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A New Bound for the Genus of a Nilpotent Group

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Introduction

Gr is the category of groups with the usual morphisms; [Gr] the class of isomorphism classes of Gr; η_0 the full subcategory of Gr consisting of all finitely generated infinite nilpotent groups with finite commutator subgroups; N_p and \hat{N}_p are the p -localization and the p -completion of the nilpotent group N .

Assume that N is a finitely generated nilpotent group. In the sense of Pickel ([1]) the genus set of N , denoted by $G_p(N)$, is the set of isomorphism classes of the finitely generated nilpotent groups M with $\hat{N}_p \simeq \hat{M}_p$ for each prime p , and $N_0 \simeq M_0$ (N_0 is the rationalization of N). In the sense of Mislin, the genus set of N , $G(N)$, is the set of the finitely generated nilpotent groups M with $M_p \simeq N_p$ for each prime p (see [2]).

In this note, we are only concerned with groups N in η_0 ; in this case, it has just been proved by Warfield in [3] that $G_p(N) = G(N)$.⁽¹⁾ This result is also an easy consequence of our proof.

This paper provides a bound for $G_p(N)$, when N is in η_0 , thus a bound for $G(N)$ (see the theorem); examples in section 3 show that this bound can be an improvement of Mislin's one ([2]).

1. Preliminaries

For a nilpotent group N , we denote its maximal torsion subgroup by TN , the p -components by T_pN and the quotient N/TN by SN .

LEMMA 1. *Let N be in η_0 , M a finitely generated nilpotent group with $\hat{M}_p \simeq \hat{N}_p$ for each prime p . Then*

(a) $TN \simeq TM$

(b) $SN \simeq SM$

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⁽¹⁾ I am very grateful to Professor Mislin for having reported me this fact before the publication of [3].

- (c) M belongs to η_0 (d) $N_0 \cong M_0$
 (e) the class of M belongs to $G_p(N)$.

Proof. (a) is proved by Pickel ([1], proposition 3.5).

(b) SN is abelian (since $[N, N]$ is included in TN) thus free abelian; then SM is also abelian as a subgroup of \widehat{SM}_p which is isomorphic to \widehat{SN}_p and SM is free abelian with the same rank since SN and SM have isomorphic finite quotients ([1], proposition 3.5).

(c) Since SM is abelian, $[M, M]$ is included in TM which is finite.

(d) $N_0 \cong (SN)_0 \cong (SM)_0 \cong M_0$.

(e) obvious, by hypothesis and d.

Among the invariants in $G_p(N)$, we have thus

- the groups TN and SN , which we shall denote simply by T and S .
- the rank k of S .
- the set Q of primes p for which T_p is non-trivial.

The lemma shows us that the class of M belongs to $G_p(N)$ if and only if $\hat{N}_p \cong \hat{M}_p$ for each prime p and that each element in $G_p(N)$ can be described as the class of the central term M of an extension:

$$T \triangleright M \twoheadrightarrow S$$

With this description, we have only to consider primes in Q , since, for $p \notin Q$, $\hat{M}_p \cong \hat{S}_p \cong \hat{N}_p$. This is the basis of the proof of the theorem; before stating it, we need two further notions.

From [4] we recall the definition of Blackburn's function for a prime p .

It is the function defined recursively by

$$d_p(0) = 0$$

$$d_p(n) = d_p(n-1) + m$$

where p^m is the highest power of p that does not exceed n .

With the aid of d_p , we introduce

$$u(p) = t(p) + d_p(\text{nil } N) \quad \text{where} \quad \exp T = \prod_{p \in Q} p^{t(p)}$$

and

$$u = \prod_{p \in Q} p^{u(p)}$$

2. The Main Result

THEOREM. *Let N be in η_0 . Then $|G_p(N)| \leq \phi(u)/2$ (ϕ is Euler's function and we assume $u > 2$).*

Proof. Since all the products have Q as their index set, we shall omit the subscripts. Define $E(X, Y)$ to be the class of equivalence classes of all the extensions

$$Y \twoheadrightarrow K \rightarrow X$$

θ is the map from $E(S, T)$ to $[Gr]$ defined by "take the isomorphism class of the central term."

We know that $G_p(N)$ is included in $Im \theta$. In particular, we can choose x_0 in $E(S, T)$ such that $\theta(x_0)$ is the class of N .

We intend to build a diagram Δ :

$$\begin{array}{ccccc}
 \hat{\Omega} = \prod GL(k, \hat{\mathbb{Z}}_p) & & & & \\
 \downarrow \delta & \searrow \lambda & & & \\
 GL(k, \mathbb{Z}/u) & \xrightarrow{\mu} & E(S, T) & \xrightarrow{\theta} & [Gr] \\
 \downarrow \gamma & & \nearrow \nu & & \\
 (\mathbb{Z}/u)^*/\{1, -1\} & & & &
 \end{array}$$

where $(\mathbb{Z}/u)^*/\{1, -1\}$ is the cokernel of $(GL(k, \mathbb{Z}) \rightarrow GL(k, \mathbb{Z}/u))$; γ, δ are the canonical epimorphisms; the two triangles are commutative; $G_p(N) = Im \nu = Im(\theta \circ \mu)$.

The result follows from $|(\mathbb{Z}/u)^*/\{1, -1\}| = \phi(u)/2$.

First step: Definition of λ

For each p in Q , there exists a map C_p from $E(S, T)$ to $E(\hat{S}_p, T_p)$ defined by completion (we know that the completion is exact). Let us denote by \hat{C} the product map. C_p can be factorized into a map τ_p from $E(S, T)$ to $E(S, T_p)$:

$$\tau_p(T \twoheadrightarrow E \rightarrow S) = (T_p \twoheadrightarrow E/T_{p'}, E \rightarrow S)$$

(p' is the complement of p in the set of prime numbers) and the p -completion \bar{C}_p , applied now to $E(S, T_p)$ see [] p. 339). We denote by τ and \bar{C} the product maps.

On the other hand, we can define:

- a map pb from $\prod E(\hat{S}_p, T_p)$ to $\prod E(S, T_p)$ by pull-back along the canonical injections of S in \hat{S}_p
- and a map PB from $\prod E(S, T_p)$ to $E(S, T)$ defined by

$$\left(\begin{array}{ccc} T_p \longrightarrow E_p \xrightarrow{\sigma_p} S \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ T_q \longrightarrow E_q \xrightarrow{\sigma_q} S \end{array} \right) \mapsto (T \longrightarrow E \longrightarrow S)$$

where E is the pull-back of the groups E_p along the morphisms σ_p .

It is easy to check that \bar{C} and pb , τ and PB , are inverses of one another. Thus:

$$E(S, T) \simeq \prod E(\hat{S}_p, T_p)$$

Now, each ω_p belonging to $GL(k, \hat{\mathbb{Z}}_p)$ acts on $E(\hat{S}_p, T_p)$ by pull-back along ω_p^{-1} . We define $\hat{\Omega} = \prod GL(k, \hat{\mathbb{Z}}_p)$ and

$$\lambda : \hat{\Omega} \rightarrow E(S, T) : (\omega_p) \mapsto \hat{C}^{-1}(\omega_p C_p(x_0))$$

Second step: Definition of μ

Consider δ to be the canonical epimorphism from $\hat{\Omega}$ onto $GL(k, \mathbb{Z}/u) \simeq \prod GL(k, \mathbb{Z}/p^{u(p)})$.

The existence of μ and the commutativity of the upper triangle of Δ are proved by the

LEMMA 2. *If ω belongs to $\ker \delta$, then $\lambda(\omega) = x_0$.*

Proof. For simplicity, if \hat{X}_p is a p -complete nilpotent group, we denote by $\hat{X}_p^{(u)}$ the subgroup of \hat{X}_p generated by the $u(p)$ -powers and by $\hat{X}_p^{/u}$ the quotient of \hat{X}_p modulo $\hat{X}_p^{(u)}$. If ω belongs to $\ker \delta$, the map induced by ω_p on $\hat{S}_p^{/u}$ is the identity. On the other hand, by our choice of $u(p)$, each element of $\hat{N}_p^{(u)}$ is a $p^{i(p)}$ -power ([4], Lemma 2), thus $\text{Im } T_p \cap \hat{N}_p^{(u)} = \{1\}$. We can

consider the commutative diagrams

$$\begin{array}{ccccc} T_p & \longrightarrow & \hat{N}_p & \twoheadrightarrow & \hat{S}_p \\ \parallel & & \downarrow & & \downarrow \\ T_p & \longrightarrow & \hat{N}_p^{/u} & \twoheadrightarrow & \hat{S}_p^{/u} \end{array}$$

where the right squares are pull-backs. The action of the ω_p 's does not modify the bottoms, neither, consequently, the tops, and thus $\lambda(\omega) = x_0$.

Third Step: $\text{Im}(\mu \circ \theta) = G_p(N)$

If the class of M belongs to $\text{Im}(\mu \circ \theta)$, M is clearly finitely generated and nilpotent, and \hat{M}_p is isomorphic to \hat{N}_p for each p ; Lemma 1 shows that the class of M belongs to $G_p(N)$.

Suppose now that the class of M belongs to $G_p(N)$. We know that there exists

$$x: T \longrightarrow M \longrightarrow S$$

and, for each p , a commutative diagram

$$\begin{array}{ccccc} \hat{C}_p x_0: T_p & \longrightarrow & \hat{N}_p & \longrightarrow & \hat{S}_p \\ \xi_p \downarrow & & \epsilon_p \downarrow & & \omega_p \downarrow \\ \hat{C}_p x: T_p & \longrightarrow & \hat{M}_p & \longrightarrow & \hat{S}_p \end{array}$$

We state that M is isomorphic to the central term of $\mu(\delta(\omega)) = \lambda(\omega)$, where $\omega = (\omega_p)$. Indeed, there exists, for each p , a commutative diagram

$$\begin{array}{ccccc} T_p & \longrightarrow & N_{\omega_p} & \longrightarrow & \hat{S}_p \\ \xi_p \downarrow & & \epsilon_p \downarrow & & \parallel \\ T_p & \longrightarrow & \hat{M}_p & & \hat{S}_p \end{array}$$

the first row being $\omega_p C_p x_0$. Applying pb and then PB , we obtain the required isomorphism.

Fourth step: Definition of ν

We can identify $\text{GL}(k, \mathbb{Z})$ with a subgroup of $\hat{\Omega}$. Now we prove:

LEMMA 3. *Let ω_1, ω_2 be in $\hat{\Omega}$ and suppose there exists η in $\text{GL}(k, \mathbb{Z})$ such that $\omega_2 = \eta \omega_1$. Then $(\theta \circ \lambda) \omega_1 = (\theta \circ \lambda) \omega_2$.*

Proof. For each p , there exists a commutative diagram

$$\begin{array}{ccccc}
 T_p & \longrightarrow & N_{\omega_{1,p}} & \longrightarrow & \hat{S}_p \\
 \parallel & & \downarrow & & \downarrow \\
 T_p & \longrightarrow & N_{\omega_{2,p}} & \longrightarrow & \hat{S}_p
 \end{array}$$

where the rows are $\omega_{1,p}C_p x_0$ and $\omega_{2,p}C_p x_0$ and η_p is the extension of η to \hat{S}_p . Such a diagram still exists after applying pb , with the same η for each p and PB produces isomorphic central terms.

In order to define ν and assure the commutativity of the lower triangle of Δ , we have to prove that $(\theta \circ \mu) \delta(\omega)$ only depends on the class of $\delta(\omega)$ modulo $\delta(\text{GL}(k, \mathbb{Z}))$. Suppose $\delta(\omega_2) = \delta(\eta) \cdot \delta(\omega_1)$. Then: $(\theta \circ \mu) \delta(\omega_2) = (\theta \circ \mu) \delta(\eta \omega_1) = (\theta \circ \lambda)(\eta \omega_1) = (\theta \circ \lambda) \omega_1$ (by Lemma 3) $= (\theta \circ \mu) \delta(\omega_1)$. The proof of the theorem is completed.

Remark. The proof can be rewritten with p -localization instead of p -completions. All differences disappear after the second step, and the third implies the equality, in η_0 , of $G_p(N)$ and $G(N)$ (see [3] for another proof.)

3. Comparison with Mislin's Result

Since the theorem gives a bound for $|G(N)|$, it is interesting to compare this bound to that established by Mislin in [2]. We adopt the improvement of Mislin's bound introduced at the end of [5] and we follow the notation of this paper as far as possible. We have to compare $u(p) = t(p) + d_p(\text{nil } N)$ to the exponents $v(p)$ of $\exp(QN)_{ab}$ where $QN = N / (ZN)^{\exp TZN}$. First, two remarks.

If N is abelian, $N = ZN$, $TN = TZN$, $\exp(QN)_{ab} = \exp TN$ and $d_p(1) = 0$ for any p : the two bounds are always equal. More generally, if we assume that TN is central (and $2 \notin Q$), then $\text{nil } N \leq 2$ and $d_p(\text{nil } N) = 0$; $(QB)_{ab}$ contains at least a copy of $\mathbb{Z} / \exp TN$, and $u(p) \leq v(p)$. In this case, we can have $u(p) < v(p)$ (see below).

Our bound can give new information, as it is shown by the following class of examples (a modified version of examples elaborated by Professor G. Frei, whom I thank here). The proof is standard and we omit it.

Consider groups N with generators $a_1, a_2, a_3, \dots, a_{2n+1}$, relations $[a_i, a_j] = 1$ except if i is odd and $j = i + 1$; in this case, $[a_i, a_{i+1}] = a_{i+2}$; we assume also that the a_i 's are of finite order for $i \in J = \{3, 5, \dots, 2n + 1\}$. From $[a_i^m, a_{i+1}] = a_{i+2}^m$ ($i = 3, 5, \dots, 2n - 1$), we conclude that the orders r_i of a_i ($i \in J$) are related by $r_{2n+1} | r_{2n-1} \cdots | r_3$. We assume $r_{2n+1} \neq 1$, and we write s_i for r_i / r_{i+2} ($i \in J$) with $r_{2n+3} = 1$. For $i \geq 2$, $\Gamma_i(N)$ is generated by $\{a_j \mid j \in J, j \geq 2i - 1\}$. Thus

N is nilpotent of class $n+1$ and its commutator subgroup is finite. ZN is generated by $a_1^{r_3}, a_2^{r_3}, a_3^{r_5}, \dots, a_{2n+1}$ and $TZN \simeq \prod_{i \in J} \mathbb{Z}/s_i$; its exponent is the LCM of the s_i 's, denoted by s . r_{2n+1} divides s such that s is not 1. $ZN^{\exp TZN}$ is the free abelian subgroup generated by $a_1^{sr_3}, a_2^{sr_3}, a_4^{sr_5}, \dots$. Finally, $(QN)_{ab}$ is the abelian group with $a_1, a_2, a_4, \dots, a_{2n}$ as generators and relations $a_1^{sr_3} = 1, a_2^{sr_3} = 1, \dots, a_{2n}^{sr_{2n+1}} = 1$; its exponent is sr_3 . On the other hand, TN is generated by $\{a_i \mid i \in J\}$. It is abelian and its exponent is r_3 .

If $x(p)$ represents the exponent of the p -component of the natural x ,

$$v(p) - u(p) = s(p) - d_p(n+1)$$

where

$$s(p) = \max (r_i(p) - r_{i+2}(p))$$

$$s(p) \geq r_{2n+1}(p);$$

$$s(p) \geq 1 \quad \text{for each } p \text{ in } Q$$

We can conclude:

- for any choice of the nilpotency class except 1 (n fixed) and any choice of $Q(2 \notin Q)$ there exists a group in η_0 for which our bound is strictly smaller than Mislin's one: it is sufficient to choose r_{2n+1} adequately.
- on the other hand, if we fix a bound for the exponent of TN , it is always possible (now choosing n) to find a group in η_0 for which Mislin's bound is better.

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