping of  $G$  onto  $\hat{\mathbb{C}} \setminus \bar{G}$  whose complex dilatation  $\phi_w/\phi_{\bar{w}}$  is bounded by  $k$  in absolute value.

*Proof.* Obviously  $\phi$  is a sense-reversing homeomorphism in  $G$ . Moreover, if  $w = re^{i\varphi}$ ,  $0 < r < R(\varphi)$ , then

$$\Phi(w) = \Phi(re^{i\varphi}) = R^2(\varphi)e^{i\varphi}/r$$

and we have for almost all  $\varphi$  in  $[0, 2\pi]$ :

$$\mu_\phi = \frac{\phi_w}{\phi_{\bar{w}}} = e^{-2i\varphi} \frac{\phi_r + \phi_\varphi/ir}{\phi_r - \phi_\varphi/ir} = -e^{-2i\varphi} \frac{R'(\varphi)}{R'(\varphi) + iR(\varphi)}$$

so that  $|\mu_\phi(w)| \leq k$  almost everywhere in  $G$  by (4).

We now prove that  $\phi$  has the ACL-property in  $G \setminus \{0\}$ . The function  $R^2(\varphi)e^{i\varphi}/r$  is absolutely continuous in  $\varphi$  with fixed  $r > 0$  because by (4)  $R'(\varphi)$  is essentially bounded and also absolutely continuous in  $r$ ,  $r \in [\delta, R(\varphi)]$ , for fixed  $\varphi$ ,  $\delta > 0$ . Thus the ACL-property holds in the  $\log w$ -plane. Since the ACL-property is invariant under composition with conformal mapping,  $\phi$  has in fact the ACL-property in  $G \setminus \{0\}$ . This ends the proof.

The condition (4) has a simple geometrical interpretation. Suppose that  $R'(\varphi)$  does exist. Then  $\Gamma$  has a tangent intersecting the radius vector at an angle  $\psi = \arctan R/R'$  and consequently

$$R'(R^2 + R'^2)^{-1/2} = \cos \psi.$$

Hence (4) means that the angle  $\psi$  is bounded away from 0 and  $\pi$  at points where the tangent does exist.

The mapping  $\phi(w)$  will be called a reflection with respect to the starshaped curve  $\Gamma$ . It is a sense-reversing homeomorphism for any starshaped Jordan curve  $\Gamma$ . Moreover, if the angle between the radius vector and the tangent of  $\Gamma$  is bounded away from 0 and  $\pi$  a.e., the reflection  $\phi(w)$  is an anti-quasiconformal mapping.

*Proof of Theorem 1.* If  $f \in S^*(\alpha)$  with  $0 \leq \alpha < 1$  then  $f$  has a continuous extension on  $\bar{\Delta}$ ,  $f(e^{i\theta})$  is absolutely continuous and  $d/d\theta f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta})$  a.e. in  $[0, 2\pi]$ , cf. [1]. Hence the definition of  $F$  in (2) makes sense. Let  $\Gamma$  be the Jordan curve  $w = f(e^{i\theta}) = R(\varphi)e^{i\varphi}$ ,  $0 \leq \theta \leq 2\pi$ . After differentiation with respect to  $\theta$  of the identity:

$$\log f(e^{i\theta}) = \log R(\varphi) + i\varphi,$$

we obtain

$$\frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} = \left[ 1 - \frac{iR'(\varphi)}{R(\varphi)} \right] \frac{d\varphi}{d\theta}$$

and hence by (1)

$$\left| \arg \left\{ 1 - i \frac{R'(\varphi)}{R(\varphi)} \right\} \right| \leq \frac{\alpha\pi}{2},$$

or

$$|R'(\varphi)| [R^2(\varphi) + R'^2(\varphi)]^{-1/2} \leq \sin \frac{\alpha\pi}{2} \quad \text{a.e.}$$

This means that  $\Gamma$  satisfies the condition (4) with  $k = \sin \alpha\pi/2$  and therefore the reflection with respect to  $\Gamma$  is anti-quasiconformal with complex dilatation bounded by  $\sin \alpha\pi/2$ . Now, the mapping  $F(z)$  for  $|z| > 1$  is composed of the following mappings: reflection in  $|z| = 1$ , conformal mapping  $f$  and a reflection with respect to  $\Gamma$ . Complex dilatation of  $F$  has the form

$$\mu_F = \left( \frac{z}{\bar{z}} \right)^2 \frac{f'(1/\bar{z}) \phi_w}{f(1/z) \phi_{\bar{w}}}.$$

Therefore  $F$  is a quasiconformal mapping in  $\{z : |z| > 1\}$  and  $|\mu_F(z)| \leq \sin \alpha\pi/2$  a.e. by Lemma 1. Obviously  $F$  as defined by (2), is a homeomorphism of the sphere  $\hat{\mathcal{C}}$  onto itself which is conformal in  $\Delta$  and quasiconformal in  $\mathcal{C} \setminus \bar{\Delta}$ . Since  $\partial\Delta$ ,  $\{\infty\}$ ,  $\{0\}$  are removable sets, cf. [4],  $F$  is quasiconformal in  $\hat{\mathcal{C}}$ .

**COROLLARY 1.** *If  $\Gamma$  is a Jordan curve starshaped with respect to  $w = 0$  and intersecting the radius vectors at an angle bounded away from 0 and  $\pi$  by  $\beta\pi/2$ ,  $0 < \beta \leq 1$ , then  $\Gamma$  is a  $k$ -circle with  $k \leq \cos \beta\pi/2$ .*

### 3. Some Applications of Theorem 1

It is well-known that, if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{in } \Delta \tag{5}$$

and

$$\sum_{n=2}^{\infty} n |a_n| \leq 1 \tag{6}$$

then  $f$  is a starlike univalent function. The condition (6) does not imply the possibility of quasiconformal extension of  $f$  (e.g.  $f(z) = z + \frac{1}{2}z^2$  satisfies (6) and obviously has no quasiconformal extension on  $\hat{\mathbb{C}}$ ).

Consider the class  $\tilde{S}(k)$  of functions  $f$  of the form (5) that satisfy the condition

$$\sum_{n=2}^{\infty} n |a_n| \leq k < 1. \tag{6'}$$

We prove

LEMMA 2. *If  $f \in \tilde{S}(k)$ , then  $f \in S^*(\alpha)$  with  $\alpha = (2/\pi) \arcsin k$ .*

*Proof.* The condition (6') implies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq k,$$

because

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|} \leq k.$$

Hence  $f$  satisfies (1) with  $\alpha = (2/\pi) \arcsin k$ .

From Lemma 2 and Theorem 1 we immediately obtain

THEOREM 2. *If  $f \in \tilde{S}(k)$  then  $f \in S_k$ .*

Another quasiconformal extension of  $f \in \tilde{S}(k)$  can be obtained in a different way, similarly as in [2].

THEOREM 2'. *Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  belong to  $\tilde{S}(k)$ . Then the mapping  $G(z)$  defined by the formula*

$$G(z) = \begin{cases} z + \sum_{n=2}^{\infty} a_n z^n & \text{for } |z| \leq 1, \\ z + \sum_{n=2}^{\infty} a_n \bar{z}^{-n} & \text{for } |z| \geq 1 \end{cases} \tag{7}$$

*is a quasiconformal extension of  $f$  onto  $\hat{\mathbb{C}}$  and  $|\mu_G(z)| \leq k$ .*

The mapping  $G$  satisfies the following condition:

$$|z_1 - z_2| (1 - k) \leq |G(z_1) - G(z_2)| \leq |z_1 - z_2| (1 + k) \tag{8}$$

for  $z_1, z_2 \in \Delta$  and also for  $z_1, z_2 \in \mathbb{C} \setminus \bar{\Delta}$ . It is well known that a function lipschitzian in  $\Delta$  has a continuous extension on  $\bar{\Delta}$  that satisfies (8) also in  $\bar{\Delta}$ . Hence  $G$  as defined by (7) is a sense-preserving homeomorphism in  $\hat{\mathcal{C}}$ . Its complex dilatation satisfies

$$|\mu_G(z)| = |G_{\bar{z}}/G_z| = \left| \sum_{n=2}^{\infty} n a_n z^{-n-1} \right| \leq \sum_{n=2}^{\infty} n |a_n| \leq k$$

in  $\mathbb{C} \setminus \bar{\Delta}$ . Since  $\partial\Delta$  is a removable set,  $G$  is a quasiconformal in  $\hat{\mathcal{C}}$ .

Let  $C(B)$  denote the subclass of  $S$  consisting of all convex functions for which  $|f(z)| \leq B, z \in \Delta$ . Next, let  $V_\lambda(B)$  denote the subclass of  $S$  consisting of all bounded functions  $|f(z)| \leq B$  for which  $f(\Delta)$  has boundary rotation at most  $\lambda\pi$ , cf. [3].

Moreover, let  $d_f$  denote the radius of the largest open disc centered at the origin which is contained in  $f(\Delta)$ .

In [1] Brañnan and Kirwan have found the following relations between  $C(B)$ ,  $V_\lambda(B)$  and  $S^*(\alpha)$ .

- (i) If  $f \in C(B)$ , then  $f \in S^*(\alpha)$  with  $\alpha = 1 - (2/\pi) \arcsin d_f/B$ .
- (ii) If  $f \in V_\lambda(B)$  and  $(\lambda - 2)\pi < 2 \arcsin (d_f/B)$ , then  $f \in S^*(\alpha)$  with  $\alpha = \lambda - 1 - (2/\pi) \arcsin (d_f/B)$ .

The above stated relations yield at once as immediate consequences of Theorem 1 the following results.

**THEOREM 3.** *If  $f \in C(B)$ , then  $f$  has a quasiconformal extension  $F$  on the whole plane defined by the formula (2) and*

$$|\mu_F(z)| \leq \sqrt{1 - \left(\frac{d_f}{B}\right)^2}.$$

**COROLLARY 2.** *If  $\Gamma$  is a convex Jordan curve contained in the annulus  $\{w : r \leq |w| \leq R\}$ , then  $\Gamma$  is a  $k$ -circle with  $k \leq \sqrt{1 - (r/R)^2}$ .*

**THEOREM 4.** *If  $f \in V_\lambda(B)$  and  $(\lambda - 2)\pi < 2 \arcsin (d_f/B)$ , then the function  $F$  defined by (2) is a quasiconformal extension of  $f$  and*

$$|\mu_F(z)| \leq \sin \left[ (\lambda - 1) \frac{\pi}{2} - \arcsin \frac{d_f}{B} \right].$$

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*Department of Mathematics*  
*Marie Curie-Skłodowska University*  
*Lublin, Poland*

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