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Atoms of Group Valued Measures

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Any real valued measure may be written (uniquely) as a sum of an atomic and of an atomless measure. This result was extended first time by J. Hoffmann-Jørgensen ([1] Theorem 6) to absolutely continuous measures with values in locally convex spaces and then by K. Musiał ([2] Theorem 1) to the more general case of group valued measures satisfying *ccc*. The aim of this paper is to extend this result once more for group valued measures satisfying *asc* (Proposition 3.2 and Theorem 3.6). Any measure satisfying *ccc* (even locally) satisfies *asc* (Proposition 3.4). If the group G is complete any G -valued measure satisfies *asc* (Propositions 3.5 and 2.1). The nature of atoms for measures not satisfying *ccc* becomes by far more complicated and so a great part of this paper is dedicated to their study.

Throughout this paper we shall denote by \mathfrak{R} a δ -ring (i.e. $\mathfrak{R} \neq \emptyset$ and for any sequence $(A_n)_{n \in \mathbb{N}}$ in \mathfrak{R} we have $\bigcap_{n \in \mathbb{N}} A_n \in \mathfrak{R}$ and $A_0 \Delta A_1 \in \mathfrak{R}$) by \mathfrak{A} a subset of \mathfrak{R} such that the union of any finite family in \mathfrak{A} belongs to \mathfrak{A} , and by G a Hausdorff topological commutative group. We consider \mathfrak{R} ordered by the inclusion relation and denote by Λ the set of lower directed nonempty subsets of $\mathfrak{R} \setminus \{\emptyset\}$. For any $\mathfrak{A} \in \Lambda$ we denote by $\mathfrak{F}(\mathfrak{A})$ the filter on \mathfrak{R} generated by the filter base

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

A system of null sets of \mathfrak{R} is a nonempty subset \mathfrak{N} of \mathfrak{R} such that: (a) any set of \mathfrak{R} belongs to \mathfrak{N} if it is contained in a set of \mathfrak{N} ; (b) the union of any countable family in \mathfrak{N} belongs to \mathfrak{N} if it belongs to \mathfrak{N} .

1. Atoms

Throughout this section we shall denote by \mathfrak{N} a system of null sets of \mathfrak{R} . We denote by $\Lambda(\mathfrak{N})$ the set of subsets $\mathfrak{A} \neq \emptyset$ of $\mathfrak{R} \setminus \mathfrak{N}$ such that the intersection of any countable family in \mathfrak{A} belongs to \mathfrak{A} . It is obvious that $\Lambda(\mathfrak{N}) \subset \Lambda$. The maximal elements of $\Lambda(\mathfrak{N})$ (for the inclusion relation) will be called *atoms* (with respect to

\mathfrak{N}). A *set-atom* (with respect to \mathfrak{N}) is a set $A \in \mathfrak{N} \setminus \mathfrak{N}$ such that for any subset B of A belonging to \mathfrak{N} we have either $B \in \mathfrak{N}$ or $A \setminus B \in \mathfrak{N}$. We say that \mathfrak{N} satisfies the *countable chain condition* (abbreviated *ccc*) if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{N} \setminus \mathfrak{N}$ (i.e. $A_\iota \cap A_{\iota'} = \emptyset$ for different elements ι, ι' of I) is countable. We say that \mathfrak{N} satisfies *locally ccc* if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{N} \setminus \mathfrak{N}$ is countable if $\bigcup_{\iota \in I} A_\iota$ is contained in a set of \mathfrak{N} .

PROPOSITION 1.1. *Assume \mathfrak{N} satisfies locally ccc and let $\mathfrak{A} \in \Lambda$. Then there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in \mathfrak{A} such that we have for any $A \in \mathfrak{A}$*

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \in \mathfrak{N}.$$

Assume that for any decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in \mathfrak{A} there exists $A \in \mathfrak{A}$ such that

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \notin \mathfrak{N}.$$

Let ω_1 be the first uncountable ordinal number. We shall construct inductively a family $(B_\xi)_{\xi < \omega_1}$ in \mathfrak{A} such that

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}$$

for any $\mu < \omega_1$. Let $\mu < \omega_1$ and assume the family $(B_\xi)_{\xi < \mu}$ with the required property is constructed. Since this family is countable and since \mathfrak{A} is lower directed there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in \mathfrak{A} such that

$$\bigcap_{n \in \mathbf{N}} A_n \subset \bigcap_{\xi < \mu} B_\xi.$$

By the hypothesis there exists $B_\mu \in \mathfrak{A}$ with

$$\bigcap_{n \in \mathbf{N}} A_n \setminus B_\mu \notin \mathfrak{N}.$$

Hence

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}.$$

The existence of the family $(\bigcap_{\xi < \mu} B_\xi \setminus B_\mu)_{\mu < \omega_1}$ contradicts the hypothesis that \mathfrak{N} satisfies locally ccc. ■

PROPOSITION 1.2. *For any $A \in \mathfrak{R} \setminus \mathfrak{N}$ we set*

$$\mathfrak{A}_A := \{B \in \mathfrak{R} \mid A \setminus B \in \mathfrak{N}\}$$

Then:

- (a) $\mathfrak{A}_A \in \Lambda(\mathfrak{N})$;
- (b) \mathfrak{A}_A is an atom $\Leftrightarrow A$ is a set-atom;
- (c) if \mathfrak{N} satisfies locally ccc then for any atom \mathfrak{A} there exists a set-atom $A \in \mathfrak{R} \setminus \mathfrak{N}$ with $\mathfrak{A}_A = \mathfrak{A}$; in particular for any $B \in \mathfrak{R}$ the set of atoms \mathfrak{A} such that $B \in \mathfrak{A}$ is countable.

(a) The intersection of any sequence in \mathfrak{A}_A belongs to \mathfrak{A}_A .

(b) Let A be a set-atom. Let $\mathfrak{A} \in \Lambda(\mathfrak{N})$ with $\mathfrak{A}_A \subset \mathfrak{A}$ and let $B \in \mathfrak{A}$. Then $A \cap B \in \mathfrak{A}$ and therefore $A \cap B \notin \mathfrak{N}$. We get $A \setminus B \in \mathfrak{N}$ and therefore $B \in \mathfrak{A}_A$. Hence $\mathfrak{A}_A = \mathfrak{A}$. Thus \mathfrak{A}_A is an atom.

Assume now \mathfrak{A}_A is an atom. If A is not a set-atom then there exists $B \in \mathfrak{R} \setminus \mathfrak{N}$ such that $B \subset A$ and $A \setminus B \notin \mathfrak{N}$. We get $\mathfrak{A}_B \in \Lambda(\mathfrak{N})$, $\mathfrak{A}_A \subset \mathfrak{A}_B$, and $B \in \mathfrak{A}_B \setminus \mathfrak{A}_A$. Hence \mathfrak{A}_A is not a maximal element of $\Lambda(\mathfrak{N})$ and this is a contradiction.

(c) Assume \mathfrak{N} satisfies locally ccc and let \mathfrak{A} be an atom. By Proposition 1.1 there exists $A \in \mathfrak{A}$ such that $\mathfrak{A} \subset \mathfrak{A}_A$. Since \mathfrak{A} is maximal in $\Lambda(\mathfrak{N})$ we deduce by (a) $\mathfrak{A} = \mathfrak{A}_A$. By (b) A is a set-atom.

Let $B \in \mathfrak{R}$ and assume that the set of atoms \mathfrak{A} such that $B \in \mathfrak{A}$ is uncountable. Let ω_1 be the first uncountable ordinal number. There exists a family $(\mathfrak{A}_\xi)_{\xi < \omega_1}$ of atoms such that $B \in \mathfrak{A}_\xi$ for any $\xi < \omega_1$ and such that $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$ for any $\xi < \eta < \omega_1$. For any $\xi < \omega_1$ let A_ξ be a set-atom such that $\mathfrak{A}_\xi = \mathfrak{A}_{A_\xi}$. Let $\xi < \eta$. If $A_\xi \cap A_\eta \notin \mathfrak{N}$ then $A_\eta \setminus A_\xi \in \mathfrak{N}$ and we get for any $C \in \mathfrak{A}_\xi$

$$A_\xi \setminus C \in \mathfrak{N}, \quad A_\eta \setminus C \in \mathfrak{N}, \quad C \in \mathfrak{A}_\eta.$$

Hence $\mathfrak{A}_\xi \subset \mathfrak{A}_\eta$ and this leads to the contradiction $\mathfrak{A}_\xi = \mathfrak{A}_\eta$. Hence $A_\xi \cap A_\eta \in \mathfrak{N}$. We get for any $\eta < \omega_1$

$$A_\eta \cap B \setminus \bigcup_{\xi < \eta} A_\xi \in \mathfrak{A}_\eta$$

Hence $(A_\eta \cap B \setminus \bigcup_{\xi < \eta} A_\xi)_{\eta < \omega_1}$ is an uncountable disjoint family of subsets of B in $\mathfrak{R} \setminus \mathfrak{N}$ and this contradicts the hypothesis that \mathfrak{N} satisfies locally ccc. ■

Remark. From (c) it follows that if \mathfrak{N} satisfies locally ccc the atoms and the set-atoms may be identified.

PROPOSITION 1.3. *Let \mathfrak{A} be an atom. Then*

- (a) *If $B \in \mathfrak{N}$ and if there exists $A \in \mathfrak{A}$ such that $A \setminus B \in \mathfrak{N}$ then $B \in \mathfrak{A}$.*
- (b) *If $B \in \mathfrak{N}$ and if $A \cap B \notin \mathfrak{N}$ for any $A \in \mathfrak{A}$ then $B \in \mathfrak{A}$.*
- (c) *If $(A_\iota)_{\iota \in I}$ is a countable family in \mathfrak{N} whose union belongs to \mathfrak{A} then there exists $\iota \in I$ with $A_\iota \in \mathfrak{A}$.*
- (d) *\mathfrak{A} is a maximal element of Λ .*

(a) The set $\{C \in \mathfrak{N} \mid \exists A \in \mathfrak{A}, A \setminus C \in \mathfrak{N}\}$ belongs to $\Lambda(\mathfrak{N})$ and contains \mathfrak{A} . Since \mathfrak{A} is a maximal element of $\Lambda(\mathfrak{N})$ this set coincides with \mathfrak{A} . Hence $B \in \mathfrak{A}$.

(b) The set $\{C \in \mathfrak{N} \mid \exists A \in \mathfrak{A}, C \supset A \cap B\}$ belongs to $\Lambda(\mathfrak{N})$ and contains \mathfrak{A} . Since \mathfrak{A} is a maximal element of $\Lambda(\mathfrak{N})$ this set coincides with \mathfrak{A} . Hence $B \in \mathfrak{A}$.

(c) Assume $A_\iota \notin \mathfrak{A}$ for any $\iota \in I$. By (b) there exists for any $\iota \in I$ a $B_\iota \in \mathfrak{A}$ such that $A_\iota \cap B_\iota \in \mathfrak{N}$. From

$$\left(\bigcup_{\iota \in I} A_\iota\right) \cap \left(\bigcap_{\iota \in I} B_\iota\right) \subset \bigcup_{\iota \in I} (A_\iota \cap B_\iota)$$

it follows $(\bigcup_{\iota \in I} A_\iota) \cap (\bigcap_{\iota \in I} B_\iota) \in \mathfrak{N} \cap \mathfrak{A}$ and this is a contradiction.

(d) Let $\mathfrak{B} \in \Lambda$ with $\mathfrak{A} \subset \mathfrak{B}$. Let $B \in \mathfrak{B} \setminus \mathfrak{A}$. By (b) there exists $A \in \mathfrak{A}$ with $A \cap B \in \mathfrak{N}$. By (a) we get $A \setminus B \in \mathfrak{A}$ and therefore

$$\emptyset = (A \setminus B) \cap B \in \mathfrak{B}$$

which is a contradiction. Hence $\mathfrak{B} = \mathfrak{A}$ and \mathfrak{A} is a maximal element of Λ . ■

COROLLARY 1.4. *Let $\mathfrak{N}', \mathfrak{N}''$ be two systems of null sets on \mathfrak{R} and let \mathfrak{A} be an atom with respect to \mathfrak{N}' . Then either \mathfrak{A} is an atom with respect to \mathfrak{N}'' or $\mathfrak{A} \cap \mathfrak{N}'' \neq \emptyset$.*

If $\mathfrak{A} \cap \mathfrak{N}'' = \emptyset$ then $\mathfrak{A} \in \Lambda(\mathfrak{N}'')$. By the Proposition 1.3(d) \mathfrak{A} is a maximal element of Λ and therefore a fortiori it is a maximal element of $\Lambda(\mathfrak{N}'')$. ■

PROPOSITION 1.5. *Let Φ be a countable set of atoms. Then there exists a disjoint family $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ in \mathfrak{R} such that $A_{\mathfrak{A}} \in \mathfrak{A}$ for any $\mathfrak{A} \in \Phi$.*

Let $\mathfrak{A}, \mathfrak{B}$ be two different atoms of Φ and let $A_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{A} \setminus \mathfrak{B}$. By Proposition 1.3(b) there exists $B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{B}$ such that

$$A_{\mathfrak{A}, \mathfrak{B}} \cap B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{N}.$$

By Proposition 1.3(a)

$$A_{\mathfrak{A}, \mathfrak{B}} \setminus B_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{A}, \quad B_{\mathfrak{A}, \mathfrak{B}} \setminus A_{\mathfrak{A}, \mathfrak{B}} \in \mathfrak{B}.$$

We set for any $\mathfrak{A} \in \Phi$

$$A_{\mathfrak{A}} := \bigcap_{\substack{\mathfrak{B} \in \Phi \\ \mathfrak{B} = \mathfrak{A}}} ((A_{\mathfrak{A}, \mathfrak{B}} \setminus B_{\mathfrak{A}, \mathfrak{B}}) \cap (B_{\mathfrak{B}, \mathfrak{A}} \setminus A_{\mathfrak{B}, \mathfrak{A}})).$$

It is obvious that $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ possesses the required properties. ■

PROPOSITION 1.6. *Let Φ be a set of atoms such that for any $A \in \mathfrak{R}$ the set $\{\mathfrak{A} \in \Phi \mid A \in \mathfrak{A}\}$ is countable. Then there exists a family $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ in \mathfrak{R} such that $A_{\mathfrak{A}} \in \mathfrak{A} \setminus \mathfrak{A}'$ for any $\mathfrak{A}, \mathfrak{A}' \in \Phi, \mathfrak{A} \neq \mathfrak{A}'$.*

Let $(B_{\mathfrak{A}})_{\mathfrak{A}} \in \mathfrak{R}$ be a family in \mathfrak{R} such that $B_{\mathfrak{A}} \in \mathfrak{A}$ for any $\mathfrak{A} \in \Phi$. Let $\mathfrak{A} \in \Phi$ and let

$$\Psi(\mathfrak{A}) := \{\mathfrak{A}' \in \Phi \mid B_{\mathfrak{A}} \in \mathfrak{A}'\}.$$

By the hypothesis $\Psi(\mathfrak{A})$ is countable. By Proposition 1.5 there exists a disjoint family $(C_{\mathfrak{A}'})_{\mathfrak{A}' \in \Psi(\mathfrak{A})}$ such that $C_{\mathfrak{A}'} \in \mathfrak{A}'$ for any $\mathfrak{A}' \in \Psi(\mathfrak{A})$. We set

$$A_{\mathfrak{A}} := B_{\mathfrak{A}} \cap C_{\mathfrak{A}}.$$

Let $\mathfrak{A}, \mathfrak{A}'$ be two different atoms of Φ . Then $A_{\mathfrak{A}} \in \mathfrak{A}$. If $\mathfrak{A}' \notin \Psi(\mathfrak{A})$ then $B_{\mathfrak{A}} \notin \mathfrak{A}'$ and therefore $A_{\mathfrak{A}} \notin \mathfrak{A}'$ (Proposition 1.3(a)). If $\mathfrak{A}' \in \Psi(\mathfrak{A})$ then

$$A_{\mathfrak{A}} \cap A_{\mathfrak{A}'} \subset C_{\mathfrak{A}} \cap C_{\mathfrak{A}'} = \emptyset$$

and this shows that $A_{\mathfrak{A}} \notin \mathfrak{A}'$. ■

2. Measures

A *measure* on \mathfrak{R} is a map μ of \mathfrak{R} into a Hausdorff topological commutative group such that for any countable disjoint family $(A_i)_{i \in I}$ in \mathfrak{R} whose union belongs to \mathfrak{R} the family $(\mu(A_i))_{i \in I}$ is summable and its sum is $\mu(\bigcup_{i \in I} A_i)$. A measure μ on \mathfrak{R} is called *\mathfrak{R} -regular* if for any $A \in \mathfrak{R}$ and for any neighbourhood V of $\mu(A)$ there exists $K \in \mathfrak{R}$ contained in A such that

$$\{\mu(B) \mid B \in \mathfrak{R}, K \subset B \subset A\} \subset V.$$

For any measure μ on \mathfrak{R} we set

$$\mathfrak{N}(\mu) := \{A \in \mathfrak{R} \mid \forall B \in \mathfrak{R}, B \subset A \Rightarrow \mu(B) = 0\}.$$

$\mathfrak{N}(\mu)$ is a system of null sets of \mathfrak{R} . We say that μ satisfies *ccc* (resp. that μ satisfies *locally ccc*) if $\mathfrak{N}(\mu)$ satisfies *ccc* (resp. satisfies *locally ccc*). It is obvious that the set of G -valued measures on \mathfrak{R} , the set of \mathfrak{R} -regular G -valued measures on \mathfrak{R} , and the set of G -valued measures on \mathfrak{R} satisfying *ccc* or satisfying *locally ccc* are subgroups of $G^{\mathfrak{R}}$. For any measure μ on \mathfrak{R} and for any $\mathfrak{A} \in \Lambda$ we denote by $\mu(\mathfrak{F}(\mathfrak{A}))$ the image of the filter $\mathfrak{F}(\mathfrak{A})$ through μ (i.e. the filter generated by the filter base $\{\mu(\mathfrak{B}) \mid \mathfrak{B} \in \mathfrak{F}(\mathfrak{A})\}$); if this filter converges we denote by $\mu_{\mathfrak{A}}$ its limit. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any measure μ satisfying *locally ccc* and for any $\mathfrak{A} \in \Lambda$. We call *atom of μ* any atom with respect to $\mathfrak{N}(\mu)$. An atom \mathfrak{A} of μ is called *improper* if $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0; otherwise we call it *proper*. Let \mathfrak{A} be an improper atom of μ . If for any $A \in \mathfrak{A}$ there exists a proper atom \mathfrak{A}' of μ such that $A \in \mathfrak{A}'$ we say that \mathfrak{A} is *of the first kind*. An improper atom which is not of the first kind will be called *of the second kind*. We call *set-atom of μ* any set-atom with respect to $\mathfrak{N}(\mu)$.

A *preorder relation* on a set I is a binary relation \leq on I such that:

- (a) $\iota \in I \Rightarrow \iota \leq \iota$;
- (b) $\iota, \iota', \iota'' \in I, \iota \leq \iota', \iota' \leq \iota'' \Rightarrow \iota \leq \iota''$.

An *upper directed preordered set* is a set I endowed with a preorder relation \leq such that for any $\iota', \iota'' \in I$ there exists $\iota \in I$ with $\iota' \leq \iota, \iota'' \leq \iota$. The section filter of an upper directed nonempty set (I, \leq) is the filter on I generated by the filter base

$$\{\{\iota \in I \mid \iota \geq \lambda\} \mid \lambda \in I\}.$$

A *net* in a set X is a pair (I, f) such that I is an upper directed preordered set and f is a map of I into X .

Let X be a topological space. An ω -*net* in X is a net (I, f) in X such that for any increasing sequence $(\iota_n)_{n \in \mathbb{N}}$ in I the sequence $(f(\iota_n))_{n \in \mathbb{N}}$ is convergent. An ω -*filter* on X is a filter \mathfrak{F} on X such that there exists an ω -net (I, f) in X such that $f(\mathfrak{F}) \subset \mathfrak{F}$, where \mathfrak{G} denotes the section filter of I . An ω *space* is a topological space for which any ω -filter converges.

PROPOSITION 2.1. *Any ω -filter on a uniform space is a Cauchy filter. Hence any complete uniform space is an ω -space.*

Let X be a uniform space, let (I, f) be an ω -net in X , and let \mathfrak{F} be the section filter of I . Let further U be an arbitrary entourage (= vicinity) of X and let V be an entourage of X such that $V \circ V^{-1} \subset U$. Assume that for any $\iota \in I$ there exists $\lambda \in I$ such that $\lambda \geq \iota$ and $(f(\iota), f(\lambda)) \notin V$. Then we may construct inductively an increasing sequence $(\iota_n)_{n \in \mathbb{N}}$ in I such that $(f(\iota_n), f(\iota_{n+1})) \notin V$ for any $n \in \mathbb{N}$. The sequence $(f(\iota_n))_{n \in \mathbb{N}}$ being convergent this is a contradiction. Hence there exists $\iota \in I$ with $(f(\iota), f(\lambda)) \in V$ for any $\lambda \in I$, $\lambda \geq \iota$. We get $(f(\iota'), f(\iota'')) \in U$ for any $\iota', \iota'' \in I$ with $\iota' \geq \iota$, $\iota'' \geq \iota$. Hence $f(\mathfrak{F})$ is a Cauchy filter. ■

PROPOSITION 2.2. *For any measure μ on \mathfrak{R} and for any $\mathfrak{A} \in \Lambda$, $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω -filter and therefore a Cauchy filter.*

Let us order \mathfrak{A} by the converse inclusion relation, let \mathfrak{G} be the section filter of \mathfrak{A} , and let $\mu \upharpoonright \mathfrak{A}$ be the restriction of μ to \mathfrak{A} . Then $(\mathfrak{A}, \mu \upharpoonright \mathfrak{A})$ is an ω -net and

$$\mu(\mathfrak{F}(\mathfrak{A})) = \mu \upharpoonright \mathfrak{A}(\mathfrak{G}).$$

Hence $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω -filter. By Proposition 2.1 it is a Cauchy filter. ■

PROPOSITION 2.3. *Let μ be a measure on \mathfrak{R} and let \mathfrak{A} be a maximal element of Λ . Then either \mathfrak{A} is an atom of μ or $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0.*

Assume that $\mu(\mathfrak{F}(\mathfrak{A}))$ does not converge to 0. By Proposition 2.2 there exist a 0-neighbourhood V and an $A \in \mathfrak{A}$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \cap V = \emptyset.$$

Let $(A_n)_{n \in \mathbb{N}}$ be a sequence in \mathfrak{A} . If $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ then there exists a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ in \mathfrak{A} with empty intersection and such that $B_0 \subset A$. It follows that $(\mu(B_n))_{n \in \mathbb{N}}$ converges to 0 and this contradicts the above relation. Hence $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$. The set

$$\left\{ B \in \mathfrak{R} \mid \exists C \in \mathfrak{A}, \left(\bigcap_{n \in \mathbb{N}} A_n \right) \cap C \subset B \right\}$$

belongs to Λ and contains \mathfrak{A} . Since \mathfrak{A} is maximal it coincides with \mathfrak{A} . We deduce $\bigcap_{n \in \mathbb{N}} A_n \in \mathfrak{A}$. Since $\mathfrak{A} \cap \mathfrak{N}(\mu) = \emptyset$ we deduce $\mathfrak{A} \in \Lambda(\mathfrak{N}(\mu))$. It is obvious that \mathfrak{A} is a maximal element of $\Lambda(\mathfrak{N}(\mu))$. Hence \mathfrak{A} is an atom of μ . ■

COROLLARY 2.4. *Let μ, ν be measures on \mathfrak{R} and let \mathfrak{A} be an atom of μ . Then either \mathfrak{A} is an atom of ν or $\nu(\mathfrak{F}(\mathfrak{A}))$ converges to 0.*

By Proposition 1.3(d) \mathfrak{A} is a maximal element of Λ and the assertion follows from the proposition. ■

PROPOSITION 2.5. *Let μ be a \mathfrak{R} -regular measure on \mathfrak{R} and let \mathfrak{A} be a proper atom of μ . Then for any $A \in \mathfrak{A}$ there exists $K \in \mathfrak{R} \cap \mathfrak{A}$ with $K \subset A$.*

By Proposition 2.3 $\mu(\mathfrak{F}(\mathfrak{A}))$ is a Cauchy filter. Since \mathfrak{A} is a proper atom of μ , 0 is not an adherent point of this filter. Hence there exist a neighbourhood V of 0 and a set $B \in \mathfrak{A}$ such that

$$\{\mu(C) \mid C \in \mathfrak{A}, C \subset B\} \cap V = \emptyset.$$

Since μ is \mathfrak{R} -regular there exists $K \in \mathfrak{R}$ such that $K \subset A \cap B$ and

$$\{\mu(C) \mid C \in \mathfrak{R}, C \subset A \cap B \setminus K\} \subset V.$$

Let $C \in \mathfrak{A}$. If $C \cap K \in \mathfrak{N}(\mu)$ then by Proposition 1.3(a) $C \setminus K \in \mathfrak{A}$ and therefore

$$A \cap B \cap (C \setminus K) \in \mathfrak{A}, \quad \mu(A \cap B \cap (C \setminus K)) \in V$$

which is a contradiction. Hence $C \cap K \notin \mathfrak{N}(\mu)$ for any $C \in \mathfrak{A}$. By Proposition 1.3(b) we get $K \in \mathfrak{A}$. ■

3. Atomic Measures

A measure possessing no proper atom is called *atomless*. If it possesses no atoms at all it is called *strictly atomless*. Any improper atom of an atomless measure is of the second kind.

PROPOSITION 3.1. *The set of atomless (resp. strictly atomless) G -valued measures on \mathfrak{R} is a subgroup of $G^{\mathfrak{R}}$.*

Let μ, ν be two G -valued measures on \mathfrak{R} and let \mathfrak{A} be an atom of $\mu - \nu$. Since

$$\mathfrak{N}(\mu) \cap \mathfrak{N}(\nu) \subset \mathfrak{N}(\mu - \nu)$$

it follows that

$$\mathfrak{A} \cap \mathfrak{N}(\mu) = \emptyset \quad \text{or} \quad \mathfrak{A} \cap \mathfrak{N}(\nu) = \emptyset$$

By Corollary 1.4 we deduce that \mathfrak{A} is an atom of either μ or ν . This shows that the set of strictly non-atomic G -valued measures is a subgroup of $G^{\mathfrak{R}}$.

Assume now that \mathfrak{A} is a proper atom of $\mu - \nu$. By Proposition 1.3(d) \mathfrak{A} is a maximal element of Λ . By Proposition 2.3, \mathfrak{A} is a proper atom of either μ or ν . Hence the set of G -valued atomless measures on \mathfrak{R} is a subgroup of $G^{\mathfrak{R}}$. ■

We say that a measure μ on \mathfrak{R} satisfies the *atom condition* (abbreviated *ac*) if for any atom \mathfrak{A} of μ , $\mu(\mathfrak{F}(\mathfrak{A}))$ is convergent; according to the general convention made above we denote by $\mu_{\mathfrak{A}}$ the limit of $\mu(\mathfrak{F}(\mathfrak{A}))$ which may be interpreted as the value of μ at \mathfrak{A} . By Proposition 1.1 any measure satisfying locally *ccc* satisfies *ac*. By Proposition 2.2 any measures with values in an ω -topological group and a fortiori in a complete topological group (Proposition 2.1) satisfies *ac*. The set of G -valued measures on \mathfrak{R} satisfying *ac* is a subgroup of $G^{\mathfrak{R}}$ (Corollary 2.4).

A measure μ on \mathfrak{R} satisfying *ac* is called *atomic* if for any $A \in \mathfrak{R}$, $\mu(A)$ is the sum of the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$, where Φ denotes the set of atoms \mathfrak{A} of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements \mathfrak{A} of Λ such that $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G -valued atomic measures on \mathfrak{R} is a subgroup of $G^{\mathfrak{R}}$. Any improper atom of an atomic measure is of the first kind. A measure which is at the same time atomic and atomless vanishes identically.

PROPOSITION 3.2. *Let μ, μ' be two atomless G -valued measures on \mathfrak{R} and let ν, ν' be two atomic G -valued measures on \mathfrak{R} . If*

$$\mu + \nu = \mu' + \nu'$$

then $\mu = \mu'$ and $\nu = \nu'$.

By Proposition 3.1, $\mu - \mu'$ is an atomless measure on \mathfrak{R} . Since it is at the same time an atomic measure it vanishes identically. ■

We say that a measure μ on \mathfrak{R} satisfying *ac* satisfies the *atomical summability condition* (abbreviated *asc*) if for any $A \in \mathfrak{R}$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable, where Φ denotes the set of atoms \mathfrak{A} of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements \mathfrak{A} for Λ for which $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G -valued measures on \mathfrak{R} satisfying *asc* is a subgroup of $G^{\mathfrak{R}}$. Any atomic measure satisfies *asc*.

PROPOSITION 3.3. *Let μ be a G -valued measure on \mathfrak{R} satisfying ac and let Φ be a countable set of atoms of μ such that $\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A} \neq \emptyset$ and such that $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$ converges. Then the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable and its sum is the limit of $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A} \in \Phi} \mathfrak{A}))$.*

Let M be a subset of $\mathbf{N} \setminus \{\emptyset\}$ and let $(\mathfrak{A}_n)_{n \in M}$ be a family of atoms of μ such $\mathfrak{A}_m \neq \mathfrak{A}_n$ for any different $m, n \in M$ and $\Phi = \{\mathfrak{A}_n \mid n \in M\}$. We set $\mathfrak{A} := \bigcap_{n \in M} \mathfrak{A}_n$. Let V be an arbitrary 0-neighbourhood in G and let $(V_n)_{n \in \mathbf{N}}$ be a sequence of 0-neighbourhoods in G such that $V_0 + V_0 - V_0 \subset V$ and such that $V_{n+1} + V_{n+1} \subset V_n$ for any $n \in \mathbf{N}$. For any $n \in M$ there exists $A_n \in \mathfrak{A}_n$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}_n, B \subset A_n\} \subset \mu_{\mathfrak{A}_n} + V_n.$$

There exists $A \in \bigcap_{n \in M} \mathfrak{A}_n$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V_0.$$

By Proposition 1.5 there exists a disjoint family $(B_n)_{n \in M}$ in \mathfrak{R} such that $B_n \in \mathfrak{A}_n$ for any $n \in M$. Then $(A_n \cap B_n \cap A)_{n \in M}$ is a disjoint family in \mathfrak{R} whose union belongs to \mathfrak{R} and therefore

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) = \sum_{n \in M} \mu(A_n \cap B_n \cap A)$$

Since $\bigcup_{n \in M} (A_n \cap B_n \cap A) \in \mathfrak{A}$ (Proposition 1.3(a)) we have

$$\mu\left(\bigcup_{n \in M} (A_n \cap B_n \cap A)\right) \in \mu_{\mathfrak{A}} + V_0.$$

For any $n \in M$ we get

$$\mu(A_n \cap B_n \cap A) \in \mu_{\mathfrak{A}_n} + V_n.$$

Let M_0 be a finite subset of M such that

$$\sum_{n \in M'} \mu(A_n \cap B_n \cap A) - \sum_{n \in M} \mu(A_n \cap B_n \cap A) \in V_0$$

for any finite subset M' of M containing M_0 . We deduce for any finite subset M' of M containing M_0 .

$$\sum_{n \in M'} \mu_{\mathfrak{A}_n} - \mu_{\mathfrak{A}} \in V_0 + V_0 - V_0 \subset V.$$

Since V is arbitrary it follows that $(\mu_{\mathfrak{A}_n})_{n \in M}$ is summable and its sum is $\mu_{\mathfrak{A}_0}$. ■

PROPOSITION 3.4. *Any measure satisfying locally ccc satisfies asc.*

Let μ be a measure on \mathfrak{R} satisfying locally ccc. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any $\mathfrak{A} \in \Lambda$; in particular μ satisfies ac. Let $A \in \mathfrak{R}$ and let Φ be the set of atoms \mathfrak{A} of μ such that $A \in \mathfrak{A}$. By Proposition 1.2(c) Φ is countable. By the preceding proposition $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable. Hence μ satisfies asc. ■

PROPOSITION 3.5. *If G is an ω -space then any G -valued measure satisfies asc.*

Let μ be a G -valued measure on \mathfrak{R} . By Proposition 2.2 for any $\mathfrak{A} \in \Lambda$ the filter $\mu(\mathfrak{F}(\mathfrak{A}))$ converges; in particular μ satisfies ac. Let $A \in \mathfrak{R}$, let Φ be the set of atoms \mathfrak{A} of μ such that $A \in \mathfrak{A}$, and let $\mathfrak{P}_c(\Phi)$ be the set of countable subsets of Φ ordered by the inclusion relation. By Proposition 3.3 for any $\Psi \in \mathfrak{P}_c(\Phi)$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Psi}$ is summable; let us denote by f the map

$$\Psi \mapsto \sum_{\mathfrak{A} \in \Psi} \mu_{\mathfrak{A}} : \mathfrak{P}_c(\Phi) \rightarrow G.$$

Then $(\mathfrak{P}_c(\Phi), f)$ is an ω -net in G . Hence if \mathfrak{F} denotes the section filter of $\mathfrak{P}_c(\Phi)$ then $f(\mathfrak{F})$ converges. We deduce that the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable. Hence μ satisfies asc. ■

THEOREM 3.6. *Let μ be a \mathfrak{R} -regular G -valued measure on \mathfrak{R} satisfying asc. We denote for any $A \in \mathfrak{R}$ by $\Phi(A)$ the set of atoms \mathfrak{A} of μ such that $A \in \mathfrak{A}$ and by μ' the map*

$$A \mapsto \sum_{\mathfrak{A} \in \Phi(A)} \mu_{\mathfrak{A}} : \mathfrak{R} \rightarrow G.$$

Then μ' (resp. $\mu - \mu'$) is an atomic (resp. atomless) \mathfrak{R} -regular measure on \mathfrak{R} absolutely continuous with respect to μ (i.e. $\mathfrak{N}(\mu) \subset \mathfrak{N}(\mu') \cap \mathfrak{N}(\mu - \mu')$). The proper atoms and the improper atoms of the first kind of μ and μ' coincide and we have

$\mu_{\mathfrak{A}} = \mu'_{\mathfrak{A}}$ for any atom \mathfrak{A} of μ . Any improper atom of μ of the second kind is an improper atom of $\mu - \mu'$ of the second kind.

Let $(A_i)_{i \in I}$ be a countable disjoint family in \mathfrak{R} whose union belongs to \mathfrak{R} . Then $(\Phi(A_i))_{i \in I}$ is a disjoint family and by Proposition 1.3(c) its union is $\Phi(\bigcup_{i \in I} A_i)$. We get

$$\mu' \left(\bigcup_{i \in I} A_i \right) = \sum_{\substack{\mathfrak{A} \in \Phi(\bigcup_{i \in I} A_i) \\ i \in I}} \mu_{\mathfrak{A}} = \sum_{i \in I} \sum_{\mathfrak{A} \in \Phi(A_i)} \mu_{\mathfrak{A}} = \sum_{i \in I} \mu'(A_i).$$

Hence μ' is a measure.

Let $A \in \mathfrak{R}$ and let U be a closed 0-neighbourhood in G . Then there exists a finite subset Ψ_0 of $\Phi(A)$ such that

$$\mu'(A) - \sum_{\mathfrak{A} \in \Psi_0} \mu_{\mathfrak{A}} \in U$$

for any finite subset Ψ of $\Phi(A)$ containing Ψ_0 . By Proposition 2.5 there exists for any $\mathfrak{A} \in \Phi(A)$ a set $K_{\mathfrak{A}} \in \mathfrak{R} \cap \mathfrak{A}$ with $K_{\mathfrak{A}} \subset A$. We set

$$K := \bigcup_{\mathfrak{A} \in \Psi_0} K_{\mathfrak{A}} \in \mathfrak{R}$$

Let B be a set of \mathfrak{R} such that $K \subset B \subset A$. Then $\Phi(B)$ is a subset of $\Phi(A)$ containing Ψ_0 (Proposition 1.3(a)) and therefore

$$\mu'(A) - \mu'(B) = \mu'(A) - \sum_{\mathfrak{A} \in \Phi(B)} \mu_{\mathfrak{A}} \in U.$$

This shows that μ' is \mathfrak{R} -regular. We deduce that $\mu - \mu'$ is a \mathfrak{R} -regular measure on \mathfrak{R} . It is obvious that μ' and $\mu - \mu'$ are absolutely continuous with respect to μ .

Let \mathfrak{A} be an atom of μ . Let U be a closed 0-neighbourhood in G and let V be a 0-neighbourhood in G such that $V - V - V \subset U$. There exists $A \in \mathfrak{A}$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \subset \mu_{\mathfrak{A}} + V.$$

Let Ψ be a finite nonempty subset of $\Phi(A) \setminus \{\mathfrak{A}\}$. Then there exists

$$B \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}' \setminus \mathfrak{A}$$

such that $B \subset A$ and

$$\left\{ \mu(C) \mid C \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}', C \subset B \right\} \subset \sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} + V.$$

Then $A \setminus B \in \mathfrak{A}$ (Proposition 1.3(c)) and therefore

$$\mu(A \setminus B) \in \mu_{\mathfrak{A}} + V,$$

$$\sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} \in \mu(B) - V = \mu(A) - \mu(A \setminus B) - V \subset V - V - V \subset U.$$

Since Ψ is arbitrary we get

$$\sum_{\mathfrak{A}' \in \Phi(A) \setminus \{\mathfrak{A}\}} \mu_{\mathfrak{A}'} \in U, \quad \mu'(A) \in \mu_{\mathfrak{A}} + U.$$

Since U is arbitrary we deduce $\mu'_{\mathfrak{A}} = \mu_{\mathfrak{A}}$. Hence the proper atoms of μ and μ' coincide (Corollary 2.4). We deduce further that the improper atoms of μ and μ' of the first kind coincide (Corollary 1.4). Moreover for any $A \in \mathfrak{R}$ we get

$$\mu'(A) = \sum_{\mathfrak{A} \in \Phi} \mu_{\mathfrak{A}} = \sum_{\mathfrak{A} \in \Phi} \mu'_{\mathfrak{A}},$$

where Φ denotes the set of maximal elements \mathfrak{A} of Λ such that $A \in \mathfrak{A}$ (Propositions 1.3(d) and 2.3). Hence μ' is an atomic measure.

Let \mathfrak{A} be an atom of $\mu - \mu'$. Then \mathfrak{A} is an atom of μ (Corollary 1.4) and by the above considerations it follows that \mathfrak{A} is an improper atom of $\mu - \mu'$. Hence $\mu - \mu'$ is atomless.

Let \mathfrak{A} be an improper atom of μ of the second kind. Then there exists $A \in \mathfrak{A}$ such that $A \in \mathfrak{N}(\mu')$ and therefore \mathfrak{A} is an atom of $\mu - \mu'$ (Corollary 1.7)). Since $\mu - \mu'$ is atomless it is an improper atom of $\mu - \mu'$ of the second kind. ■

Example. We want to give an example of a locally convex space E and of an E -valued measure on a σ -algebra of sets, possessing an improper atom of the second kind. Let X be a set. For any $A \subset X \times [0, 1]$ and for any $x \in X$ we set

$$A(x) := \{y \in [0, 1] \mid (x, y) \in A\}.$$

We denote by \mathfrak{R} the set of $A \subset X \times [0, 1]$ such that: (a) $A(x)$ is a Borel set for any $x \in X$; (b) the set $\{x \in X \mid A(x) \neq \emptyset \text{ and } A(x) \neq [0, 1]\}$ is countable. It is obvious

that \mathfrak{A} is a σ -algebra of subsets of $X \times [0, 1]$. For any $A \in \mathfrak{A}$ we denote by $\mu(A)$ the map

$$x \mapsto \lambda(A(x)) : X \rightarrow \mathbf{R},$$

where λ denotes the Lebesgue measure on $[0, 1]$. It is easy to see that μ is an atomless \mathbf{R}^X -valued measure on \mathfrak{A} . Let \mathfrak{F} be a non-trivial ultrafilter on X such that the intersection of any countable family in \mathfrak{F} belongs to \mathfrak{F} (we assume that such an ultrafilter exists). We set

$$\mathfrak{U} := \{A \in \mathfrak{A} \mid \{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}\}.$$

Then $\mathfrak{U} \in \Lambda(\mu)$. Let $\mathfrak{U}' \in \Lambda(\mu)$ with $\mathfrak{U} \subset \mathfrak{U}'$, let $A \in \mathfrak{U}'$, and let

$$X_0 := \{x \in X \mid A(x) \neq \emptyset\}$$

If $X_0 \notin \mathfrak{F}$ then $X \setminus X_0 \in \mathfrak{F}$ and therefore $(X \setminus X_0) \times [0, 1] \in \mathfrak{U}$ and this leads to the contradictory relation

$$\emptyset = A \cap ((X \setminus X_0) \times [0, 1]) \in \mathfrak{U}'.$$

Hence $X_0 \in \mathfrak{F}$. Since $A \in \mathfrak{A}$ the set

$$\{x \in X \mid A(x) \neq \emptyset \text{ and } A(x) \neq [0, 1]\}$$

is countable and therefore it does not belong to \mathfrak{F} . Hence

$$\{x \in X \mid A(x) = [0, 1]\} \in \mathfrak{F}$$

and we deduce successively $A \in \mathfrak{U}$, $\mathfrak{U} = \mathfrak{U}'$ and \mathfrak{U} is an atom of μ . The measure μ being atomless \mathfrak{U} is an improper atom of the second kind.

Example. Let X be an uncountable set, let $\mathfrak{P}(X)$ be the set of subsets of X , and for any $A \in \mathfrak{P}(X)$ let l_A be the characteristic function of A . Then

$$A \mapsto l_A : \mathfrak{P}(X) \rightarrow \mathbf{R}^X$$

is an example of a measure satisfying *asc* and not satisfying *ccc*.

REFERENCES

- [1] HOFFMANN-JØRGENSEN, J., *Vector measures*, Math. Scand., 28 (1971), 5–32.
- [2] MUSIAŁ, K., *Absolute continuity and the range of group valued measure*, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys. 21 (1973), 105–113.

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