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# Atoms of Group Valued Measures

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Any real valued measure may be written (uniquely) as a sum of an atomic and of an atomless measure. This result was extended first time by J. Hoffmann-Jørgensen ([1] Theorem 6) to absolutely continuous measures with values in locally convex spaces and then by K. Musiał ([2] Theorem 1) to the more general case of group valued measures satisfying *ccc*. The aim of this paper is to extend this result once more for group valued measures satisfying *asc* (Proposition 3.2 and Theorem 3.6). Any measure satisfying *ccc* (even locally) satisfies *asc* (Proposition 3.4). If the group  $G$  is complete any  $G$ -valued measure satisfies *asc* (Propositions 3.5 and 2.1). The nature of atoms for measures not satisfying *ccc* becomes by far more complicated and so a great part of this paper is dedicated to their study.

Throughout this paper we shall denote by  $\mathfrak{R}$  a  $\delta$ -ring (i.e.  $\mathfrak{R} \neq \emptyset$  and for any sequence  $(A_n)_{n \in \mathbb{N}}$  in  $\mathfrak{R}$  we have  $\bigcap_{n \in \mathbb{N}} A_n \in \mathfrak{R}$  and  $A_0 \Delta A_1 \in \mathfrak{R}$ ) by  $\mathfrak{A}$  a subset of  $\mathfrak{R}$  such that the union of any finite family in  $\mathfrak{A}$  belongs to  $\mathfrak{A}$ , and by  $G$  a Hausdorff topological commutative group. We consider  $\mathfrak{R}$  ordered by the inclusion relation and denote by  $\Lambda$  the set of lower directed nonempty subsets of  $\mathfrak{R} \setminus \{\emptyset\}$ . For any  $\mathfrak{A} \in \Lambda$  we denote by  $\mathfrak{F}(\mathfrak{A})$  the filter on  $\mathfrak{R}$  generated by the filter base

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

A system of null sets of  $\mathfrak{R}$  is a nonempty subset  $\mathfrak{N}$  of  $\mathfrak{R}$  such that: (a) any set of  $\mathfrak{R}$  belongs to  $\mathfrak{N}$  if it is contained in a set of  $\mathfrak{N}$ ; (b) the union of any countable family in  $\mathfrak{N}$  belongs to  $\mathfrak{N}$  if it belongs to  $\mathfrak{R}$ .

## 1. Atoms

Throughout this section we shall denote by  $\mathfrak{N}$  a system of null sets of  $\mathfrak{R}$ . We denote by  $\Lambda(\mathfrak{N})$  the set of subsets  $\mathfrak{A} \neq \emptyset$  of  $\mathfrak{R} \setminus \mathfrak{N}$  such that the intersection of any countable family in  $\mathfrak{A}$  belongs to  $\mathfrak{A}$ . It is obvious that  $\Lambda(\mathfrak{N}) \subset \Lambda$ . The maximal elements of  $\Lambda(\mathfrak{N})$  (for the inclusion relation) will be called *atoms* (with respect to

$\mathfrak{N}$ ). A *set-atom* (with respect to  $\mathfrak{N}$ ) is a set  $A \in \mathfrak{R} \setminus \mathfrak{N}$  such that for any subset  $B$  of  $A$  belonging to  $\mathfrak{N}$  we have either  $B \in \mathfrak{N}$  or  $A \setminus B \in \mathfrak{N}$ . We say that  $\mathfrak{N}$  satisfies the *countable chain condition* (abbreviated *ccc*) if any disjoint family  $(A_\iota)_{\iota \in I}$  in  $\mathfrak{R} \setminus \mathfrak{N}$  (i.e.  $A_\iota \cap A_{\iota'} = \emptyset$  for different elements  $\iota, \iota'$  of  $I$ ) is countable. We say that  $\mathfrak{N}$  satisfies *locally ccc* if any disjoint family  $(A_\iota)_{\iota \in I}$  in  $\mathfrak{R} \setminus \mathfrak{N}$  is countable if  $\bigcup_{\iota \in I} A_\iota$  is contained in a set of  $\mathfrak{R}$ .

**PROPOSITION 1.1.** *Assume  $\mathfrak{N}$  satisfies locally ccc and let  $\mathfrak{A} \in \Lambda$ . Then there exists a decreasing sequence  $(A_n)_{n \in \mathbf{N}}$  in  $\mathfrak{A}$  such that we have for any  $A \in \mathfrak{A}$*

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \in \mathfrak{N}.$$

Assume that for any decreasing sequence  $(A_n)_{n \in \mathbf{N}}$  in  $\mathfrak{A}$  there exists  $A \in \mathfrak{A}$  such that

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \notin \mathfrak{N}.$$

Let  $\omega_1$  be the first uncountable ordinal number. We shall construct inductively a family  $(B_\xi)_{\xi < \omega_1}$  in  $\mathfrak{A}$  such that

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}$$

for any  $\mu < \omega_1$ . Let  $\mu < \omega_1$  and assume the family  $(B_\xi)_{\xi < \mu}$  with the required property is constructed. Since this family is countable and since  $\mathfrak{A}$  is lower directed there exists a decreasing sequence  $(A_n)_{n \in \mathbf{N}}$  in  $\mathfrak{A}$  such that

$$\bigcap_{n \in \mathbf{N}} A_n \subset \bigcap_{\xi < \mu} B_\xi.$$

By the hypothesis there exists  $B_\mu \in \mathfrak{A}$  with

$$\bigcap_{n \in \mathbf{N}} A_n \setminus B_\mu \notin \mathfrak{N}.$$

Hence

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}.$$

The existence of the family  $(\bigcap_{\xi < \mu} B_\xi \setminus B_\mu)_{\mu < \omega_1}$  contradicts the hypothesis that  $\mathfrak{N}$  satisfies locally *ccc*. ■

PROPOSITION 1.2. For any  $A \in \mathfrak{R} \setminus \mathfrak{N}$  we set

$$\mathfrak{A}_A := \{B \in \mathfrak{R} \mid A \setminus B \in \mathfrak{N}\}$$

Then:

(a)  $\mathfrak{A}_A \in \Lambda(\mathfrak{N})$ ;

(b)  $\mathfrak{A}_A$  is an atom  $\Leftrightarrow A$  is a set-atom;

(c) if  $\mathfrak{N}$  satisfies locally ccc then for any atom  $\mathfrak{A}$  there exists a set-atom  $A \in \mathfrak{R} \setminus \mathfrak{N}$  with  $\mathfrak{A}_A = \mathfrak{A}$ ; in particular for any  $B \in \mathfrak{R}$  the set of atoms  $\mathfrak{A}$  such that  $B \in \mathfrak{A}$  is countable.

(a) The intersection of any sequence in  $\mathfrak{A}_A$  belongs to  $\mathfrak{A}_A$ .

(b) Let  $A$  be a set-atom. Let  $\mathfrak{A} \in \Lambda(\mathfrak{N})$  with  $\mathfrak{A}_A \subset \mathfrak{A}$  and let  $B \in \mathfrak{A}$ . Then  $A \cap B \in \mathfrak{A}$  and therefore  $A \cap B \notin \mathfrak{N}$ . We get  $A \setminus B \in \mathfrak{N}$  and therefore  $B \in \mathfrak{A}_A$ . Hence  $\mathfrak{A}_A = \mathfrak{A}$ . Thus  $\mathfrak{A}_A$  is an atom.

Assume now  $\mathfrak{A}_A$  is an atom. If  $A$  is not a set-atom then there exists  $B \in \mathfrak{R} \setminus \mathfrak{N}$  such that  $B \subset A$  and  $A \setminus B \notin \mathfrak{N}$ . We get  $\mathfrak{A}_B \in \Lambda(\mathfrak{N})$ ,  $\mathfrak{A}_A \subset \mathfrak{A}_B$ , and  $B \in \mathfrak{A}_B \setminus \mathfrak{A}_A$ . Hence  $\mathfrak{A}_A$  is not a maximal element of  $\Lambda(\mathfrak{N})$  and this is a contradiction.

(c) Assume  $\mathfrak{N}$  satisfies locally ccc and let  $\mathfrak{A}$  be an atom. By Proposition 1.1 there exists  $A \in \mathfrak{A}$  such that  $\mathfrak{A} \subset \mathfrak{A}_A$ . Since  $\mathfrak{A}$  is maximal in  $\Lambda(\mathfrak{N})$  we deduce by (a)  $\mathfrak{A} = \mathfrak{A}_A$ . By (b)  $A$  is a set-atom.

Let  $B \in \mathfrak{R}$  and assume that the set of atoms  $\mathfrak{A}$  such that  $B \in \mathfrak{A}$  is uncountable. Let  $\omega_1$  be the first uncountable ordinal number. There exists a family  $(\mathfrak{A}_\xi)_{\xi < \omega_1}$  of atoms such that  $B \in \mathfrak{A}_\xi$  for any  $\xi < \omega_1$  and such that  $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$  for any  $\xi < \eta < \omega_1$ . For any  $\xi < \omega_1$  let  $A_\xi$  be a set-atom such that  $\mathfrak{A}_\xi = \mathfrak{A}_{A_\xi}$ . Let  $\xi < \eta$ . If  $A_\xi \cap A_\eta \notin \mathfrak{N}$  then  $A_\eta \setminus A_\xi \in \mathfrak{N}$  and we get for any  $C \in \mathfrak{A}_\xi$

$$A_\xi \setminus C \in \mathfrak{N}, \quad A_\eta \setminus C \in \mathfrak{N}, \quad C \in \mathfrak{A}_\eta.$$

Hence  $\mathfrak{A}_\xi \subset \mathfrak{A}_\eta$  and this leads to the contradiction  $\mathfrak{A}_\xi = \mathfrak{A}_\eta$ . Hence  $A_\xi \cap A_\eta \in \mathfrak{N}$ . We get for any  $\eta < \omega_1$

$$A_\eta \cap B \setminus \bigcup_{\xi < \eta} A_\xi \in \mathfrak{A}_\eta$$

Hence  $(A_\eta \cap B \setminus \bigcup_{\xi < \eta} A_\xi)_{\eta < \omega_1}$  is an uncountable disjoint family of subsets of  $B$  in  $\mathfrak{R} \setminus \mathfrak{N}$  and this contradicts the hypothesis that  $\mathfrak{N}$  satisfies locally ccc. ■

*Remark.* From (c) it follows that if  $\mathfrak{N}$  satisfies locally ccc the atoms and the set-atoms may be identified.

**PROPOSITION 1.3.** *Let  $\mathfrak{A}$  be an atom. Then*

- (a) *If  $B \in \mathfrak{R}$  and if there exists  $A \in \mathfrak{A}$  such that  $A \setminus B \in \mathfrak{R}$  then  $B \in \mathfrak{A}$ .*
- (b) *If  $B \in \mathfrak{R}$  and if  $A \cap B \notin \mathfrak{R}$  for any  $A \in \mathfrak{A}$  then  $B \in \mathfrak{A}$ .*
- (c) *If  $(A_\iota)_{\iota \in I}$  is a countable family in  $\mathfrak{R}$  whose union belongs to  $\mathfrak{A}$  then there exists  $\iota \in I$  with  $A_\iota \in \mathfrak{A}$ .*
- (d)  *$\mathfrak{A}$  is a maximal element of  $\Lambda$ .*

(a) The set  $\{C \in \mathfrak{R} \mid \exists A \in \mathfrak{A}, A \setminus C \in \mathfrak{R}\}$  belongs to  $\Lambda(\mathfrak{R})$  and contains  $\mathfrak{A}$ . Since  $\mathfrak{A}$  is a maximal element of  $\Lambda(\mathfrak{R})$  this set coincides with  $\mathfrak{A}$ . Hence  $B \in \mathfrak{A}$ .

(b) The set  $\{C \in \mathfrak{R} \mid \exists A \in \mathfrak{A}, C \supset A \cap B\}$  belongs to  $\Lambda(\mathfrak{R})$  and contains  $\mathfrak{A}$ . Since  $\mathfrak{A}$  is a maximal element of  $\Lambda(\mathfrak{R})$  this set coincides with  $\mathfrak{A}$ . Hence  $B \in \mathfrak{A}$ .

(c) Assume  $A_\iota \notin \mathfrak{A}$  for any  $\iota \in I$ . By (b) there exists for any  $\iota \in I$  a  $B_\iota \in \mathfrak{A}$  such that  $A_\iota \cap B_\iota \in \mathfrak{R}$ . From

$$\left(\bigcup_{\iota \in I} A_\iota\right) \cap \left(\bigcap_{\iota \in I} B_\iota\right) \subset \bigcup_{\iota \in I} (A_\iota \cap B_\iota)$$

it follows  $(\bigcup_{\iota \in I} A_\iota) \cap (\bigcap_{\iota \in I} B_\iota) \in \mathfrak{R} \cap \mathfrak{A}$  and this is a contradiction.

(d) Let  $\mathfrak{B} \in \Lambda$  with  $\mathfrak{A} \subset \mathfrak{B}$ . Let  $B \in \mathfrak{B} \setminus \mathfrak{A}$ . By (b) there exists  $A \in \mathfrak{A}$  with  $A \cap B \in \mathfrak{R}$ . By (a) we get  $A \setminus B \in \mathfrak{A}$  and therefore

$$\emptyset = (A \setminus B) \cap B \in \mathfrak{B}$$

which is a contradiction. Hence  $\mathfrak{B} = \mathfrak{A}$  and  $\mathfrak{A}$  is a maximal element of  $\Lambda$ . ■

**COROLLARY 1.4.** *Let  $\mathfrak{N}'$ ,  $\mathfrak{N}''$  be two systems of null sets on  $\mathfrak{R}$  and let  $\mathfrak{A}$  be an atom with respect to  $\mathfrak{N}'$ . Then either  $\mathfrak{A}$  is an atom with respect to  $\mathfrak{N}''$  or  $\mathfrak{A} \cap \mathfrak{N}'' \neq \emptyset$ .*

If  $\mathfrak{A} \cap \mathfrak{N}'' = \emptyset$  then  $\mathfrak{A} \in \Lambda(\mathfrak{N}'')$ . By the Proposition 1.3(d)  $\mathfrak{A}$  is a maximal element of  $\Lambda$  and therefore a fortiori it is a maximal element of  $\Lambda(\mathfrak{N}'')$ . ■

**PROPOSITION 1.5.** *Let  $\Phi$  be a countable set of atoms. Then there exists a disjoint family  $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  in  $\mathfrak{R}$  such that  $A_{\mathfrak{A}} \in \mathfrak{A}$  for any  $\mathfrak{A} \in \Phi$ .*

Let  $\mathfrak{A}$ ,  $\mathfrak{B}$  be two different atoms of  $\Phi$  and let  $A_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{A} \setminus \mathfrak{B}$ . By Proposition 1.3(b) there exists  $B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{B}$  such that

$$A_{\mathfrak{A}, \mathfrak{B}} \cap B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{R}.$$

By Proposition 1.3(a)

$$A_{\mathfrak{A}, \mathfrak{B}} \setminus B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{A}, \quad B_{\mathfrak{A}, \mathfrak{B}} \setminus A_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{B}.$$

We set for any  $\mathfrak{A} \in \Phi$

$$A_{\mathfrak{A}} := \bigcap_{\substack{\mathfrak{B} \in \Phi \\ \mathfrak{B} = \mathfrak{A}}} ((A_{\mathfrak{A}, \mathfrak{B}} \setminus B_{\mathfrak{A}, \mathfrak{B}}) \cap (B_{\mathfrak{B}, \mathfrak{A}} \setminus A_{\mathfrak{B}, \mathfrak{A}})).$$

It is obvious that  $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  possesses the required properties. ■

**PROPOSITION 1.6.** *Let  $\Phi$  be a set of atoms such that for any  $A \in \mathfrak{R}$  the set  $\{\mathfrak{A} \in \Phi \mid A \in \mathfrak{A}\}$  is countable. Then there exists a family  $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  in  $\mathfrak{R}$  such that  $A_{\mathfrak{A}} \in \mathfrak{A} \setminus \mathfrak{A}'$  for any  $\mathfrak{A}, \mathfrak{A}' \in \Phi, \mathfrak{A} \neq \mathfrak{A}'$ .*

Let  $(B_{\mathfrak{A}})_{\mathfrak{A}}$  be a family in  $\mathfrak{R}$  such that  $B_{\mathfrak{A}} \in \mathfrak{A}$  for any  $\mathfrak{A} \in \Phi$ . Let  $\mathfrak{A} \in \Phi$  and let

$$\Psi(\mathfrak{A}) := \{\mathfrak{A}' \in \Phi \mid B_{\mathfrak{A}} \in \mathfrak{A}'\}.$$

By the hypothesis  $\Psi(\mathfrak{A})$  is countable. By Proposition 1.5 there exists a disjoint family  $(C_{\mathfrak{A}'})_{\mathfrak{A}' \in \Psi(\mathfrak{A})}$  such that  $C_{\mathfrak{A}'} \in \mathfrak{A}'$  for any  $\mathfrak{A}' \in \Psi(\mathfrak{A})$ . We set

$$A_{\mathfrak{A}} := B_{\mathfrak{A}} \cap C_{\mathfrak{A}}.$$

Let  $\mathfrak{A}, \mathfrak{A}'$  be two different atoms of  $\Phi$ . Then  $A_{\mathfrak{A}} \in \mathfrak{A}$ . If  $\mathfrak{A}' \notin \Psi(\mathfrak{A})$  then  $B_{\mathfrak{A}} \notin \mathfrak{A}'$  and therefore  $A_{\mathfrak{A}} \notin \mathfrak{A}'$  (Proposition 1.3(a)). If  $\mathfrak{A}' \in \Psi(\mathfrak{A})$  then

$$A_{\mathfrak{A}} \cap A_{\mathfrak{A}'} \subset C_{\mathfrak{A}} \cap C_{\mathfrak{A}'} = \emptyset$$

and this shows that  $A_{\mathfrak{A}} \notin \mathfrak{A}'$ . ■

## 2. Measures

A *measure on  $\mathfrak{R}$*  is a map  $\mu$  of  $\mathfrak{R}$  into a Hausdorff topological commutative group such that for any countable disjoint family  $(A_i)_{i \in I}$  in  $\mathfrak{R}$  whose union belongs to  $\mathfrak{R}$  the family  $(\mu(A_i))_{i \in I}$  is summable and its sum is  $\mu(\bigcup_{i \in I} A_i)$ . A measure  $\mu$  on  $\mathfrak{R}$  is called  *$\mathfrak{R}$ -regular* if for any  $A \in \mathfrak{R}$  and for any neighbourhood  $V$  of  $\mu(A)$  there exists  $K \in \mathfrak{R}$  contained in  $A$  such that

$$\{\mu(B) \mid B \in \mathfrak{R}, K \subset B \subset A\} \subset V.$$

For any measure  $\mu$  on  $\mathfrak{R}$  we set

$$\mathfrak{N}(\mu) := \{A \in \mathfrak{R} \mid \forall B \in \mathfrak{R}, B \subset A \Rightarrow \mu(B) = 0\}.$$

$\mathfrak{N}(\mu)$  is a system of null sets of  $\mathfrak{R}$ . We say that  $\mu$  satisfies *ccc* (resp. that  $\mu$  satisfies *locally ccc*) if  $\mathfrak{N}(\mu)$  satisfies *ccc* (resp. satisfies *locally ccc*). It is obvious that the set of  $G$ -valued measures on  $\mathfrak{R}$ , the set of  $\mathfrak{R}$ -regular  $G$ -valued measures on  $\mathfrak{R}$ , and the set of  $G$ -valued measures on  $\mathfrak{R}$  satisfying *ccc* or satisfying *locally ccc* are subgroups of  $G^{\mathfrak{R}}$ . For any measure  $\mu$  on  $\mathfrak{R}$  and for any  $\mathfrak{A} \in \Lambda$  we denote by  $\mu(\mathfrak{F}(\mathfrak{A}))$  the image of the filter  $\mathfrak{F}(\mathfrak{A})$  through  $\mu$  (i.e. the filter generated by the filter base  $\{\mu(\mathfrak{B}) \mid \mathfrak{B} \in \mathfrak{F}(\mathfrak{A})\}$ ); if this filter converges we denote by  $\mu_{\mathfrak{A}}$  its limit. By Proposition 1.1  $\mu(\mathfrak{F}(\mathfrak{A}))$  converges for any measure  $\mu$  satisfying *locally ccc* and for any  $\mathfrak{A} \in \Lambda$ . We call *atom of  $\mu$*  any atom with respect to  $\mathfrak{N}(\mu)$ . An atom  $\mathfrak{A}$  of  $\mu$  is called *improper* if  $\mu(\mathfrak{F}(\mathfrak{A}))$  converges to 0; otherwise we call it *proper*. Let  $\mathfrak{A}$  be an improper atom of  $\mu$ . If for any  $A \in \mathfrak{A}$  there exists a proper atom  $\mathfrak{A}'$  of  $\mu$  such that  $A \in \mathfrak{A}'$  we say that  $\mathfrak{A}$  is *of the first kind*. An improper atom which is not of the first kind will be called *of the second kind*. We call *set-atom of  $\mu$*  any set-atom with respect to  $\mathfrak{N}(\mu)$ .

A *preorder relation* on a set  $I$  is a binary relation  $\leq$  on  $I$  such that:

- (a)  $\iota \in I \Rightarrow \iota \leq \iota$ ;
- (b)  $\iota, \iota', \iota'' \in I, \iota \leq \iota', \iota' \leq \iota'' \Rightarrow \iota \leq \iota''$ .

An *upper directed preordered set* is a set  $I$  endowed with a preorder relation  $\leq$  such that for any  $\iota', \iota'' \in I$  there exists  $\iota \in I$  with  $\iota' \leq \iota, \iota'' \leq \iota$ . The section filter of an upper directed nonempty set  $(I, \leq)$  is the filter on  $I$  generated by the filter base

$$\{\{\iota \in I \mid \iota \geq \lambda\} \mid \lambda \in I\}.$$

A *net* in a set  $X$  is a pair  $(I, f)$  such that  $I$  is an upper directed preordered set and  $f$  is a map of  $I$  into  $X$ .

Let  $X$  be a topological space. An  $\omega$ -*net in  $X$*  is a net  $(I, f)$  in  $X$  such that for any increasing sequence  $(\iota_n)_{n \in \mathbf{N}}$  in  $I$  the sequence  $(f(\iota_n))_{n \in \mathbf{N}}$  is convergent. An  $\omega$ -*filter on  $X$*  is a filter  $\mathfrak{F}$  on  $X$  such that there exists an  $\omega$ -net  $(I, f)$  in  $X$  such that  $f(\mathfrak{F}) \subset \mathfrak{F}$ , where  $\mathfrak{G}$  denotes the section filter of  $I$ . An  $\omega$  *space* is a topological space for which any  $\omega$ -filter converges.

**PROPOSITION 2.1.** *Any  $\omega$ -filter on a uniform space is a Cauchy filter. Hence any complete uniform space is an  $\omega$ -space.*

Let  $X$  be a uniform space, let  $(I, f)$  be an  $\omega$ -net in  $X$ , and let  $\mathfrak{F}$  be the section filter of  $I$ . Let further  $U$  be an arbitrary entourage (= vicinity) of  $X$  and let  $V$  be an entourage of  $X$  such that  $V \circ V^{-1} \subset U$ . Assume that for any  $\iota \in I$  there exists  $\lambda \in I$  such that  $\lambda \geq \iota$  and  $(f(\iota), f(\lambda)) \notin V$ . Then we may construct inductively an increasing sequence  $(\iota_n)_{n \in \mathbf{N}}$  in  $I$  such that  $(f(\iota_n), f(\iota_{n+1})) \notin V$  for any  $n \in \mathbf{N}$ . The sequence  $(f(\iota_n))_{n \in \mathbf{N}}$  being convergent this is a contradiction. Hence there exists  $\iota \in I$  with  $(f(\iota), f(\lambda)) \in V$  for any  $\lambda \in I, \lambda \geq \iota$ . We get  $(f(\iota'), f(\iota'')) \in U$  for any  $\iota', \iota'' \in I$  with  $\iota' \geq \iota, \iota'' \geq \iota$ . Hence  $f(\mathfrak{F})$  is a Cauchy filter. ■

**PROPOSITION 2.2.** *For any measure  $\mu$  on  $\mathfrak{R}$  and for any  $\mathfrak{A} \in \Lambda, \mu(\mathfrak{F}(\mathfrak{A}))$  is an  $\omega$ -filter and therefore a Cauchy filter.*

Let us order  $\mathfrak{A}$  by the converse inclusion relation, let  $\mathfrak{G}$  be the section filter of  $\mathfrak{A}$ , and let  $\mu | \mathfrak{A}$  be the restriction of  $\mu$  to  $\mathfrak{A}$ . Then  $(\mathfrak{A}, \mu | \mathfrak{A})$  is an  $\omega$ -net and

$$\mu(\mathfrak{F}(\mathfrak{A})) = \mu | \mathfrak{A}(\mathfrak{G}).$$

Hence  $\mu(\mathfrak{F}(\mathfrak{A}))$  is an  $\omega$ -filter. By Proposition 2.1 it is a Cauchy filter. ■

**PROPOSITION 2.3.** *Let  $\mu$  be a measure on  $\mathfrak{R}$  and let  $\mathfrak{A}$  be a maximal element of  $\Lambda$ . Then either  $\mathfrak{A}$  is an atom of  $\mu$  or  $\mu(\mathfrak{F}(\mathfrak{A}))$  converges to 0.*

Assume that  $\mu(\mathfrak{F}(\mathfrak{A}))$  does not converge to 0. By Proposition 2.2 there exist a 0-neighbourhood  $V$  and an  $A \in \mathfrak{A}$  such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \cap V = \emptyset.$$

Let  $(A_n)_{n \in \mathbf{N}}$  be a sequence in  $\mathfrak{A}$ . If  $\bigcap_{n \in \mathbf{N}} A_n = \emptyset$  then there exists a decreasing sequence  $(B_n)_{n \in \mathbf{N}}$  in  $\mathfrak{A}$  with empty intersection and such that  $B_0 \subset A$ . It follows that  $(\mu(B_n))_{n \in \mathbf{N}}$  converges to 0 and this contradicts the above relation. Hence  $\bigcap_{n \in \mathbf{N}} A_n \neq \emptyset$ . The set

$$\left\{ B \in \mathfrak{R} \mid \exists C \in \mathfrak{A}, \left( \bigcap_{n \in \mathbf{N}} A_n \right) \cap C \subset B \right\}$$

belongs to  $\Lambda$  and contains  $\mathfrak{A}$ . Since  $\mathfrak{A}$  is maximal it coincides with  $\mathfrak{A}$ . We deduce  $\bigcap_{n \in \mathbf{N}} A_n \in \mathfrak{A}$ . Since  $\mathfrak{A} \cap \mathfrak{N}(\mu) = \emptyset$  we deduce  $\mathfrak{A} \in \Lambda(\mathfrak{N}(\mu))$ . It is obvious that  $\mathfrak{A}$  is a maximal element of  $\Lambda(\mathfrak{N}(\mu))$ . Hence  $\mathfrak{A}$  is an atom of  $\mu$ . ■

**COROLLARY 2.4.** *Let  $\mu, \nu$  be measures on  $\mathfrak{A}$  and let  $\mathfrak{A}$  be an atom of  $\mu$ . Then either  $\mathfrak{A}$  is an atom of  $\nu$  or  $\nu(\mathfrak{F}(\mathfrak{A}))$  converges to 0.*

By Proposition 1.3(d)  $\mathfrak{A}$  is a maximal element of  $\Lambda$  and the assertion follows from the proposition. ■

**PROPOSITION 2.5.** *Let  $\mu$  be a  $\mathfrak{R}$ -regular measure on  $\mathfrak{A}$  and let  $\mathfrak{A}$  be a proper atom of  $\mu$ . Then for any  $A \in \mathfrak{A}$  there exists  $K \in \mathfrak{R} \cap \mathfrak{A}$  with  $K \subset A$ .*

By Proposition 2.3  $\mu(\mathfrak{F}(\mathfrak{A}))$  is a Cauchy filter. Since  $\mathfrak{A}$  is a proper atom of  $\mu$ , 0 is not an adherent point of this filter. Hence there exist a neighbourhood  $V$  of 0 and a set  $B \in \mathfrak{A}$  such that

$$\{\mu(C) \mid C \in \mathfrak{A}, C \subset B\} \cap V = \emptyset.$$

Since  $\mu$  is  $\mathfrak{R}$ -regular there exists  $K \in \mathfrak{R}$  such that  $K \subset A \cap B$  and

$$\{\mu(C) \mid C \in \mathfrak{A}, C \subset A \cap B \setminus K\} \subset V.$$

Let  $C \in \mathfrak{A}$ . If  $C \cap K \in \mathfrak{N}(\mu)$  then by Proposition 1.3(a)  $C \setminus K \in \mathfrak{A}$  and therefore

$$A \cap B \cap (C \setminus K) \in \mathfrak{A}, \quad \mu(A \cap B \cap (C \setminus K)) \in V$$

which is a contradiction. Hence  $C \cap K \notin \mathfrak{N}(\mu)$  for any  $C \in \mathfrak{A}$ . By Proposition 1.3(b) we get  $K \in \mathfrak{A}$ . ■

### 3. Atomic Measures

A measure possessing no proper atom is called *atomless*. If it possesses no atoms at all it is called *strictly atomless*. Any improper atom of an atomless measure is of the second kind.

**PROPOSITION 3.1.** *The set of atomless (resp. strictly atomless)  $G$ -valued measures on  $\mathfrak{A}$  is a subgroup of  $G^{\mathfrak{A}}$ .*

Let  $\mu, \nu$  be two  $G$ -valued measures on  $\mathfrak{A}$  and let  $\mathfrak{A}$  be an atom of  $\mu - \nu$ . Since

$$\mathfrak{N}(\mu) \cap \mathfrak{N}(\nu) \subset \mathfrak{N}(\mu - \nu)$$

it follows that

$$\mathfrak{A} \cap \mathfrak{N}(\mu) = \emptyset \quad \text{or} \quad \mathfrak{A} \cap \mathfrak{N}(\nu) = \emptyset$$

By Corollary 1.4 we deduce that  $\mathfrak{A}$  is an atom of either  $\mu$  or  $\nu$ . This shows that the set of strictly non-atomic  $G$ -valued measures is a subgroup of  $G^{\mathfrak{R}}$ .

Assume now that  $\mathfrak{A}$  is a proper atom of  $\mu - \nu$ . By Proposition 1.3(d)  $\mathfrak{A}$  is a maximal element of  $\Lambda$ . By Proposition 2.3,  $\mathfrak{A}$  is a proper atom of either  $\mu$  or  $\nu$ . Hence the set of  $G$ -valued atomless measures on  $\mathfrak{R}$  is a subgroup of  $G^{\mathfrak{R}}$ . ■

We say that a measure  $\mu$  on  $\mathfrak{R}$  satisfies the *atom condition* (abbreviated *ac*) if for any atom  $\mathfrak{A}$  of  $\mu$ ,  $\mu(\mathfrak{F}(\mathfrak{A}))$  is convergent; according to the general convention made above we denote by  $\mu_{\mathfrak{A}}$  the limit of  $\mu(\mathfrak{F}(\mathfrak{A}))$  which may be interpreted as the value of  $\mu$  at  $\mathfrak{A}$ . By Proposition 1.1 any measure satisfying locally *ccc* satisfies *ac*. By Proposition 2.2 any measures with values in an  $\omega$ -topological group and a fortiori in a complete topological group (Proposition 2.1) satisfies *ac*. The set of  $G$ -valued measures on  $\mathfrak{R}$  satisfying *ac* is a subgroup of  $G^{\mathfrak{R}}$  (Corollary 2.4).

A measure  $\mu$  on  $\mathfrak{R}$  satisfying *ac* is called *atomic* if for any  $A \in \mathfrak{R}$ ,  $\mu(A)$  is the sum of the family  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ , where  $\Phi$  denotes the set of atoms  $\mathfrak{A}$  of  $\mu$  such that  $A \in \mathfrak{A}$ . By Propositions 1.3(d) and 2.3 we may replace  $\Phi$  in the above definition by the set of maximal elements  $\mathfrak{A}$  of  $\Lambda$  such that  $A \in \mathfrak{A}$ . From this remark it follows immediately that the set of  $G$ -valued atomic measures on  $\mathfrak{R}$  is a subgroup of  $G^{\mathfrak{R}}$ . Any improper atom of an atomic measure is of the first kind. A measure which is at the same time atomic and atomless vanishes identically.

**PROPOSITION 3.2.** *Let  $\mu, \mu'$  be two atomless  $G$ -valued measures on  $\mathfrak{R}$  and let  $\nu, \nu'$  be two atomic  $G$ -valued measures on  $\mathfrak{R}$ . If*

$$\mu + \nu = \mu' + \nu'$$

*then  $\mu = \mu'$  and  $\nu = \nu'$ .*

By Proposition 3.1,  $\mu - \mu'$  is an atomless measure on  $\mathfrak{R}$ . Since it is at the same time an atomic measure it vanishes identically. ■

We say that a measure  $\mu$  on  $\mathfrak{R}$  satisfying *ac* satisfies the *atomical summability condition* (abbreviated *asc*) if for any  $A \in \mathfrak{R}$  the family  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  is summable, where  $\Phi$  denotes the set of atoms  $\mathfrak{A}$  of  $\mu$  such that  $A \in \mathfrak{A}$ . By Propositions 1.3(d) and 2.3 we may replace  $\Phi$  in the above definition by the set of maximal elements  $\mathfrak{A}$  for  $\Lambda$  for which  $A \in \mathfrak{A}$ . From this remark it follows immediately that the set of  $G$ -valued measures on  $\mathfrak{R}$  satisfying *asc* is a subgroup of  $G^{\mathfrak{R}}$ . Any atomic measure satisfies *asc*.

**PROPOSITION 3.3.** *Let  $\mu$  be a  $G$ -valued measure on  $\mathfrak{A}$  satisfying ac and let  $\Phi$  be a countable set of atoms of  $\mu$  such that  $\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A} \neq \emptyset$  and such that  $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$  converges. Then the family  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  is summable and its sum is the limit of  $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$ .*

Let  $M$  be a subset of  $\mathbf{N} \setminus \{\emptyset\}$  and let  $(\mathfrak{A}_n)_{n \in M}$  be a family of atoms of  $\mu$  such  $\mathfrak{A}_m \neq \mathfrak{A}_n$  for any different  $m, n \in M$  and  $\Phi = \{\mathfrak{A}_n \mid n \in M\}$ . We set  $\mathfrak{A} := \bigcap_{n \in M} \mathfrak{A}_n$ . Let  $V$  be an arbitrary 0-neighbourhood in  $G$  and let  $(V_n)_{n \in \mathbf{N}}$  be a sequence of 0-neighbourhoods in  $G$  such that  $V_0 + V_0 - V_0 \subset V$  and such that  $V_{n+1} + V_{n+1} \subset V_n$  for any  $n \in \mathbf{N}$ . For any  $n \in M$  there exists  $A_n \in \mathfrak{A}_n$  such that

$$\{\mu(B) \mid B \in \mathfrak{A}_n, B \subset A_n\} \subset \mu_{\mathfrak{A}_n} + V_n.$$

There exists  $A \in \bigcap_{n \in M} \mathfrak{A}_n$  such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V_0.$$

By Proposition 1.5 there exists a disjoint family  $(B_n)_{n \in M}$  in  $\mathfrak{A}$  such that  $B_n \in \mathfrak{A}_n$  for any  $n \in M$ . Then  $(A_n \cap B_n \cap A)_{n \in M}$  is a disjoint family in  $\mathfrak{A}$  whose union belongs to  $\mathfrak{A}$  and therefore

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) = \sum_{n \in M} \mu(A_n \cap B_n \cap A)$$

Since  $\bigcup_{n \in M} (A_n \cap B_n \cap A) \in \mathfrak{A}$  (Proposition 1.3(a)) we have

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) \in \mu_{\mathfrak{A}} + V_0.$$

For any  $n \in M$  we get

$$\mu(A_n \cap B_n \cap A) \in \mu_{\mathfrak{A}_n} + V_n.$$

Let  $M_0$  be a finite subset of  $M$  such that

$$\sum_{n \in M'} \mu(A_n \cap B_n \cap A) - \sum_{n \in M} \mu(A_n \cap B_n \cap A) \in V_0$$

for any finite subset  $M'$  of  $M$  containing  $M_0$ . We deduce for any finite subset  $M'$  of  $M$  containing  $M_0$ .

$$\sum_{n \in M'} \mu_{\mathfrak{A}_n} - \mu_{\mathfrak{A}} \in V_0 + V_0 - V_0 \subset V.$$

Since  $V$  is arbitrary it follows that  $(\mu_{\mathfrak{A}_n})_{n \in M}$  is summable and its sum is  $\mu_{\mathfrak{A}_0}$ . ■

**PROPOSITION 3.4.** *Any measure satisfying locally ccc satisfies asc.*

Let  $\mu$  be a measure on  $\mathfrak{R}$  satisfying locally ccc. By Proposition 1.1  $\mu(\mathfrak{F}(\mathfrak{A}))$  converges for any  $\mathfrak{A} \in \Lambda$ ; in particular  $\mu$  satisfies ac. Let  $A \in \mathfrak{R}$  and let  $\Phi$  be the set of atoms  $\mathfrak{A}$  of  $\mu$  such that  $A \in \mathfrak{A}$ . By Proposition 1.2(c)  $\Phi$  is countable. By the preceding proposition  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  is summable. Hence  $\mu$  satisfies asc. ■

**PROPOSITION 3.5.** *If  $G$  is an  $\omega$ -space then any  $G$ -valued measure satisfies asc.*

Let  $\mu$  be a  $G$ -valued measure on  $\mathfrak{R}$ . By Proposition 2.2 for any  $\mathfrak{A} \in \Lambda$  the filter  $\mu(\mathfrak{F}(\mathfrak{A}))$  converges; in particular  $\mu$  satisfies ac. Let  $A \in \mathfrak{R}$ , let  $\Phi$  be the set of atoms  $\mathfrak{A}$  of  $\mu$  such that  $A \in \mathfrak{A}$ , and let  $\mathfrak{B}_c(\Phi)$  be the set of countable subsets of  $\Phi$  ordered by the inclusion relation. By Proposition 3.3 for any  $\Psi \in \mathfrak{B}_c(\Phi)$  the family  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Psi}$  is summable; let us denote by  $f$  the map

$$\Psi \mapsto \sum_{\mathfrak{A} \in \Psi} \mu_{\mathfrak{A}} : \mathfrak{B}_c(\Phi) \rightarrow G.$$

Then  $(\mathfrak{B}_c(\Phi), f)$  is an  $\omega$ -net in  $G$ . Hence if  $\mathfrak{F}$  denotes the section filter of  $\mathfrak{B}_c(\Phi)$  then  $f(\mathfrak{F})$  converges. We deduce that the family  $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$  is summable. Hence  $\mu$  satisfies asc. ■

**THEOREM 3.6.** *Let  $\mu$  be a  $\mathfrak{R}$ -regular  $G$ -valued measure on  $\mathfrak{R}$  satisfying asc. We denote for any  $A \in \mathfrak{R}$  by  $\Phi(A)$  the set of atoms  $\mathfrak{A}$  of  $\mu$  such that  $A \in \mathfrak{A}$  and by  $\mu'$  the map*

$$A \mapsto \sum_{\mathfrak{A} \in \Phi(A)} \mu_{\mathfrak{A}} : \mathfrak{R} \rightarrow G.$$

*Then  $\mu'$  (resp.  $\mu - \mu'$ ) is an atomic (resp. atomless)  $\mathfrak{R}$ -regular measure on  $\mathfrak{R}$  absolutely continuous with respect to  $\mu$  (i.e.  $\mathfrak{N}(\mu) \subset \mathfrak{N}(\mu') \cap \mathfrak{N}(\mu - \mu')$ ). The proper atoms and the improper atoms of the first kind of  $\mu$  and  $\mu'$  coincide and we have*

$\mu_{\mathfrak{A}} = \mu'_{\mathfrak{A}}$  for any atom  $\mathfrak{A}$  of  $\mu$ . Any improper atom of  $\mu$  of the second kind is an improper atom of  $\mu - \mu'$  of the second kind.

Let  $(A_i)_{i \in I}$  be a countable disjoint family in  $\mathfrak{R}$  whose union belongs to  $\mathfrak{R}$ . Then  $(\Phi(A_i))_{i \in I}$  is a disjoint family and by Proposition 1.3(c) its union is  $\Phi(\bigcup_{i \in I} A_i)$ . We get

$$\mu' \left( \bigcup_{i \in I} A_i \right) = \sum_{\substack{\mathfrak{A} \in \Phi(\bigcup_{i \in I} A_i) \\ i \in I}} \mu_{\mathfrak{A}} = \sum_{i \in I} \sum_{\mathfrak{A} \in \Phi(A_i)} \mu_{\mathfrak{A}} = \sum_{i \in I} \mu'(A_i).$$

Hence  $\mu'$  is a measure.

Let  $A \in \mathfrak{R}$  and let  $U$  be a closed 0-neighbourhood in  $G$ . Then there exists a finite subset  $\Psi_0$  of  $\Phi(A)$  such that

$$\mu'(A) - \sum_{\mathfrak{A} \in \Psi_0} \mu_{\mathfrak{A}} \in U$$

for any finite subset  $\Psi$  of  $\Phi(A)$  containing  $\Psi_0$ . By Proposition 2.5 there exists for any  $\mathfrak{A} \in \Phi(A)$  a set  $K_{\mathfrak{A}} \in \mathfrak{R} \cap \mathfrak{A}$  with  $K_{\mathfrak{A}} \subset A$ . We set

$$K := \bigcup_{\mathfrak{A} \in \Psi_0} K_{\mathfrak{A}} \in \mathfrak{R}$$

Let  $B$  be a set of  $\mathfrak{R}$  such that  $K \subset B \subset A$ . Then  $\Phi(B)$  is a subset of  $\Phi(A)$  containing  $\Psi_0$  (Proposition 1.3(a)) and therefore

$$\mu'(A) - \mu'(B) = \mu'(A) - \sum_{\mathfrak{A} \in \Phi(B)} \mu_{\mathfrak{A}} \in U.$$

This shows that  $\mu'$  is  $\mathfrak{R}$ -regular. We deduce that  $\mu - \mu'$  is a  $\mathfrak{R}$ -regular measure on  $\mathfrak{R}$ . It is obvious that  $\mu'$  and  $\mu - \mu'$  are absolutely continuous with respect to  $\mu$ .

Let  $\mathfrak{A}$  be an atom of  $\mu$ . Let  $U$  be a closed 0-neighbourhood in  $G$  and let  $V$  be a 0-neighbourhood in  $G$  such that  $V - V - V \subset U$ . There exists  $A \in \mathfrak{A}$  such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V.$$

Let  $\Psi$  be a finite nonempty subset of  $\Phi(A) \setminus \{\mathfrak{A}\}$ . Then there exists

$$B \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}' \setminus \mathfrak{A}$$

such that  $B \subset A$  and

$$\left\{ \mu(C) \mid C \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}', C \subset B \right\} \subset \sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} + V.$$

Then  $A \setminus B \in \mathfrak{A}$  (Proposition 1.3(c)) and therefore

$$\mu(A \setminus B) \in \mu_{\mathfrak{A}} + V,$$

$$\sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} \in \mu(B) - V = \mu(A) - \mu(A \setminus B) - V \subset V - V - V \subset U.$$

Since  $\Psi$  is arbitrary we get

$$\sum_{\mathfrak{A}' \in \Phi(A) \setminus \{\mathfrak{A}\}} \mu_{\mathfrak{A}'} \in U, \quad \mu'(A) \in \mu_{\mathfrak{A}} + U.$$

Since  $U$  is arbitrary we deduce  $\mu_{\mathfrak{A}'} = \mu_{\mathfrak{A}}$ . Hence the proper atoms of  $\mu$  and  $\mu'$  coincide (Corollary 2.4). We deduce further that the improper atoms of  $\mu$  and  $\mu'$  of the first kind coincide (Corollary 1.4). Moreover for any  $A \in \mathfrak{A}$  we get

$$\mu'(A) = \sum_{\mathfrak{A}' \in \Phi} \mu_{\mathfrak{A}'} = \sum_{\mathfrak{A}' \in \Phi} \mu_{\mathfrak{A}'},$$

where  $\Phi$  denotes the set of maximal elements  $\mathfrak{A}$  of  $\Lambda$  such that  $A \in \mathfrak{A}$  (Propositions 1.3(d) and 2.3). Hence  $\mu'$  is an atomic measure.

Let  $\mathfrak{A}$  be an atom of  $\mu - \mu'$ . Then  $\mathfrak{A}$  is an atom of  $\mu$  (Corollary 1.4) and by the above considerations it follows that  $\mathfrak{A}$  is an improper atom of  $\mu - \mu'$ . Hence  $\mu - \mu'$  is atomless.

Let  $\mathfrak{A}$  be an improper atom of  $\mu$  of the second kind. Then there exists  $A \in \mathfrak{A}$  such that  $A \in \mathfrak{N}(\mu')$  and therefore  $\mathfrak{A}$  is an atom of  $\mu - \mu'$  (Corollary 1.7)). Since  $\mu - \mu'$  is atomless it is an improper atom of  $\mu - \mu'$  of the second kind. ■

*Example.* We want to give an example of a locally convex space  $E$  and of an  $E$ -valued measure on a  $\sigma$ -algebra of sets, possessing an improper atom of the second kind. Let  $X$  be a set. For any  $A \subset X \times [0, 1]$  and for any  $x \in X$  we set

$$A(x) := \{y \in [0, 1] \mid (x, y) \in A\}.$$

We denote by  $\mathfrak{A}$  the set of  $A \subset X \times [0, 1]$  such that: (a)  $A(x)$  is a Borel set for any  $x \in X$ ; (b) the set  $\{x \in X \mid A(x) \neq \emptyset \text{ and } A(x) \neq [0, 1]\}$  is countable. It is obvious

that  $\mathfrak{A}$  is a  $\sigma$ -algebra of subsets of  $X \times [0, 1]$ . For any  $A \in \mathfrak{A}$  we denote by  $\mu(A)$  the map

$$x \mapsto \lambda(A(x)): X \rightarrow \mathbf{R},$$

where  $\lambda$  denotes the Lebesgue measure on  $[0, 1]$ . It is easy to see that  $\mu$  is an atomless  $\mathbf{R}^X$ -valued measure on  $\mathfrak{A}$ . Let  $\mathfrak{F}$  be a non-trivial ultrafilter on  $X$  such that the intersection of any countable family in  $\mathfrak{F}$  belongs to  $\mathfrak{F}$  (we assume that such an ultrafilter exists). We set

$$\mathfrak{A} := \{A \in \mathfrak{A} \mid \{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}\}.$$

Then  $\mathfrak{A} \in \Lambda(\mu)$ . Let  $\mathfrak{A}' \in \Lambda(\mu)$  with  $\mathfrak{A} \subset \mathfrak{A}'$ , let  $A \in \mathfrak{A}'$ , and let

$$X_0 := \{x \in X \mid A(x) \neq \emptyset\}$$

If  $X_0 \notin \mathfrak{F}$  then  $X \setminus X_0 \in \mathfrak{F}$  and therefore  $(X \setminus X_0) \times [0, 1] \in \mathfrak{A}$  and this leads to the contradictory relation

$$\emptyset = A \cap ((X \setminus X_0) \times [0, 1]) \in \mathfrak{A}'.$$

Hence  $X_0 \in \mathfrak{F}$ . Since  $A \in \mathfrak{A}$  the set

$$\cdot \{x \in X \mid A(x) \neq \emptyset \text{ and } A(x) \neq [0, 1]\}$$

is countable and therefore it does not belong to  $\mathfrak{F}$ . Hence

$$\{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}$$

and we deduce successively  $A \in \mathfrak{A}$ ,  $\mathfrak{A} = \mathfrak{A}'$  and  $\mathfrak{A}$  is an atom of  $\mu$ . The measure  $\mu$  being atomless  $\mathfrak{A}$  is an improper atom of the second kind.

*Example.* Let  $X$  be an uncountable set, let  $\mathfrak{B}(X)$  be the set of subsets of  $X$ , and for any  $A \in \mathfrak{B}(X)$  let  $l_A$  be the characteristic function of  $A$ . Then

$$A \mapsto l_A : \mathfrak{B}(X) \rightarrow \mathbf{R}^X$$

is an example of a measure satisfying *asc* and not satisfying *ccc*.

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