Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 51 (1976)

Artikel: The Range of Atomless Group Valued Measures

Autor: Constantinescu, Corneliu

DOI: https://doi.org/10.5169/seals-39438

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 30.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

The Range of Atomless Group Valued Measures

CORNELIU CONSTANTINESCU

We prove the following results: (1) the range of an atomless group valued measure satisfying ccc is pathwise connected (Corollary 6; generalization of [2] Theorem 4); (2) the closure of the range of an atomless group valued measure is connected if it is compact (Theorem 3).

A δ -ring is a nonempty set \Re such that for any sequence $(A_n)_{n\in\mathbb{N}}$ in \Re we have $\bigcap_{n\in\mathbb{N}}A_n\in\Re$ and $A_0\triangle A_1\in\Re$. If moreover $\bigcup_{n\in\mathbb{N}}A_n\in\Re$ we call \Re a σ -ring. A semi-value on a commutative group G is a map p of G into \mathbb{R}_+ such that

$$p(0) = 0,$$
 $p(x + y) \le p(x) + p(y),$ $p(-x) = p(x)$

for any $x, y \in G$. Any family of semi-values on a commutative group G defines a group topology on G and any such topology is defined by the family of continuous semi-values.

Let \Re be a δ -ring and let G be a Hausdorff topological commutative group. A G-valued measure on \Re is a map μ of \Re into G such that for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in \Re whose union belongs to \Re we have

$$\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)=\sum_{n\in\mathbb{N}}\mu(A_n).$$

We set

$$\mathfrak{N}(\mu) := \{ A \in \mathfrak{R} \mid \forall B \in \mathfrak{R}, \ B \subset A \Rightarrow \mu(B) = 0 \}.$$

We say that μ satisfies locally ccc if any disjoint family in $\Re \backslash \Re(\mu)$ is countable if its union is contained in a set of \Re . Let $\Lambda(\mu)$ be the set of subsets $\Re \neq \phi$ of $\Re \backslash \Re(\mu)$ such that the intersection of any countable family in \Re belongs to \Re . The maximal elements of $\Lambda(\mu)$ (for the inclusion relation) will be called *atoms of* μ . Let \Re be an atom of μ and let $\Re(\Re)$ be the filter on \Re generated by the filter base

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

An atom $\mathfrak A$ of μ is called *improper* if $\mu(\mathfrak F(\mathfrak A))$ converges to 0; otherwise we call it proper. A measure possessing no proper atoms is called *atomless*.

Throughout this paper we shall denote by \Re a δ -ring and by G a Hausdorff topological commutative group. We consider \Re o Jered by the inclusion relation and denote by Λ the set of lower directed nonempty subsets of $\Re \setminus \{\phi\}$. For any $\mathfrak{A} \in \Lambda$ we denote by $\mathfrak{F}(\mathfrak{A})$ the filter on \Re generated by the filter base

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

PROPOSITION 1. Let μ be an atomless G-valued measure, let p be a continuous semi-value on G, and let u be the canonical map $G \to G/P^{-1}(0)$. Then $u \circ u$ is an atomless measure satisfying locally ccc.

 $p^{-1}(0)$ is a closed subgroup of G, $G/p^{-1}(0)$ is a Hausdorff topological commutative group, and $u \circ \mu$ is a measure on a δ -ring. Since $G/p^{-1}(0)$ possesses a coarser metrizable topology $u \circ \mu$ satisfies locally ccc. From $\Re(\mu) \subset \Re(u \circ \mu)$ we deduce by [1] Corollary 1.4 that $u \circ \mu$ is atomless.

PROPOSITION 2. Let μ be an atomless G-valued measure on \Re , let p be a continuous semi-value on G, and let $A \in \Re$. Then there exists an increasing map $B:[0,1] \to \Re$ such that $B(0) = \phi$, B(1) = A and such that $\mu \circ B$ is continuous with respect to the topology on G defined by p.

By Proposition 1 and [3] Proposition 2 there exists for any $n \in \mathbb{N}$ a family $(A_{n,i})_{0 < i \le k_n}$ of pairwise disjoint sets of \Re whose union is A and such that for any natural number $i \in]0, k_n]$ and for any $A' \in \Re$ contained in $A_{n,i}$ we have

$$p(\mu(A')) \leq \frac{1}{n}$$
.

We may even assume $k_n \ge 2$ for any $n \in \mathbb{N}$. We set for any $n \in \mathbb{N}$

$$l_n:=\prod_{m\leq n}k_m,$$

for any $i \in \mathbb{N}$, $0 < i \le l_0$,

$$A_{0,i}':=\bigcup_{j\leqslant i}A_{0,j},$$

and for any $n \in \mathbb{N}$, $A'_{n,0} := \phi$. We construct inductively for any $n \in \mathbb{N} \setminus \{0\}$ a family $(A'_{n,i})_{0 \le i \le l_n}$ by setting for any $i \in \mathbb{N}$, $0 \le i \le l_n$,

$$A'_{n,i} := A'_{n-1,i'} \cup (A'_{n-1,i'+1} \cap \left(\bigcup_{j \leqslant i-i'k_n} A_{n,j}\right),$$

where i' denotes the greatest natural number such that $i'k_n < i$. It can be shown inductively that the following properties hold for any $n \in \mathbb{N}$:

- (a) $A'_{n, l_n} = A$;
- (b) $0 < i \le j \le l_n \Rightarrow A'_{n,i} \subset A'_{n,i}$;

(c)
$$0 < i \le l_n$$
, $0 < j \le l_{n+1}$, $\frac{i}{l_n} = \frac{j}{l_{n+1}} \Rightarrow A'_{n,i} = A'_{n+1,j}$;
(d) $0 < i \le l_n$, $A' \in \Re$, $A' \subset A'_{n,i} \setminus A'_{n,i-1} \Rightarrow p(\mu(A')) \le \frac{1}{n}$.

(d)
$$0 < i \le l_n$$
, $A' \in \mathfrak{R}$, $A' \subset A'_{n,i} \setminus A'_{n,i-1} \Rightarrow p(\mu(A')) \le \frac{1}{n}$.

Let r be a rational number, $0 \le r \le 1$ for which there exists $n \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $0 \le i \le l_n$ and $(i/l_n) = r$. By (c) we may set

$$B(r) := A'_{n,i}$$

We have $B(0) = \phi$ and (by a)) B(1) = A. By b) $B(r) \subset B(r')$ for any $0 \le r \le r' \le 1$. This last property allows us to extend the domain of B by setting for any $\alpha \in [0, 1]$

$$B(\alpha) := \bigcap_{r \geq \alpha} B(r) \in \mathfrak{R}.$$

By d) the map $\mu \circ B$ is continuous with respect to the topology on G defined by p.

THEOREM 3. Let μ be an atomless measure on \Re such that for any $A \in \Re$ the set $\{\mu(B) \mid B \in \Re, B \subset A\}$ is compact (resp. relatively compact). Then $\mu(\Re)$ (resp. the closure of $\mu(\Re)$ is connected.

Let G be the target of μ , let $A \in \Re$, and let

$$\mathfrak{R}' := \{ B \in \mathfrak{R} \mid B \subset A \}.$$

By Proposition 2 for any continuous semi-value p on G there exists a map

$$f:[0,1] \to \overline{\mu(\mathfrak{R}')}$$

continuous with respect to the topology on $\overline{\mu(\Re')}$ defined by p and such that $\underline{f(0)} = 0$, $\underline{f(1)} = \mu(A)$. Hence $\mu(A)$ belongs to the connected component of 0 in $\overline{\mu(\Re')}$ (N. Bourbaki, nouvelle édition, TG II p. 32, Proposition 6). It follows that $\mu(A)$ belongs to the connected component of 0 in $\mu(\Re)$ (resp. $\overline{\mu(\Re)}$). Since A is arbitrary $\mu(\Re)$ (resp. $\overline{\mu(\Re)}$) is connected.

PROPOSITION 4. Let μ be an atomless G-valued measure on \Re , let A be an increasing map of [0, 1] into \Re , and let p be a continuous semi-value on G. Then there exists an increasing map B of [0, 1] into \Re such that

$$A([0,1]) \subset B([0,1])$$

and such that $\mu \circ B$ is continuous with respect to the topology on G defined by p.

Let G_p be the group G endowed with the topology defined by p, let M be the topological group $G_p/p^{-1}(0)$ and let u be the canonical map $G \to M$. By Proposition 1 $u \circ \mu$ is an atomless measure satisfying locally ccc. Let T be the set of $\alpha \in [0, 1]$ at which $u \circ \mu \circ A$ is not continuous from the left. For any $\alpha \in T$ we have

$$A(\alpha)\setminus\bigcup_{\beta<\alpha}A(\beta)\not\in\mathfrak{N}(u\circ\mu).$$

It follows that T is countable. Let $\alpha \in T$. By Proposition 2 there exists for any $\alpha \in T$ an increasing map A_{α} of [0, 1] into \Re such that

$$A_{\alpha}(0) = \phi, \qquad A_{\alpha}(1) = A(\alpha) \setminus \bigcup_{\beta < \alpha} A(\beta),$$

and such that $\mu \circ A_{\alpha}$ is continuous as a map in G_p . Let us endow the set

$$C := \{(\alpha, \beta) \in [0, 1] \times [0, 1] \mid \alpha \in T \text{ or } \beta = 0\}$$

with the lexicographical order relation. It is easy to see that C is order complete and contains a countable infinite subset which is dense in order. Moreover for any $a, b \in C$ with a < b there exists $c \in C$ with a < c < b. From these properties we

deduce that there exists a bijective map $\psi:[0,1] \to C$ which is an isomorphism of ordered sets. Let $t \in [0,1]$ and let $(\alpha,\beta) = \psi(t)$. If $\alpha \notin T$ we set

$$B(t) := A(\alpha);$$

if $\alpha \in T$ we set

$$B(t) := A_{\alpha}(\beta) \cup \Big(\bigcup_{\gamma < \alpha} A(\gamma)\Big).$$

Then B is an increasing map of [0, 1] into \Re such that

$$A([0,1]) \subset B([0,1])$$

and such that $\mu \circ B$ is continuous from the left as a map in G_p . Moreover if A is continuous from the right then B is continuous from the right.

If we repeat the same construction starting with B instead of A and replacing the continuity from the left by the continuity from the right we get a map with the required properties.

THEOREM 5. Let μ be an atomless G-valued measure on \Re satisfying locally ccc and let $A \in \Re$. Then there exists an increasing map $B:[0,1] \to \Re$ such that $B(0) = \phi$, B(1) = A and such that $\mu \circ B$ is continuous.

Assume the contrary and let ω_1 be the first uncountable ordinal. We construct inductively a family $(p_{\xi})_{\xi<\omega_1}$ of continuous semi-values on G and a family $(B_{\xi})_{\xi<\omega_1}$ of increasing maps of [0,1] into \Re such that we have for any $\xi<\omega_1$:

- (a) $B_{\xi}(0) = \phi$, $B_{\xi}(1) = A$;
- (b) $\mu \circ B_{\xi}$ is continuous with respect to the topology on G defined by $\{p_{\eta} \mid \eta < \xi\}$ and it is not continuous with respect to the topology on G defined by p_{η} ;
- (c) $\bigcup_{\eta < \xi} B_{\eta}$ ([0, 1]) $\subset B_{\xi}$ ([0, 1]).

Let $\xi < \omega_1$ and assume the families were constructed for all ordinals strictly smaller than ξ . The set

$$C = \bigcup_{\eta < \xi} B_{\eta}([0, 1])$$

is linearly ordered with respect to the inclusion relation and contains a countable subset which is dense in order. Hence there exists a subset M of [0, 1] and a

bijection $\psi: M \to C$ which is an isomorphism of ordered sets. We may easily extend ψ to an increasing map of [0, 1] to \Re . By Proposition 4 there exists an increasing map B_{ξ} of [0, 1] into \Re such that

$$\psi([0,1]) \subset B_{\varepsilon}([0,1])$$

and such that $\mu \circ B_{\xi}$ is continuous with respect to the topology on G defined by $\{p_{\eta} \mid \eta < \xi\}$. Since ϕ , $A \in \psi$ ([0, 1]) we may assume $B_{\xi}(0) = \phi$ and $B_{\xi}(1) = A$. Hence B_{ξ} fulfills a) and b). By the hypothesis of the proof $\mu \circ B_{\xi}$ is not continuous. Hence there exists a continuous semi-value p_{ξ} on G such that $\mu \circ B_{\xi}$ is not continuous with respect to the topology on G defined by p_{ξ} .

We set for any $\xi < \omega_1$ and for any $\alpha \in [0, 1]$

$$\bar{B}_{\xi}(\alpha) := A \cap \left(\bigcap_{\beta > \alpha} B_{\xi}(\beta)\right) \setminus \left(\bigcup_{\gamma < \alpha} B_{\xi}(\gamma)\right),$$

$$M_{\xi}:=G/p_{\xi}^{-1}(0),$$

and denote by φ_{ξ} the canonical map $G \to M_{\xi}$. By c) two sets of the type $\bar{B}_{\xi}(\alpha)$ either are disjoint or one of them is included in the other one. By b) there exists for any $\xi < \omega_1$ an $\alpha(\xi) \in [0, 1]$ such that $\bar{B}_{\xi}(\alpha(\xi)) \notin \Re(\varphi_{\xi} \circ \mu)$. By b) for any $\eta < \xi$ we have $\bar{B}_{\xi}(\alpha(\xi)) \in \Re(\varphi_{\eta} \circ \mu)$. Let us denote by M_0 (resp. M_1) the set of $\xi < \omega_1$ for which the set

$$\{\eta < \omega_1 \mid \bar{B}_{\eta}(\alpha(\eta)) \subset \bar{B}_{\xi}(\alpha(\xi))\}$$

is countable (resp. uncountable). We set for any $\xi \in M_0$

$$C_{\xi} := \bar{B}_{\xi}(\alpha(\xi)) \setminus \bigcup_{\substack{\eta < \omega_1 \\ \eta > \xi}} \bar{B}_{\eta}(\alpha(\eta)).$$

Since $(C_{\xi})_{\xi \in M_0}$ is a family of pairwise disjoint sets of $\Re \backslash \Re(\mu)$ and since μ satisfies locally $\operatorname{ccc} M_0$ is countable. We may therefore construct a strictly increasing family $(\zeta(\xi))_{\xi < \omega_1}$ of elements of M_1 such that

$$\bar{B}_{\zeta(\eta)}(\alpha(\zeta(\eta))) \subset \bar{B}_{\zeta(\xi)}(\alpha(\zeta(\xi)))$$

for any ξ , η such that $\xi < \eta < \omega_1$. Then

$$(\bar{B}_{\zeta(\xi)}(\alpha(\zeta(\xi))) \setminus \bar{B}_{\zeta(\xi+1)}(\alpha(\zeta(\xi+1))))_{\xi < \omega_1}$$

is a family of pairwise disjoint sets of $\Re \setminus \Re(\mu)$ contained in A and this contradicts the hypothesis that μ satisfies locally ccc.

COROLLARY 6. If μ is an atomless measure on \Re satisfying locally ccc then $\mu(\Re)$ is pathwise connected.

Remark. D. Landers ([2] Theorem 4) showed that $\mu(\Re)$ is pathwise connected if there exists an atomless submeasure $\lambda : \Re \to [0, \infty[$ dominating μ . In this case μ is atomless and satisfies locally ccc (since λ satisfies locally ccc and $\Re(\lambda) \subset \Re(\mu)$).

REFERENCES

- [1] CONSTANTINESCU, C. Atoms of group valued measures. Comment. Math. Helvetici, 51 (1976), 191-205.
- [2] LANDERS, D. Connectedness properties of the range of vector and semi-measures. Manuscripta Math., 9 (1973), 105-112.
- [3] MUSIAŁ, K. Absolute continuity and the range of group valued measure. Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys., 21 (1973). 105-113.

ETH Mathematisches Seminar 8006 Zürich, Switzerland

Received July 3, 1975