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Projective k-invariants

MICHEAL N. DYER

1. Introduction

Let π be a group. A (π, m) -complex X is a finite connected m-dimensional CW complex having fundamental group π and trivial homotopy modules $\pi_i(X) = 0$ in dimensions i = 2, ..., m-1. A π -module π_m is said to be topologically realizable if $\pi_m \approx \pi_m(X)$ for some (π, m) -complex X. The classification problem for (π, m) -complexes is the problem of describing the set HT (π, m) of homotopy types of (π, m) -complexes.

For π a finite group of order n, $H^{m+1}(\pi; \pi_m) \cong Z_n$ as a ring. An important aspect in this classification is the boundary operator $\partial: Z_n^* = \text{Units}(H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z_{\pi}$, the (reduced) projective class group of the integral group ring Z_{π} , associated with the Milnor Mayer-Vietoris sequence in algebraic K-theory [10].

This arises as follows. The cellular chain complex $C_*(\tilde{X})$ of the universal cover \tilde{X} is a truncated resolution of the trivial π -module Z:

$$0 \longrightarrow \pi_m \longrightarrow C_m(\tilde{X}) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\epsilon} Z \longrightarrow 0.$$

The algebraic m-type T(X) of X is the triple $(\pi, \pi_m(X), k(X))$ where $k(X) \in H^{m+1}(\pi, \pi_m)$ is the k-invariant which arises by comparing the truncated resolution above with a standard resolution (see section 6; also [5], [6]). One can show that $k(X) \in \text{Units } (H^{m+1}(\pi; \pi_m))$; furthermore any $k \in \mathbb{Z}_n^*$ can be the k-invariant of a finitely generated truncated projective resolution

(*)
$$\mathcal{P}_k: 0 \to \pi_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to Z \to 0.$$

Also the assignment $(\pi, \pi_m, k) \rightarrow \text{Euler}$ characteristic $\chi(\mathcal{P}_k) = \sum_{i=0}^m (-1)^i [P_i]$ ([P] is the class of the projective P in $\tilde{K}_0 Z \pi$) is the negative of the Milnor boundary ∂ . Then (π, π_m, k) $(k \in \mathbb{Z}_n^*, m \ge 3)$ is the m-type of a (π, m) -complex iff $k \in \ker \partial$ [4].

The purpose of this paper is to generalize the above to groups other than finite groups.

- **1.1.** THEOREM. Let π be a group and m be an integer $m \ge 0$ such that $H^{m+1}(\pi; Z\pi) = 0$. Let π_m be any finitely generated topologically realizable π -module. Then
- (a) $H^{m+1}(\pi; \pi_m)$ has the structure of a ring with identity such that the units $U(H^{m+1}(\pi, \pi_m))$ are the projective k-invariants, i.e., those k-invariants realizable by a resolution of the form (*).
- (b) The function $\chi_m: U(H^{m+1}(\pi; \pi_m)) \to \tilde{K}_0 Z \pi$ which assigns to each $k \in U$ the Euler characteristic of a truncated resolution \mathcal{P}_k realizing the m-type (π, π_m, k) is a homomorphism.

We say that an m-type (π, π_m, k) comes from a (π, m) -complex if there exists a (π, m) -complex X such that $T(X) \cong (\pi, \pi_m, k)$ in the appropriate sense (see [4], [6] for a definition).

1.2. COROLLARY. If $m \ge 3$ and $H^{m+1}(\pi; Z\pi) = 0$, then $\ker \chi_m$ is the set of k-invariants which come from (π, m) -complexes.

The corollary follows from a theorem of J. Milnor [11, theorem 3.1] concerning the realizability of a resolution by a (π, m) -complex.

DEFINITION. The subgroup im $\chi_m \subset \tilde{K}_0 Z \pi$ is called the Swan subgroup of $\tilde{K}_0 Z \pi$ in dimension m.

If π is a finite group of order n, let $N = \sum_{x \in \pi} x \in Z\pi$ be the norm element. The left ideal (p, N) of $Z\pi$ is projective provided p is prime to n. For π finite, im $\chi_m = \text{im } \partial = \{[(p, N)] \in \tilde{K}_0 Z\pi \mid 1 \le p < n, (p, n) = 1\}$. If π is a (Poincaré) duality group of cohomological dimension m, then im $\chi_{m-i} = 0$ $(2 \le i \le m)$.

The Swan subgroup im χ_m is important because the Wall obstruction of any CW complex having fundamental group π and realizable π_m , which is dominated by a (π', m) -complex lies in im χ_m [12].

The organization of the paper is as follows. Let R be a ring. Section 2 gives certain constructions associated with the exact sequence of R-modules $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$. We say that P is K-projective if ∂ : End $(K) \rightarrow$ Ext (C, K) is surjective. Section 3 gives conditions under which Ext (C, K) inherits a ring structure from End (K), provided P is K-projective. Section 4 shows that elements in End (K) which determine K-projective extensions are right units in Ext (C, K). Section 5 studies conditions under which each K-projective element in End (K) is a unit in Ext (C, K). Theorem 1 is proved in section 6. In an appendix we study conditions under which $H^i(\pi; Z\pi) = 0$.

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2. Extensions as Pushouts and Pull-backs.

Let R be a ring. All modules are left R-modules. Let C be a given R-module and $\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of R-modules.

It is shown in [9, page 66] that given any module homomorphism $k: K \to K'$ there exists a module kP and a homomorphism $k\beta: P \to kP$ such that the following diagram commutes

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

Here the bottom row is exact also. kP is defined as the pushout of i and k.

Furthermore, given any module homomorphism $s: C \rightarrow C$, there exists a module Ps and a homomorphism $\beta s: Ps \rightarrow P$ such that the following diagram commutes

$$0 \longrightarrow K \xrightarrow{i^{s}} Ps \xrightarrow{j^{s}} C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

Ps is defined to be the pullback of j and s.

3. $\operatorname{Ext}_{R}(C, K)$ as a Ring.

Let R be a ring and

$$\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

be an exact sequence of (left) R-modules.

DEFINITION We say that P is K-projective if

$$i^*$$
: Ext¹_R $(P, K) \rightarrow$ Ext¹_R (K, K)

is a monomorphism.

Of course, it follows from the long exact sequence for $\operatorname{Ext}_R^i(-, K)$ [9, page 74] associated with ξ that P is K-projective iff the boundary operator $\partial : \operatorname{End}_R(K) \to \operatorname{Ext}_R^1(C, K)$ is surjective. Here $\partial(k)$ equals the equivalence class of the extension kP for any $k \in \operatorname{End}(K)$. If $\operatorname{Ext}_R(P, K) = 0$, then P is K-projective; in particular, any projective R-module is K-projective.

3.1. THEOREM. If $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ is an exact sequence of R-modules with P K-projective, then the boundary operator ∂ induces an isomorphism

$$\bar{\partial}: \frac{\operatorname{End}_{R}(K)}{i^{*}(\operatorname{Hom}_{R}(P,K))} \to \operatorname{Ext}_{R}^{1}(C,K).$$

For each $k \in \text{End}(K)$, let $\{k\}$ denote the element $\partial(k)$ in $\text{Ext}^1_R(C, K)$.

End (K) has a ring structure under composition. The question is: when is $B = i^* \operatorname{Hom}(P, K)$ a two-sided ideal? If we denote the composition $K \xrightarrow{\alpha} K \xrightarrow{\beta} K$ by $\beta \alpha$, then

$$B = {\alpha : K \rightarrow K \mid \alpha \text{ extends to a map } \alpha' : P \rightarrow K}$$

is always a left ideal. For, if $\alpha \in B$, $\beta \in \text{End}(K)$ and $\alpha' \in \text{Hom}(P, K)$ extends α , then $\beta \alpha'$ extends $\beta \alpha$. Thus B is a right ideal and $B \neq \text{End}(K)$ implies that Ext(C, K) is a ring with identity.

We will now delineate a sequence of sufficient conditions that imply that B is a right ideal.

3.2. (C). The composition in End(K) is commutative modulo B.

- **3.3.** (RE). Each homomorphism in End(K) extends to a homomorphism in End(P).
- **3.4.** (E). Each homomorphism in Hom(K, P) extends to a homomorphism in End(P).

Note that the following implications hold:

- $(E) \Rightarrow (RE) \Rightarrow B$ is a right ideal $\Leftarrow (C)$.
- **3.5.** If $\operatorname{Ext}(C, P) = 0$, then (E) is true. This follows because $\operatorname{Ext}(C, P) = 0$ implies $i^* : \operatorname{End}(P) \to \operatorname{Hom}(K, P)$ is surjective. If $\operatorname{Ext}(P, P) = 0$, then (E) is equivalent to $\operatorname{Ext}(C, P) = 0$. In particular, this is true if P is projective.
- **3.6.** Also, one can easily see that (RE) iff the boundary homomorphism $\partial: \operatorname{End}(C) \to \operatorname{Ext}(C, K)$ is surjective iff $j_*: \operatorname{Ext}(C, P) \to \operatorname{Ext}(C, C)$ is a monomorphism.

Note that Ext(C, K) is cyclic automatically implies (C).

We may call P C-injective if j_* : $\operatorname{Ext}(C, P) \to \operatorname{Ext}(C, C)$ is a monomorphism. Thus $\operatorname{Ext}(C, K)$ has a ring structure as above if P is C-injective and K-projective. More generally, we may proceed as follows: let P be K-projective.

DEFINITION. Let $\operatorname{Ext}(C, K)_K$ denote the subset of $\operatorname{Ext}(C, K)$ such that $\{k\} \in \operatorname{Ext}(C, K)_K$ iff $Bk \subset B$.

It is clear that

- (a) $\operatorname{Ext}(C, K)_K$ is a subgroup of $\operatorname{Ext}(C, K)$.
- (b) $\operatorname{Ext}(C, K)_K$ is a ring with identity under composition.
- (c) The image of the center of $\operatorname{End}(K)$ is contained in $\operatorname{Ext}(C,K)_K$.

 $\operatorname{Ext}(C, K)_K$ is called the maximal K-ring of $\operatorname{Ext}(C, K)$.

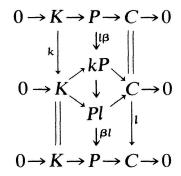
Let ∂_C : End $(C) \to \text{Ext}(C, K)$ be the boundary operator in the exact sequence for $\text{Ext}^i(C, -)$ associated with the extension $\xi: 0 \to K \to P \to C \to 0$. $\partial_C(r)$ is the equivalence class of the extension Pr (see 2.2).

3.7. PROPOSITION.

- (a) End (C) always induces a ring structure on the subgroup im $\partial_C = CExt(C, K)$.
 - (b) $_{C}$ Ext(C, K) is a subring of Ext(C, K) $_{K}$
 - (c) If ∂C is surjective, then $_{C}\text{Ext}(C, K) \cong \text{Ext}(C, K)_{K}$ as rings.

Proof.

(a) P is K-projective implies that im $\{j_*: \operatorname{Hom}(C, P) \to \operatorname{End}(C)\}$ is a two-sided ideal. This follows because each homomorphism in $\operatorname{End}(C)$ extends to a homomorphism in $\operatorname{End}(P)$. Consider $l \in \operatorname{End}(C)$ and the extension Pl. Then P is K-projective implies that there exists a $k \in \operatorname{End}(K)$ such kP and Pl are equivalent extensions. Thus there is an isomorphism $\alpha: kP \to Pl$ such that the following diagram commutes:



(b) Any $\{k\} \in \text{Ext}(C, K) \ (k \in \text{End}(K))$ which is in the image of ∂_C clearly satisfies $Bk \subset B$. Let $\partial_C(l) = \{k\}$. Then we may choose an extension as in (a) so that the following commutes

$$0 \to K \to P \to C \to 0$$

$$\downarrow \downarrow \qquad \downarrow \downarrow \downarrow \downarrow$$

$$0 \to K \to P \to C \to 0$$

Now $\alpha \in B$ iff α extends the zero map $0: C \rightarrow C$, i.e., the following diagram commutes:

$$0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$$

$$\stackrel{\alpha}{\downarrow} \qquad \stackrel{\beta}{\downarrow} \stackrel{\beta}{\downarrow} \qquad \stackrel{0}{\downarrow} 0$$

$$0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$$

But $\alpha \in B$ and $\{k\} \in \text{im } \partial_C \text{ implies that } \alpha \circ k \text{ extends } 0 \circ l = 0.$ Thus (b) is proved.

(c) follows easily from (a) and (b). We only note that the ring isomorphism is given by the correspondence $\partial_C(l) \mapsto \{k\}$ where $k \in \text{End}(K)$ extends $l \in \text{End}(C)$. This completes 3.7.

Note that ∂_C is surjective iff condition (RE).

We now give a simple example to show that B is not always a right ideal. Let R = Z and let the basic extension be given by

$$0 \longrightarrow Z \oplus Z \xrightarrow{i} Z \oplus Z \xrightarrow{j} Z_3 \oplus Z_2 \longrightarrow 0$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$K \qquad P \qquad C$$

where *i* has matrix $\binom{3}{0} \binom{0}{2}$ with respect to the natural bases. Then $B \subset \operatorname{End}(Z \oplus Z)$ is the set of all 2×2 matrices $\binom{a_{11}}{a_{21}} \binom{a_{12}}{a_{22}}$ over Z with the first column divisible by 3, the second by 2. $\operatorname{Ext}(C, K) \cong Z_3^2 \oplus Z_2^2$. Representatives of the cosets modulo B are given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \begin{array}{l} 0 \le a_{i1} \le 2 \\ 0 \le a_{i2} \le 1 \end{array}, \quad i = 1, 2 \right\}$$

It is easy to check that only the diagonal matrices in \Re have the property that $B \circ k \subset B$. Hence $\operatorname{Ext}(C, K)_K \cong Z_3 \oplus Z_2 \hookrightarrow \operatorname{Ext}(C, K)$ by embedding in the first and fourth coordinates.

4. K-Projective k-Invariants

Throughout this section we assume that i^* : End $(K) \rightarrow \text{Ext}(C, K)$ is surjective; i.e., that P is K-projective.

DEFINITION. The class $\{k\} \in \operatorname{Ext}(C, K)$ determined by $k \in \operatorname{End}(K)$ is called the k-invariant of the extension kP. A k-invariant $\{k\}$ is called K-projective if kP is a K-projective R-module. An element $k \in \operatorname{End}(K)$ is also called K-projective if $\{k\}$ is K-projective. Let $\mathscr{P}_K(\operatorname{Ext}(C, K))$ denote the set of K-projective k-invariants in $\operatorname{Ext}(C, K)$, $\mathscr{P}_K(\operatorname{End}(K))$ the set of K-projective elements $\operatorname{End}(K)$.

DEFINITION. Let E be a ring with identity. An element $\alpha \in E$ is a right unit if there exists $\beta \in E$ such that $\beta \alpha = 1$. The set of (right) units of E is denoted by (R)U(E).

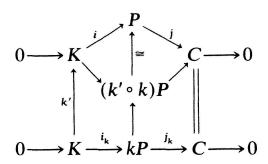
For each $\alpha \in E$, let α^* denote the abelian group homomorphism $E \to E$ given by right multiplication by α . α is a right unit iff α^* is surjective.

4.1. THEOREM. Let Ext(C, K) inherit a ring structure from End(K). $\{k\}$ is a K-projective k-invariant iff $\{k\}$ is a right unit.

Proof. Suppose that k is K-projective. Then $\partial_k : \operatorname{End}(K) \to \operatorname{Ext}(C, K)$ $(\partial_k(\alpha) = (\alpha \circ k)P, \alpha \in \operatorname{End}(K))$ is surjective. Thus there is a $k' \in \operatorname{End}(K)$ such that $(k' \circ k)P$ is equivalent to P as extensions. Hence $k' \circ k - 1 \in B$, and k is a right unit.

If $k' \circ k - 1 \in B$, we will show that kP is K-projective. P and $(k' \circ k)P$ are

equivalent extensions, so there is a commutative diagram



Call the resulting map $\beta: kP \rightarrow P$. Apply Ext (-, K) to this diagram to obtain the commutative diagram:

$$\operatorname{Ext}(C, K) \xrightarrow{j^{*}} \operatorname{Ext}(P, K) \xrightarrow{i^{*}} \operatorname{Ext}(K, K)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad$$

Thus $j_k^* = \beta^* j^* = 0$ because $j^* = 0$. Thus i_k^* is a monomorphism. This completes 4.1.

4.2. THEOREM. If $\{k \circ k'\} = \{k \circ k'\} = \{1\}$ in Ext (C, K), then Ext (kP, M) = 0 iff Ext (P, M) = 0, where M is an R-module.

If we were to define the "degree of projectivity" of k by the class of modules \mathcal{M}_k such that $M \in \mathcal{M}_k$ iff $\operatorname{Ext}(kP, M) = 0$, then the above says that $\{k\}$ is a unit implies that $\mathcal{M}_k = \mathcal{M}_1$; i.e., kP is "just as projective" as P is.

Proof. Because $k' \circ k - 1 \in B$, the argument of (4.1) implies the existence of the following commutative diagram:

$$0 \longrightarrow K \xrightarrow{i_{k}} kP \xrightarrow{j_{k}} C \longrightarrow 0$$

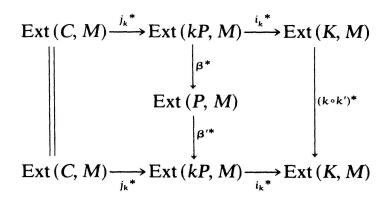
$$\downarrow k \qquad \qquad \downarrow \beta \qquad \qquad \parallel$$

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

$$\downarrow k \qquad \qquad \downarrow \beta \qquad \qquad \parallel$$

$$0 \longrightarrow K \xrightarrow{i_{k}} kP \xrightarrow{j_{k}} C \longrightarrow 0$$

Now $k \circ k' = 1 + \alpha' \circ i$, where $\alpha' \in \text{Hom}(P, K)$. Let M be any R-module such that Ext(P, M) = 0. Apply the functor Ext(-, M) to the above diagram.



The rows are exact at $\operatorname{Ext}(kP, M)$. $(k \circ k')^* = (1 + \alpha' \circ i)^* = 1 + (\alpha' \circ i)^* = 1$, since $(\alpha' \circ i)^* = 0$. Thus $\beta'^* \circ \beta^*$ is an isomorphism. Then $\operatorname{Ext}(P, M) = 0$ implies $\operatorname{Ext}(kP, M) = 0$. A similar argument shows the converse. This completes (4.2).

Since the set of right units is a semigroup under composition, the following is clear.

4.3. COROLLARY. Let $\operatorname{Ext}(C, K)$ have a ring structure as above. Then the set $\mathcal{P}_k(\operatorname{Ext}(C, K))$ of K-projective k-invariants is a semigroup with identity under composition. \mathcal{P}_k is a group iff each K-projective k-invariant is a unit.

5. k-Invariants as Units.

In this section we will study conditions under which right units are units in the ring Ext(C, K). We continue our assumption that P is K-projective. We also assume in this section that B is a right ideal.

DEFINITION. For each $k \in \text{End}(K)$, let $B_k = \text{im} \{ \text{Hom}(kP, K) \rightarrow \text{End}(K) \} = \text{ker} \{ \partial_k : \text{End}(K) \rightarrow \text{Ext}(C, K) \}$, where $\partial_k(\alpha) = (\alpha \circ k) P (\alpha \in \text{End}(K))$.

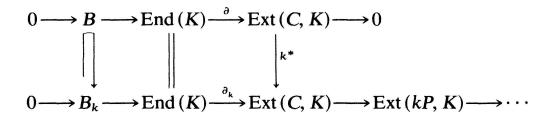
5.1. LEMMA. $B = \text{im} \{ \text{Hom} (P, K) \rightarrow \text{End} (K) \}$ is a right ideal iff $B \subseteq B_k$ for all $k \in \text{End} (K)$.

Proof. Let $\alpha \in B$. For any $k \in \text{End}(K)$, $\alpha \circ k \in B$ since B is a right ideal. Thus $(\alpha \circ k)P \cong \alpha(kP)$ is trivial implies that $\alpha \in B_k$. Conversely, if $B \subseteq B_k$ for all $k \in \text{End}(K)$, then let $\alpha \in B$, and consider $\alpha \circ k$ $(k \in \text{End}(K))$. $\alpha \in B_k$ implies $\alpha(kP) \cong (\alpha \circ k)P \cong K \times C$ which in turn implies that $\alpha \circ k \in B$.

We say that $\{k\} \in \text{Ext}(C, K)$ is a right zero divisor if there exists a $\{k'\} \neq 0$ such that $\{k' \circ k\} = 0$.

5.2. PROPOSITION. $\{k\} \in \text{Ext}(C, K)$ is not a right zero divisor iff $B = B_k$. If k is K-projective, then $\{k\}$ is a unit iff $B = B_k$.

Proof. For each $k \in \text{End}(K)$, let $k^* : \text{Ext}(C, K) \to \text{Ext}(C, K)$ be the function defined by right multiplication by $\{k\}$. It is a homomorphism of the underlying abelian group structure. Thus $\{k\}$ is not a right zero divisor iff k^* is a monomorphism. But k^* is a monomorphism iff $B = B_k$ follows from the following commutative diagram:



Here $\partial(\alpha) = \alpha P$, $\partial_k(\alpha) = \alpha(kP) = (\alpha \circ k)P$ and the horizontal sequences are exact. Furthermore, k^* is an isomorphism implies that ∂_k is surjective and hence $B = B_k$. $B = B_k$ together with ∂_k surjective implies k^* is an isomorphism.

5.3. LEMMA. Let $k \in \text{End}(K)$ and suppose there exists $k' \in \text{End}(K)$ such that $k' \circ k - 1 \in B$. Then $B = B_{k'}$.

Proof. Consider the homomorphisms k^* , k'^* as in the proof of (5.2). The composite $k^* \circ k'^* = (k' \circ k)^* = 1$. Thus k'^* is a monomorphism and, by (5.2), $B = B_{k'}$.

We will now give several conditions under which K-projective k-invariants are units. Clearly, if Ext(C, K) is commutative or has no zero divisors, then every right unit is a unit. Furthermore a theorem of N. Jacobson [7] shows that any ring having right units which are not units must be very large. The following is just a restatement of theorem 1 of [7].

5.4. THEOREM. If E = Ext(C, K) has either the ascending or descending chain condition for principal right ideals generated by idempotent elements, then right units are units.

Thus it follows that if E is finitely generated as a left (or right) E module, then right units are units in E. For example, if R is commutative and K is a finitely generated R-module, then $\operatorname{Ext}(C, K)$ is a finitely generated R-module and hence, by (5.4), right units are units.

Now let P be a projective R-module and consider any exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\iota_1} P_1 \xrightarrow{j_1} K \longrightarrow 0$$

of R-modules where P_1 is projective. The boundary operator

$$\partial: \operatorname{Ext}^{1}(C, K) \to \operatorname{Ext}^{2}(C, K_{1}) = \operatorname{Ext}^{1}(K, K_{1})$$

is given by $\partial(\{k\}) = \text{class of the extension } P_1 k \text{ (see 2.2)}.$

- **5.5.** THEOREM. If $\partial: \operatorname{Ext}^1(C, K) \to \operatorname{Ext}^2(C, K_1)$ is a monomorphism, then projective k-invariants are units in $\operatorname{Ext}(C, K)$.
- **5.6.** COROLLARY. If $\operatorname{Ext}(C, R) = 0$ and K is finitely generated as an R-module, then projective k-invariants are units in $\operatorname{Ext}(C, K)$.

The proof of (5.5) is postponed to (6.13). The corollary follows because K is finitely generated implies P_1 may be chosen to be finitely generated. Ext (C, R) = 0 then yields Ext $(C, P_1) = 0$ and this implies that ∂ is a monomorphism.

6. The k-Invariant of a Truncated Resolution.

Let M be an R-module. Choose a projective resolution

$$\mathscr{F}(M): \cdots \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

of M, where each C_i is projective R-module. $\mathscr{F}(M)$ is called the base resolution; each $\pi_m = \ker \partial_m (m \ge 0)$ is called an M-realizable R-module. If M = Z, the trivial R-module, then π_m is realizable means it is Z-realizable. We say that a resolution \mathscr{F} is of finite type if each C_i is a finitely generated R-module.

Let

$$\mathscr{G}(M): \cdots \longrightarrow G_m \xrightarrow{g_m} G_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \longrightarrow 0$$

be a (not necessarily projective) resolution of M. Let π'_m denote $\ker g_m$. The k-invariant of \mathcal{G} in dimension m relative to \mathcal{F} is the element $\{k\} \in \operatorname{Ext}_R^{m+1}(M, \pi'_m)$ determined by a chain map $f: \mathcal{F}(M) \to \mathcal{G}(M)$ covering the identity on M. Thus f is a sequence of maps making the following diagram commute:

quence of maps making the following diagram commute:
$$C_{m+1} \xrightarrow{\partial_{m+1}} C_m \xrightarrow{\partial_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

$$\downarrow^k \qquad \downarrow^{f_m} \qquad \downarrow^{f_{m-1}} \qquad \downarrow^{f_0} \qquad \downarrow^{f_0} \qquad \downarrow^{g_0} \qquad \downarrow^{$$

The map $k = f_m \circ \partial_{m+1} : C_{m+1} \to \pi'_m$ determines an element $\{k\} \in \operatorname{Ext}_R^{m+1}(M, \pi'_m)$. This is well-defined by a standard argument [5].

6.1. LEMMA. For each $m \ge 0$ and each element $\bar{k} \in \operatorname{Ext}_{R}^{m+1}(M, \pi'_{m}) \exists a$ resolution $\mathcal{G}_{\bar{k}}$ of M realizing \bar{k} . If C_{i} (i = 0, 1, ..., m) and π'_{m} are finitely generated, then $\mathcal{G}_{\bar{k}}^{(m)}$ may be chosen to be of finite type.

Proof. Consider $k: C_{m+1} \to \pi'_m$ realizing \bar{k} ; $k \cdot \partial_{m+2} = 0$ implies that k defines a map $k': \pi_m \to \pi'_m$. Use the construction of section 2 to build

$$0 \longrightarrow \pi_{m} \longrightarrow C_{m} \longrightarrow \pi_{m-1} \longrightarrow 0$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$\pi'_{m} \xrightarrow{i'} k' C_{m} \xrightarrow{j'} \pi_{m-1} \longrightarrow 0$$

Then the m-skeleton $\mathscr{G}_{\bar{k}}^{(m)}$ is given by

$$0 \longrightarrow \pi'_m \xrightarrow{i'} k' C_m \xrightarrow{\partial'_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

where ∂'_m is the composite $k'C_m \xrightarrow{j'} \pi_{m-1} \subset C_{m-1}$. This completes 6.1.

DEFINITION. An element $k \in \operatorname{Ext}^{m+1}(M, \pi'_m)$ is called *projective* if k can be realized as the k-invariant of a truncated projective resolution:

$$\mathcal{P}_{k}^{(m)}: 0 \to \pi'_{m} \to P_{m} \to P_{m-1} \to \cdots \to P_{0} \to M \to 0$$

when compared with the base resolution $\mathcal{F}(M)$. The set of projective k-invariants of $\operatorname{Ext}^{m+1}(M, \pi'_m)$ is denoted by $\mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi'_m))$.

6.2. THEOREM. Let M be any R-module and π_m be M-realizable for $m \ge 0$. Then

(a)
$$\operatorname{Ext}_{R}^{m+1}(M, \pi_{m}) \cong \frac{\operatorname{End}(\pi_{m})}{\operatorname{im}\operatorname{Hom}(C_{m}, \pi_{m})}$$
.

(b) If $B^m = \operatorname{im} \{ \operatorname{Hom} (C_m, \pi_m) \to \operatorname{End} (\pi_m) \}$ is a right ideal, then $\operatorname{Ext}^{m+1}(M, \pi_m)$ has a ring structure induced from that of $\operatorname{End} (\pi_m)$ such that the projective k-invariants lie between the units and right units of $\operatorname{Ext}^{m+1}(M, \pi_m)$:

$$U(\operatorname{Ext}^{m+1}(M, \pi_m)) \subset \mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m)) \subset RU(\operatorname{Ext}^{m+1}(M, \pi_m)).$$

(c) If B^m is a right ideal, $\mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m)) = U(\operatorname{Ext}^{m+1}(M, \pi_m))$, and each C_i

(i = 0, 1, ..., m + 1) a finitely generated free module, then the function

$$\chi_m: \mathscr{P}(\operatorname{Ext}^{m+1}(M; \pi_m)) \to \tilde{K}_0 R$$

which assigns to each $k \in \mathcal{P}$ the Euler characteristic $\chi_m(\mathcal{P}_k^{(m)}) = \sum_{i=0}^m (-1)^i [P_i] \in \tilde{K}_0 R$ of $\mathcal{P}_k^{(m)}$ is a homomorphism.

- Note. (1) Theorem 6.2 is theorem 1.1 in the case $R = Z\pi$ and M = Z. This follows because $H^{m+1}(\pi; Z\pi) = 0$ and C_m finitely generated implies that $H^{m+1}(\pi; C_m) = 0$. Thus $H^{m+1}(\pi; \pi_m)$ is a ring (3.5) and by (5.6) right units are units because π_m is finitely generated.
- (2) It follows from [11, theorem 3.1] that if $m \ge 3$, any π -module π_m realizable by a truncated *free* resolution over Z is topologically realizable as well.
- (3) It follows from (4.1) that the set \mathcal{P}_{π_m} of π_m -projective k-invariants is equal to the set of right units of $\operatorname{Ext}^{m+1}(M; \pi_m)$. Furthermore, (4.2) implies that any unit in $\operatorname{Ext}^{m+1}(M, \pi_m)$ must be projective. We do not know whether in general \mathcal{P} is distinct from U or RU (see 5.4).

The following lemma is useful in the subsequent work:

6.3. LEMMA OF COCKCROFT-SWAN [3, Appendix]. Let $\xi_i^{(m)}: 0 \to \pi_m \to E_m^i \to P_{m-1}^i \to \cdots \to P_0^i \to M \to 0$ (i = 1, 2) be resolutions of M with each P_i^i $(j = 0, 1, \ldots, m-1)$ projective. Let $f: \xi_1^{(m)} \to \xi_2^{(m)}$ be a chain map covering the identity on M and inducing an isomorphism on π_m . Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

Note the similarity between this and Schanuel's lemma [11].

6.4. COROLLARY. Let $\xi_1^{(m)}$ be projective (i.e., E_m^1 is projective) and suppose $k(\xi_1^{(m)}) = k(\xi_2^{(m)})$ when compared to \mathcal{F} . Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \cdots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \cdots$$

and hence $\xi_2^{(m)}$ is projective also.

Proof. By standard obstruction arguments, there exists a chain map $\xi_1^{(m)} \to \xi_2^{(m)}$ inducing the identity on M and π_m . Then apply (6.3).

Proof of 6.2. We will only show that if $\mathcal{P} = U$, then $\chi: \mathcal{P} \to \tilde{K}_0 R$ is a homomorphism. Let $k, k' \in \text{End}(\pi_m)$ represent projective k-invariants in $\text{Ext}^{m+1}(M; \pi_m)$. We will show that

$$(k' \circ k)C_m \oplus C_m \oplus C_{m+1} \cong kC_m \oplus k'C_m \oplus C_{m+1}.$$

Let $\partial k' \in \text{End}(\pi_{m+1})$ be any map determined by extending k':

$$0 - \pi_{m+1} \rightarrow C_{m+1} \rightarrow \pi_m \rightarrow 0$$

$$\downarrow^{\beta'_{m+1}} \downarrow^{k'}$$

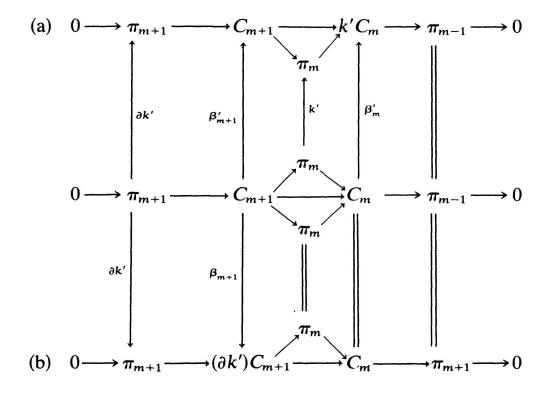
$$0 \rightarrow \pi_{m+1} \rightarrow C_{m+1} \rightarrow \pi_m \rightarrow 0$$

The correspondence $\{k'\} \rightarrow \{\partial k'\}$ gives the boundary homomorphism

$$\partial: \operatorname{Ext}^{m+1}(M; \pi_m) \to \operatorname{Ext}^{m+2}(M; \pi_{m+1}).$$

6.5. LEMMA. Let $k' \in \text{End}(\pi_m)$ be projective. Then $(\partial k')C_{m+1} \oplus k'C_m \cong C_m \oplus C_{m+1}$. Hence $(\partial k')C_{m+1}$ is projective and $[(\partial k')C_{m+1}] + [k'C_m] = 0$ in \tilde{K}_0R .

Proof. Consider the resolutions

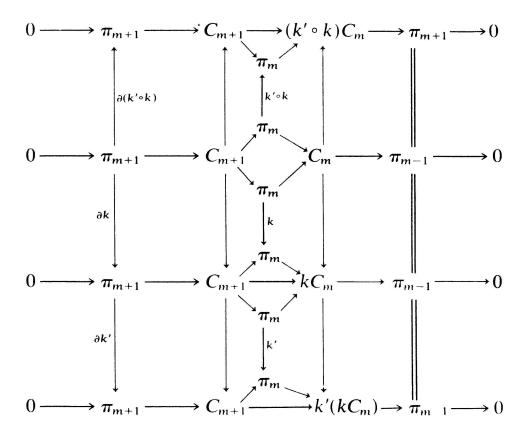


These resolutions (a) and (b) necessarily have the same k-invariant, (a) is projective; hence (b) is also projective by lemma 6.4. $(\partial k')C_{m+1} \oplus k'C_m \cong C_{m+1} \oplus C_m$ follows from (6.4).

6.6. LEMMA. k is projective and $k' \circ k - 1 \in B^m$ implies $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$.

Proof. Realize the k-invariant $\{\partial(k' \circ k)\} = \{\partial k' \circ \partial k\} \in \operatorname{Ext}^{m+2}(M; \pi_{m+1})$ in

three ways:

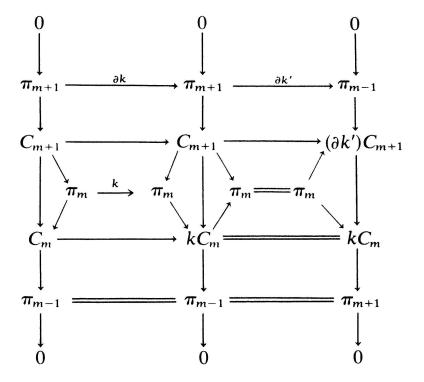


It follows that

$$(k' \circ k)C_m \cong k'(kC_m)$$

via a map inducing identity on π_{m-1} and π_m because the k-invariants are the same. Thus $\{\partial(k'\circ k)\}=\{\partial k'\circ \partial k\}$. Note that $k'\circ k$ is projective because it is a unit.

Furthermore, the following also has k-invariant $\partial k' \circ \partial k$:



Thus, by another application of lemma 6.4, we have $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$. (6.5) and (6.6) taken together prove (c).

CONJECTURE (see [11, lemma 6.1 (c)]).

$$(k' \circ k)C_m \oplus C_m \cong kC_m \oplus k'C_m$$
.

Let $\partial: \operatorname{Ext}^{m+1}(M, \pi_m) \to \operatorname{Ext}^{m+2}(M, \pi_{m+1})$ be the boundary operator in the coefficient exact sequence associated with the functor $\operatorname{Ext}^i(M, -)$ and the exact sequence

$$0 \to \pi_{m+1} \to C_{m+1} \to \pi_m \to 0.$$

The previous proof shows that ∂ is a ring homomorphism, provided the domain and range are rings.

Furthermore, we see that because C_i is finitely generated and free for $i = 0, \ldots, m+1$, then im $\chi_m \subset \text{im } \chi_{m+1}$. This follows from the commutative diagram:

$$\mathcal{P}(\operatorname{Ext}^{m+2}(M, \pi_{m+1})) \xrightarrow{\tilde{K}_0 R} \tilde{K}_0 R$$

$$\mathcal{P}(\operatorname{Ext}^{m+1}(M, \pi_m))$$

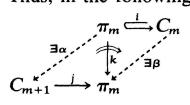
The conditions of section 3 have obvious analogs in this setting:

- **6.7.** (C(m)). The composition in End (π_m) is commutative modulo B^m .
- **6.8.** (RE_m) . Each map $k \in \operatorname{End}(\pi_m)$ extends to a map in $\operatorname{End}(C_m) \Leftrightarrow \partial : \operatorname{Ext}^m(M, \pi_{m-1}) \to \operatorname{Ext}^{m+1}(M, \pi_m)$ is surjective $\Leftrightarrow \operatorname{Ext}^{m+1}(M; C_m) \to \operatorname{Ext}^{m+1}(M; \pi_{m-1})$ is monic.
- **6.9.** (E_m) . Each map $f \in \text{Hom}(\pi_m, C_m)$ extends to a map in $\text{End}(C_m) \Leftrightarrow \text{Ext}^1(\pi_{m-1}, C_m) = \text{Ext}^{m+1}(M; C_m) = 0$

Again: $(E_m) \Rightarrow (RE_m) \Rightarrow B^m$ is a right ideal $\Leftarrow (C(m))$

At the present writing, I know of no examples where C(m) is not satisfied. We can "dualize" RE_m as follows:

6.10. (RE^m) . Any map $k \in \operatorname{End}(\pi_m)$ which coextends to C_{m+1} extends to C_m . Thus, in the following diagram,



the existence of α such that $j \circ \alpha = k$ implies the existence of a β such that $\beta \circ i = k$. The converse is always true because C_m is projective.

- **6.11.** PROPOSITION. Any map $k \in \text{End}(\pi_m)$ which coextends to C_{m+1} extends to C_m iff $\partial: \text{Ext}^{m+1}(M, \pi_m) \to \text{Ext}^{m+2}(M, \pi_{m+1})$ is a monomorphism iff $i_*: \text{Ext}^{m+1}(M, \pi_{m+1}) \to \text{Ext}^{m+1}(M, C_{m+1})$ is surjective.
- **6.12.** PROPOSITION. If each $k \in \text{End}(\pi_m)$ which coextends to C_{m+1} also extends to C_m , then $\text{Ext}^{m+1}(M; \pi_m)$ is a ring.

Proof. Let $k, \bar{k} \in \text{End}(\pi_m)$, let k extend to C_m . We must show that $k \circ \bar{k}$ extends to C_m . But k extends to C_m implies that k coextends to C_{m+1} by (6.10). Thus $k \circ k'$ coextends to C_{m+1} . But condition RE^m implies that $k \circ k'$ extends to C_m . This proves (6.12).

Note that $(RE_m) \Leftarrow (E_m) \Rightarrow (RE^m)$.

Notice that it follows from (6.6) that if $\{k\} \in \operatorname{Ext}^m(M, \pi_{m-1})$ is projective and $\{k' \circ k\} = 1$, then $\{\partial k'\} \in \operatorname{Ext}^{m+1}(M; \pi_m)$ is projective. Also, (6.5) implies that $\partial \{k\}$ is projective if $\{k\}$ is.

6.13. COROLLARY. If $\partial: \operatorname{Ext}^{m+1}(M; \pi_m) \to \operatorname{Ext}^{m+2}(M; \pi_{m+1})$ is a monomorphism (condition RE^m), then each projective k-invariant is a unit.

Proof. Let $\{k\} \in \operatorname{Ext}^{m+1}(M; \pi_m)$ be projective. By (5.3), there is a $k' \in \operatorname{End}(\pi_m)$ such that $k' \circ k' - 1 \in B^m$. Thus $\partial k' \circ \partial k - 1 \in B^{m+1}$. By (6.6), $\{\partial k'\}$ is projective. By (5.3) again, $\{\partial k \circ \partial k'\} = 1 = \{\partial k' \circ \partial k\}$. Since ∂ is a monomorphism, im ∂ a ring, and $\partial \{k \circ k'\} = \{\partial k \circ \partial k'\}$, then $\{k \circ k'\} = 1 = \{k' \circ k\}$. This completes (6.13).

The proof of the following corollary is similar to 6.13.

6.14. COROLLARY. If $\partial|_{\mathscr{P}}: \mathscr{P}(\operatorname{Ext}^m(M, \pi_{m-1}) \to \mathscr{P}(\operatorname{Ext}^{m+1}(M, \pi_m)))$ is surjective, then each projective k-invariant in $\operatorname{Ext}^{m+1}(M, \pi_m)$ is a unit.

Questions. (a) If M = Z, is B^m always a right ideal? For example, if $A(\pi)$ is the augmentation ideal in $Z\pi$, is $H^1(\pi; A(\pi))$ a ring?

(b) If B^m is a right ideal, is $\mathfrak{P}(\operatorname{Ext}^{m+1}(M; \pi_m))$ a semigroup under composition?

Appendix: Groups Having $H^i(\pi; Z\pi) = 0$

We will give some results that show that the hypothesis of theorem 1.1 is often satisfied.

- (a) If π is a finite group, then $H^i(\pi; Z\pi) = 0$ (i > 0). This follows because any projective π -module is weakly injective.
- (b) If π is a (Poincare) duality group with cohomological dimension m, then $H^i(\pi; Z\pi) = 0$ $(i \neq m)$ [1].
- (c) If F is a free abelian group of countable rank, then $H^{i}(F; ZF) = 0$ for all $i \ge 0$.
- (d) [1, Proposition 3.1] If S is a subgroup of G with finite index (not necessarily normal), then $H^i(S; ZS) \cong H^i(G; ZG)$ as right S-modules. Thus if S < G such that $[G:S] < \infty$, then $H^k(S; ZS) = 0 \Leftrightarrow H^k(G; ZG) = 0$.

For example, if $0 \rightarrow C \rightarrow G \rightarrow T \rightarrow 0$ is an exact sequence of groups where C is a group of cohomological dimension n and T is finite, then $H^{i}(G; ZG) = 0$ for i > n. Thus, any finitely generated abelian group A of rank n has $H^{i}(A; ZA) = 0$ for $i \ne n$.

(e) The following theorem is an easy consequence of the spectral sequence of a group extension: Let $1 \rightarrow N \rightarrow \pi \rightarrow G \rightarrow 1$ be an exact sequence of groups. Let N be finite. Then $H^i(\pi; Z\pi) \cong H^i(G; ZG)$ for all i > 0.

For example, if π is an extension of a finite group by a duality group of cohomological dimension n, then $H^i(\pi; Z\pi) = 0$ for $i \neq n$. Also any one relator group G[8] is such that $H^i(G; ZG) = 0$ for $i \geq 3$.

(f) We say that a group π has property \mathcal{P}^n if $H^i(\pi; Z\pi) = 0$, 0 < i < n. The functor $H^*(\pi, -)$ is strongly additive if it commutes with arbitrary direct sums. For example, if π admits a projective resolution of finite type

$$\cdots \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow Z \rightarrow 0$$

of the trivial π -module Z (i.e., each P_i is a finitely generated projective π -module), then $H^*(\pi; -)$ is strongly additive. The following is then true: Let $1 \to A \to \pi \to B \to 1$ be an exact sequence of groups such that $H^*(A; -)$ is strongly additive. Then A has \mathcal{P}^i and B has \mathcal{P}^j implies that π has \mathcal{P}^k , where $k = \min(i, j)$.

(g) Let n(G) denote the smallest integer $\leq \infty$ such that $H^i(G; ZG) = 0$ for all i > n(G). Let \mathcal{L} be the class of all groups G such that n(G) is finite. It follows easily from (d) and (e) that \mathcal{L} contains all polycyclic (= soluble with maximum condition on subgroups) groups. More generally, if \mathcal{L} is a class of groups, we say that a group G is poly (\mathcal{L}) if there exists a *finite* sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1$$

such that $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} is a member of \mathcal{A} . Let fcd denote the class of

groups of finite cohomological dimension. By the use of (d) and (e) one may show the following:

THEOREM. If G is poly (finitely generated abelian) or poly (finite or fcd) then G is a member of \mathcal{L} .

Furthermore, it follows from [13, page 138] that \mathcal{L} is closed under finite sums. It is closed under infinite sums provided that each of the summands G_i has $n(G_i) < k$, k being independent of i. \mathcal{L} is closed under amalgamated sums by [2]. If $G = \bigcup_{i \in \mathbb{Z}} G_i$ is a countable union of subgroups G_i such that $n(G_i) \le M < \infty$ for all $i \in \infty$, then $n(G) \le M + 1$ (R. Bieri). Thus any countable torsion group G has $n(G) \le 1$, because G is the countable union of finite subgroups. There are simple examples to show that \mathcal{L} is not closed under arbitrary direct limits.

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