Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 51 (1976)

Artikel: Convolution and Quasiconformal Extension

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DOI: https://doi.org/10.5169/seals-39431

Comment. Math. Helvetici 51 (1976) 99–104 Birkhäuser Verlag, Basel

Convolution and Quasiconformal Extension

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Introduction and Notations

The celebrated Hadamard's theorem on the multiplication of singularities (cf. e.g. [2], p. 82) has given rise to the investigation of the convolution (or Hadamard's product) $h=f*g$ of two power series $f(z)=\sum a_n z^n$, $g(z)=\sum b_n z^n$, $h(z)$ being defined as $\sum a_n b_n z^n$.

Let C be the class of normalized convex univalent functions. Some time ago G. Pólya and I. J. Schoenberg [6] conjectured that C is closed under convolution. In other words, if $f \in C$ and $g \in C$, then also $f*g \in C$. This conjecture has been lately proved by St. Ruscheweyh and T. Sheil-Small [8]. T. J. Suffridge has also given an alternative proof of Pólya–Schoenberg conjecture, [9]. It is therefore quite a natural problem to look after another classes of functions closed under convolution.

In this paper we shall be concerned with the class Σ_k, $0 \leqslant k < 1$, of sense-preserving homeomorphisms f of the extended plane $\hat{\mathbf{C}}$ onto itself whose restriction $f|\Delta^*$ to $\Delta^* = \{z : |z| > 1\}$ is a regular and univalent function with the Laurent series expansion

$$f(z)=z+\sum_{n=0}^{\infty} a_n z^{-n}, \quad |z|>1, \tag{1}$$

whereas f is a Q-quasiconformal mapping ($Q=(1+k)/(1-k)$) in $\hat{\mathbf{C}}$.

Let Σ be the class of functions regular and univalent in Δ^*, with the normalization (1). We say that the function $f \in \Sigma$ belongs to Σ_k, if there exists a Q-quasiconformal homeomorphism g of $\hat{\mathbf{C}}$ onto itself ($Q=(1+k)/(1-k)$) such that the restriction $g|\Delta^* = f$. Obviously the extension g of f is not necessarily unique. Let f and g be the elements of Σ_{k_1} and Σ_{k_2}, with the expansion (1) and (2), resp.:

$$g(z)=z+\sum_{n=0}^{\infty} b_n z^{-n}, \quad |z|>1, \tag{2}$$

and let $h=f*g$ be the convolution of f and g:

$$h(z)=z+\sum_{n=0}^{\infty} a_n b_n z^{-n}, \quad |z|>1. \tag{3}$$

In this paper we are going to prove that $h\in\Sigma_{k_1k_2}$ (Theorem 3). Hence it follows that Σ_k is closed under convolution. In the limiting case $k\to 1$ we obtain the convolution theorem for the class Σ proved by M. S. Robertson, [7]. Our proof will be based on the area theorem for Σ_k obtained independently by R. Kühnau [3] and O. Lehto [5] quoted here as Lemma 1 and on a simple observation (Theorem 1) yielding a sufficient condition for $f\in\Sigma$ to be a member of Σ_k. Theorem 1 gives an explicit construction of a quasiconformal extension of f as well. Its counterpart for the analogous class S_k of normalized, regular and univalent functions in the unit disk Δ that have a Q quasiconformal extension to $\hat{\mathbf{C}}$ is Theorem 2. On the other hand, some other sufficient conditions for f to have a quasiconformal extension to $\hat{\mathbf{C}}$ can be derived from Theorem 1 as its corollaries. The author is much indebted to Professor Pfluger for his suggestions and criticism.

Some sufficient conditions for quasiconformal extension

Our starting point in this section is the following

THEOREM 1. *Suppose that $\omega(z)$ is a function analytic in the unit disk Δ such that $|\omega'(z)|\leqslant 1$ in Δ. Then $f(z)=z+\omega(1/z)$ is a function of the class Σ. Moreover, if $|\omega'(z)|\leqslant k<1$, then f can be extended to a Q-quasiconformal mapping of the whole plane onto itself with $Q=(1+k)/(1-k)$. A Q-quasiconformal extension has the form*

$$f(z)=z+\omega(\bar{z}),\qquad z\in\Delta. \tag{4}$$

Proof. We first prove the univalence of $f(1/\zeta)$, $\zeta\in\Delta$, under the assumption $|\omega'(\zeta)|\leqslant 1$. We have for $\zeta_1,\zeta_2\in\Delta-\{0\}$, $\zeta_1\neq\zeta_2$:

$$|f(1/\zeta_1)-f(1/\zeta_2)|=|1/\zeta_1-1/\zeta_2+\omega(\zeta_1)-\omega(\zeta_2)|\geqslant$$
$$\geqslant|\zeta_2-\zeta_1|/|\zeta_1\zeta_2|-\left|\int_{\zeta_1}^{\zeta_2}\omega'(\zeta)\,d\zeta\right|\geqslant|\zeta_2-\zeta_1|\left(\frac{1}{|\zeta_1\zeta_2|}-1\right)>0$$

and this proves that $f\in\Sigma$.

Assume now that $|\omega'(\zeta)|\leqslant k<1$. Then we have:

$$|\omega(\zeta_2)-\omega(\zeta_1)|\leqslant k|\zeta_2-\zeta_1|, \tag{5}$$

and, consequently

$$|f(1/\zeta_1)-f(1/\zeta_2)|\geqslant|\zeta_2-\zeta_1|\left(\frac{1}{|\zeta_1\zeta_2|}-k\right)>0. \tag{6}$$

From (5) it follows that ω has a continuous extension on $\bar{\Delta}$ which also satisfies (5) on $\partial\Delta$. Therefore we can repeat our previous argument with $|\zeta_1|=|\zeta_2|=1$, $|\zeta_1|\neq|\zeta_2|$, and obtain from (6):

$$|f(\zeta_1)-f(\zeta_2)|\geqslant|\bar{\zeta}_1-\bar{\zeta}_2|(1-k)=|\zeta_1-\zeta_2|(1-k)>0. \tag{7}$$

This proves that the image line Γ_1 of $|z|=1$ is a Jordan curve. Moreover, a similar reasoning gives:

$$|f(\zeta_1)-f(\zeta_2)|\leqslant|\bar{\zeta}_1-\bar{\zeta}_2|(1+k)=|\zeta_1-\zeta_2|(1+k), \tag{8}$$

for $|\zeta_1|=|\zeta_2|=1$ which obviously implies that Γ_1 is a rectifiable Jordan curve of length at most $2\pi(1+k)$.

Consider now the mapping f as given by the formula (4) in Δ. Formal derivatives $f_z, f_{\bar{z}}$ have the form: $f_z=1, f_{\bar{z}}=\omega'(\bar{z})$ so that the complex dilatation $\mu(z)$ of f satisfies $|\mu(z)|=|f_{\bar{z}}|\leqslant k<1$, whereas the Jacobian $J(f)=|f_z|^2-|f_{\bar{z}}|^2\geqslant 1-k^2>0$ in Δ. If we take an arbitrary point a not on Γ_1, then the index $n(\Gamma_1, a)=0, 1$, whereas $J(f)>0$. Hence, by the argument principle for C_1-mappings in the plane, f is a sense-preserving homeomorphism in $\bar{\Delta}$. Its complex dilatation satisfies $|\mu(z)|\leqslant k<1$, hence f is Q-quasiconformal in Δ. Thus we see that the mapping

$$f(z)=\begin{cases} z+\omega(1/z), & |z|\geqslant 1,\\ z+\omega(\bar{z}), & |z|\leqslant 1,\end{cases} \tag{9}$$

is a sense -preserving homeomorphism of $\hat{\mathbf{C}}$ onto itself. Since $\partial\Delta$ is an analytic Jordan curve, it is a removable set for f, cf. [4], and this means that f as defined by (9) is a Q-quasiconformal mapping in $\hat{\mathbf{C}}$, i.e. $f\in\Sigma_k$.

Theorem 1 has some interesting corollaries.

COROLLARY 1. *If f has the form* (1) *in Δ^* and*

$$\sum_{n=1}^{\infty} n|a_n|\leqslant k<1, \tag{10}$$

then $f\in\Sigma_k$. The mapping

$$f(z)=\begin{cases} z+\sum_{n=0}^{\infty} a_n z^{-n}, & |z|\geqslant 1,\\ z+\sum_{n=0}^{\infty} a_n \bar{z}^{n}, & |z|\leqslant 1,\end{cases} \tag{11}$$

is Q-quasiconformal in the whole plane, $Q=(1+k)/(1-k)$.

In this case the function ω of Theorem 1 has the form $\omega(z)=\sum_{n=0}^{\infty}a_n z^n$.

COROLLARY 2. *If f with the Laurent series expansion* (1) *is regular in Δ^* and if there exists k, $0\leqslant k<1$, such that*

$$|f'(z)-1|\leqslant k|z|^{-2}, \quad z\in\Delta^*, \tag{12}$$

then $f\in\Sigma_k$.

In fact, $f(z)=z+\omega(1/z)$, with ω regular in Δ by (1). We have $f'(z)-1=-z^{-2}\omega'(\zeta)$, $\zeta=1/z$, and (12) implies $|\omega'(\zeta)|\leqslant k$. This reduces the case to Theorem 1.

Note that for any $f\in\Sigma_k$ we have, cf. [3], p. 85,

$$|f'(z)-1|\leqslant\left(1-\frac{1}{|z|^2}\right)^{-k}-1=\frac{k}{|z|^2}+\frac{k(k+1)}{2}\frac{1}{|z|^4}+\cdots \tag{13}$$

so that the replacement of the right hand side in the necessary condition (13) by its leading term yields the sufficient condition (12). The limiting case $k\to 1$ leads to a sufficient condition for univalence due to Aksent'ev [1].

We can also consider the class S_k, $0\leqslant k<1$, of functions

$$F(z)=z+A_2z^2+A_3z^3+\cdots \tag{14}$$

regular and univalent in Δ which admit a Q-quasiconformal extension to the whole plane with $Q=(1+k)/(1-k)$. If $F\in S_k$, and $f(z)=1/F(1/z)$, $z\in\Delta^*$, then obviously $f\in\Sigma_k$. The condition (12) can be restated as a sufficient condition for F to be a member of S_k. Thus we have

COROLLARY 3. *If $F(z)$ has the form* (14) *and if there exists k, $0\leqslant k<1$, such that*

$$\left|\frac{F'(z)}{F^2(z)}-\frac{1}{z^2}\right|\leqslant k \quad \textit{for all} \quad z\in\Delta, \tag{15}$$

then $F\in S_k$.

We can also derive from Corollary 3 another sufficient condition for F to have a Q-quasiconformal extension to $\hat{\mathbb{C}}$ that does not require the normalization (14). We have

THEOREM 2. *If $G(z)$ is regular in the unit disk Δ and if there exists k, $0 \leqslant k < 1$, and $\zeta \in \Delta$ such that*

$$\left|\frac{G'(z)\,G'(\zeta)}{(G(z)-G(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right| \leqslant \frac{k}{|1-z\bar{\zeta}|^2} \tag{16}$$

for all $z \in \Delta$, then G *has a Q-quasiconformal extension to $\hat{\mathbf{C}}$ with $Q=(1+k)/(1-k)$.*

Proof. If (16) holds for some $\zeta \in \Delta$, then evidently $G'(\zeta) \neq 0$, otherwise we obtain a contradiction as $z \to \zeta$. Consider

$$F(w)=\frac{G\left(\dfrac{w+\zeta}{1+w\bar{\zeta}}\right)-G(\zeta)}{(1-|\zeta|^2)\,G'(\zeta)}.$$

Obviously F is regular in Δ and, moreover, $F(0)=0$, $F'(0)=1$, so that F has the form (14). Now,

$$\frac{F'(w)}{F(w)^2}-\frac{1}{w^2}=(1-z\bar{\zeta})^2\left[\frac{G'(z)\,G'(\zeta)}{(G(z)-G(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right], \tag{17}$$

where $z=(w+\zeta)/(1+w\bar{\zeta})$. Hence from (16) and (17) it follows that F satisfies (15). Consequently, $F \in S_k$ and therefore G has in fact a Q-quasiconformal extension to $\hat{\mathbf{C}}$.

As shown by Kühnau [3], the condition

$$\left|\frac{G'(z)\,G'(\zeta)}{(G(z)-G(\zeta))^2}-\frac{1}{(z-\zeta)^2}\right| \leqslant \frac{k}{(1-|z|^2)(1-|\zeta|^2)}=\frac{k}{|1-z\bar{\zeta}|^2-|z-\zeta|^2}, \quad z, \zeta \in \Delta, \tag{18}$$

is necessarily satisfied by all G regular in Δ that have a Q-quasiconformal extension to $\hat{\mathbf{C}}$. We see that the sufficient condition (16) arises by dropping the term $|z-\zeta|^2$ in the denominator of the r.h.s. in the necessary condition (18).

Convolution theorem

THEOREM 3. *If f and g with Laurent expansions* (1) *and* (2) *belong to Σ_{k_1} and Σ_{k_2}, resp., then the convolution* (3) *belongs to $\Sigma_{k_1 k_2}$.*

We first quote the area theorem for Σ_k as

LEMMA 1 [3], [5]. *If f has the Laurent series expansion* (1) *and $f \in \Sigma_k$, then*

$$\sum_{n=1}^{\infty} n\,|a_n|^2 \leqslant k^2. \tag{19}$$

The sign of equality holds for

$$f(z)=\begin{cases} z+k/z, & |z|\geqslant 1, \\ z+k\bar{z}, & |z|\leqslant 1, \end{cases} \tag{20}$$

and its rotations.

Proof of Theorem 3. Suppose that $f\in\Sigma_{k_1}$, $g\in\Sigma_{k_2}$ and (1), (2) hold. Then by Lemma 1 we have:

$$\sum_{n=1}^{\infty} n|a_n|^2\leqslant k_1^2, \qquad \sum_{n=1}^{\infty} n|b_n|^2\leqslant k_2^2. \tag{21}$$

Hence by Cauchy-Schwarz inequality and by (21):

$$\sum_{n=1}^{\infty} n|a_n b_n| = \sum_{n=1}^{\infty} \sqrt{n}|a_n|\sqrt{n}|b_n| \leqslant \left(\sum_{n=1}^{\infty} n|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty} n|b_n|^2\right)^{1/2}\leqslant k_1k_2.$$

Thus $h=f*g$ satisfies the condition (10) with $k=k_1k_2$. Corollary 1 shows that $h\in\Sigma_{k_1k_2}$ and this ends the proof.

The order k_1k_2 of quasiconformality of $f*g$ is best possible which is readily seen by considering f and g of the form (20).

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Received July 7, 1975