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A New Bound for the Genus of a Nilpotent Group

CLAUDE LEMAIRE

Introduction

Gr is the category of groups with the usual morphisms; [Gr] the class of isomorphism classes of Gr; η_0 the full subcategory of Gr consisting of all finitely generated infinite nilpotent groups with finite commutator subgroups; N_p and \hat{N}_p are the p-localization and the p-completion of the nilpotent group N.

Assume that N is a finitely generated nilpotent group. In the sense of Pickel ([1]) the genus set of N, denoted by $G_p(N)$, is the set of isomorphism classes of the finitely generated nilpotent groups M with $\hat{N}_p \simeq \hat{M}_p$ for each prime p, and $N_0 \simeq M_0$ (N_0 is the rationalization of N). In the sense of Mislin, the genus set of N, G(N), is the set of the finitely generated nilpotent groups M with $M_p \simeq N_p$ for each prime p (see [2]).

In this note, we are only concerned with groups N in η_0 ; in this case, it has just been proved by Warfield in [3] that $G_p(N) = G(N)$. This result is also an easy consequence of our proof.

This paper provides a bound for $G_p(N)$, when N is in η_0 , thus a bound for G(N) (see the theorem); examples in section 3 show that this bound can be an improvement of Mislin's one ([2]).

1. Preliminaries

For a nilpotent group N, we denote its maximal torsion subgroup by TN, the p-components by T_pN and the quotient N/TN by SN.

LEMMA 1. Let N be in η_0 , M a finitely generated nilpotent group with $\hat{M}_p \simeq \hat{N}_p$ for each prime p. Then

(a)
$$TN \simeq TM$$

(b)
$$SN \approx SM$$

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⁽¹⁾ I am very grateful to Professor Mislin for having reported me this fact before the publication of [3].

(c) M belongs to η_0

(d)
$$N_0 \approx M_0$$

(e) the class of M belongs to $G_p(N)$.

Proof. (a) is proved by Pickel ([1], proposition 3.5).

- (b) SN is abelian (since [N, N] is included in TN) thus free abelian; then SM is also abelian as a subgroup of \widehat{SM}_p which is isomorphic to \widehat{SN}_p and SM is free abelian with the same rank since SN and SM have isomorphic finite quotients ([1], proposition 3.5).
 - (c) Since SM is abelian, $[M, \dot{M}]$ is included in TM which is finite.
 - (d) $N_0 \simeq (SN)_0 \simeq (SM)_0 \simeq M_0$.
 - (e) obvious, by hypothesis and d.

Among the invariants in $G_p(N)$, we have thus

- —the groups TN and SN, which we shall denote simply by T and S.
- —the rank k of S.
- —the set Q of primes p for which T_p is non-trivial.

The lemma shows us that the class of M belongs to $G_p(N)$ if and only if $\hat{N}_p \simeq \hat{M}_p$ for each prime p and that each element in $G_p(N)$ can be described as the class of the central term M of an extension:

$$T \longrightarrow M \longrightarrow S$$

With this description, we have only to consider primes in Q, since, for $p \notin Q$, $\hat{M}_p \simeq \hat{S}_p \simeq \hat{N}_p$. This is the basis of the proof of the theorem; before stating it, we need two further notions.

From [4] we recall the definition of Blackburn's function for a prime p. It is the function defined recursively by

$$d_p(0) = 0$$
$$d_p(n) = d_p(n-1) + m$$

where p^m is the highest power of p that does not exceed n.

With the aid of d_p , we introduce

$$u(p) = t(p) + d_p(\text{nil } N)$$
 where $\exp T = \prod_{p \in Q} p^{t(p)}$

and

$$u = \prod_{p \in O} p^{u(p)}$$

2. The Main Result

THEOREM. Let N be in η_0 . Then $|G_p(N)| \le \phi(u)/2$ (ϕ is Euler's function and we assume u > 2).

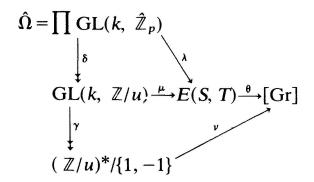
Proof. Since all the products have Q as their index set, we shall omit the subscripts. Define E(X, Y) to be the class of equivalence classes of all the extensions

$$Y \longrightarrow K \longrightarrow X$$

 θ is the map from E(S, T) to [Gr] defined by "take the isomorphism class of the central term."

We know that $G_p(N)$ is included in Im θ . In particular, we can choose x_0 in E(S, T) such that $\theta(x_0)$ is the class of N.

We intend to build a diagram Δ :



where $(\mathbb{Z}/u)^*/\{1, -1\}$ is the cokernel of $(GL(k, \mathbb{Z}) \longrightarrow GL(k, \mathbb{Z}/u))$; γ , δ are the canonical epimorphism; the two triangles are commutative; $G_p(N) = \text{Im } \nu = \text{Im } (\theta \circ \mu)$.

The result follows from $|(\mathbb{Z}/u)^*/\{1, -1\}| = \phi(u)/2$.

First step: Definition of λ

For each p in Q, there exists a map C_p from E(S, T) to $E(\hat{S}_p, T_p)$ defined by completion (we know that the completion is exact). Let us denote by \hat{C} the product map. C_p can be factorized into a map τ_p from E(S, T) to $E(S, T_p)$:

$$\tau_p(T \longmapsto E \longrightarrow S) = (T_p \longmapsto E/T_{p'}, E \longrightarrow S)$$

(p' is the complement of p in the set of prime numbers) and the p-completion \bar{C}_p , applied now to $E(S, T_p)$ see [] p. 339). We denote by τ and \bar{C} the product maps.

On the other hand, we can define:

a map pb from $\prod E(\hat{S}_p, T_p)$ to $\prod E(S, T_p)$ by pull-back along the canonical injections of S in \hat{S}_p

and a map PB from $\prod E(S, T_p)$ to E(S, T) defined by

where E is the pull-back of the groups E_p along the morphisms σ_p .

It is easy to check that \bar{C} and pb, τ and PB, are inverses of one another. Thus:

$$E(S, T) \simeq \prod E(\hat{S}_p, T_p)$$

Now, each ω_p belonging to $GL(k, \hat{\mathbb{Z}}_p)$ acts on $E(\hat{S}_p, T_p)$ by pull-back along ω_p^{-1} . We define $\hat{\Omega} = \prod_{p} GL(k, \hat{\mathbb{Z}}_p)$ and

$$\lambda: \hat{\Omega} \to E(S, T): (\omega_p) \mapsto \hat{C}^{-1}(\omega_p C_p(x_0))$$

Second step: Definition of μ

Consider δ to be the canonical epimorphism from $\hat{\Omega}$ onto $GL(k, \mathbb{Z}/u) \simeq \prod GL(k, \mathbb{Z}/p^{u(p)})$.

The existence of μ and the commutativity of the upper triangle of Δ are proved by the

LEMMA 2. If ω belongs to ker δ , then $\lambda(\omega) = x_0$.

Proof. For simplicity, if \hat{X}_p is a p-complete nilpotent group, we denote by $\hat{X}_p^{(u)}$ the subgroup of \hat{X}_p generated by the u(p)-powers and by $\hat{X}_p^{/u}$ the quotient of \hat{X}_p modulo $\hat{X}_p^{(u)}$. If ω belongs to ker δ , the map induced by ω_p on $\hat{S}_p^{/u}$ is the identity. On the other hand, by our choice of u(p), each element of $\hat{N}_p^{(u)}$ is a $p^{t(p)}$ -power ([4], Lemma 2), thus Im $T_p \cap \hat{N}_p^{(u)} = \{1\}$. We can

consider the commutative diagrams

$$T_{p} \longrightarrow \hat{N}_{p} \longrightarrow \hat{S}_{p}$$

$$\downarrow \qquad \qquad \downarrow$$

$$T_{p} \longrightarrow \hat{N}_{p}^{\prime u} \longrightarrow \hat{S}_{p}^{\prime u}$$

where the right squares are pull-backs. The action of the ω_p 's does not modify the bottoms, neither, consequently, the tops, and thus $\lambda(\omega) = x_0$.

Third Step: Im
$$(\mu \circ \theta) = G_p(N)$$

If the class of M belongs to $\text{Im}(\mu \circ \theta)$, M is clearly finitely generated and nilpotent, and \hat{M}_p is isomorphic to \hat{N}_p for each p; Lemma 1 shows that the class of M belongs to $G_p(N)$.

Suppose now that the class of M belongs to $G_p(N)$. We know that there exists

$$x: T \longrightarrow M \longrightarrow S$$

and, for each p, a commutative diagram

$$\hat{C}_{p}x_{0}: T_{p} \longrightarrow \hat{N}_{p} \longrightarrow \hat{S}_{p}$$

$$\stackrel{\xi_{p}|_{l}}{\longleftarrow} \stackrel{\epsilon_{p}|_{l}}{\longleftarrow} \stackrel{\omega_{p}|_{l}}{\longrightarrow} \hat{S}_{p}$$

$$\hat{C}_{p}x: T_{p} \longrightarrow \hat{M}_{p} \longrightarrow \hat{S}_{p}$$

We state that M is isomorphic to the central term of $\mu(\delta(\omega)) = \lambda(\omega)$, where $\omega = (\omega_p)$. Indeed, there exists, for each p, a commutative diagram

the first row being $\omega_p C_p x_0$. Applying pb and then PB, we obtain the required isomorphism.

Fourth step: Definition of ν

We can identify $GL(k, \mathbb{Z})$ with a subgroup of $\hat{\Omega}$. Now we prove:

LEMMA 3. Let ω_1 , ω_2 be in $\hat{\Omega}$ and suppose there exists η in $GL(k, \mathbb{Z})$ such that $\omega_2 = \eta \omega_1$. Then $(\theta \circ \lambda) \omega_1 = (\theta \circ \lambda) \omega_2$.

Proof. For each p, there exists a commutative diagram

$$T_{p} \longmapsto N_{\omega_{1,p}} \longrightarrow \hat{S}_{p}$$

$$\downarrow \downarrow \qquad \qquad \downarrow^{l}$$

$$T_{p} \longmapsto N_{\omega_{2,p}} \longrightarrow \hat{S}_{p}$$

where the rows are $\omega_{1,p}C_px_0$ and $\omega_{2,p}C_px_0$ and η_p is the extension of η to \hat{S}_p . Such a diagram still exists after applying pb, with the same η for each p and PB produces isomorphic central terms.

In order to define ν and assure the commutativity of the lower triangle of Δ , we have to prove that $(\theta \circ \mu) \, \delta(\omega)$ only depends on the class of $\delta(\omega)$ modulo $\delta(GL(k, \mathbb{Z}))$. Suppose $\delta(\omega_2) = \delta(\eta) \cdot \delta(\omega_1)$. Then: $(\theta \circ \mu) \, \delta(\omega_2) = (\theta \circ \mu) \, \delta(\eta \omega_1) = (\theta \circ \lambda)(\eta \omega_1) = (\theta \circ \lambda)\omega_1$ (by Lemma 3) = $(\theta \circ \mu) \, \delta(\omega_1)$. The proof of the theorem is completed.

Remark. The proof can be rewritten with p-localization instead of p-completions. All differences disappear after the second step, and the third implies the equality, in η_0 , of $G_p(N)$ and G(N) (see [3] for another proof.)

3. Comparison with Mislin's Result

Since the theorem gives a bound for |G(N)|, it is interesting to compare this bound to that established by Mislin in [2]. We adopt the improvement of Mislin's bound introduced at the end of [5] and we follow the notation of this paper as far as possible. We have to compare $u(p) = t(p) + d_p(\text{nil } N)$ to the exponents v(p) of $\exp(QN)_{ab}$ where $QN = N/(ZN)^{\exp TZN}$. First, two remarks.

If N is abelian, N = ZN, TN = TZN, $\exp(QN)_{ab} = \exp TN$ and $d_p(1) = 0$ for any p: the two bounds are always equal. More generally, if we assume that TN is central (and $2 \notin Q$), then nil $N \le 2$ and $d_p(\text{nil } N) = 0$; $(QB)_{ab}$ contains at least a copy of $\mathbb{Z}/\exp TN$, and $u(p) \le v(p)$. In this case, we can have u(p) < v(p) (see below).

Our bound can give new information, as it is shown by the following class of examples (a modified version of examples elaborated by Professor G. Frei, whom I thank here). The proof is standard and we omit it.

Consider groups N with generators $a_1, a_2, a_3, \ldots, a_{2n+1}$, relations $[a_i, a_j] = 1$ except if i is odd and j = i+1; in this case, $[a_i, a_{i+1}] = a_{i+2}$; we assume also that the a_i 's are of finite order for $i \in J = \{3, 5, \ldots, 2n+1\}$. From $[a_i^m, a_{i+1}] = a_{i+2}^m (i = 3, 5, \ldots, 2n-1)$, we conclude that the orders r_i of $a_i (i \in J)$ are related by $r_{2n+1} | r_{2n-1} \cdots | r_3$. We assume $r_{2n+1} \neq 1$, and we write s_i for $r_i/r_{i+2} (i \in J)$ with $r_{2n+3} = 1$. For $i \geq 2$, $\Gamma_i(N)$ is generated by $\{a_j | j \in J, j \geq 2i-1\}$. Thus

N is nilpotent of class n+1 and its commutator subgroup is finite. ZN is generated by $a_1^{r_3}$, $a_2^{r_3}$, $a_3^{r_5}$, ..., a_{2n+1} and $TZN = \prod_{i \in J} \mathbb{Z}/s_i$; its exponent is the LCM of the s_i 's, denoted by s. r_{2n+1} divides s such that s is not 1. $ZN^{\exp TZN}$ is the free abelian subgroup generated by $a_1^{sr_3}$, $a_2^{sr_3}$, $a_4^{sr_5}$, Finally, $(QN)_{ab}$ is the abelian group with a_1 , a_2 , a_4 , ..., a_{2n} as generators and relations $a_1^{sr_3} = 1$, $a_2^{sr_3} = 1$, ..., $a_{2n}^{sr_2} = 1$; its exponent is sr_3 . On the other hand, s0 is generated by s1 is abelian and its exponent is s3.

If x(p) represents the exponent of the p-component of the natural x,

$$v(p) - u(p) = s(p) - d_p(n+1)$$

where

$$s(p) = \max (r_i(p) - r_{i+2}(p))$$

$$s(p) \ge r_{2n+1}(p);$$

$$s(p) \ge 1 \quad \text{for each} \quad p \quad \text{in} \quad Q$$

We can conclude:

- —for any choice of the nilpotency class except 1 (n fixed) and any choice of $Q(2 \notin Q)$ there exists a group in η_0 for which our bound is strictly smaller than Mislin's one: it is sufficient to choose r_{2n+1} adequately.
- —on the other hand, if we fix a bound for the exponent of TN, it is always possible (now choosing n) to find a group in η_0 for which Mislin's bound is better.

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