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**Autor:** Eckmann, Beno  
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# Aspherical Manifolds and Higher-Dimensional Knots

BENO ECKMANN

E. Dyer and R. Vasquez [3] proved that the complement of a higher-dimensional knot  $S^{n-2} \subset S^n$ ,  $n \geq 4$ , is never aspherical unless the knot group is infinite cyclic (and hence, for  $n \geq 5$ , the imbedding is unknotted<sup>1)</sup>). In the present note we give a simple proof of this fact based on some remarks concerning compact  $\partial$ -manifolds. By the same method we show that the complement of a link in  $S^n$ ,  $n \geq 4$ , is never aspherical.

Let  $X$  be a compact  $n$ -dimensional  $\partial$ -manifold,  $G = \pi_1(X)$  its fundamental group. If  $\partial X$  is connected, let  $G_0$  be the image of  $\pi_1(\partial X)$  in  $G$ . Using the connection between  $G_0$  and the boundary  $\partial \tilde{X}$  of the universal cover of  $X$  we first note that  $H^{n-1}(X; \mathbb{Z}G) = 0$  if and only if  $G_0 = G$ . If, moreover,  $X$  is aspherical, we show that  $\text{cd } G_0 < n - 1$  implies  $G_0 = G$  (and vice-versa). Since in the case of a knot-complement in  $S^n$ ,  $n \geq 4$ , the image  $G_0$  is infinite cyclic, the Dyer-Vasquez result follows. Actually the asphericity is used here in a weak form only, cf. 3.2 below. – In the case where  $\partial X$  is not connected, and if  $X$  is aspherical, then for at least one of the components of  $\partial X$  one has  $\text{cd } G_0 = n - 1$ . This immediately implies the sphericity of higher-dimensional links.

## 1. The Fundamental Group of a $\partial$ -Manifold

1.1. Let  $X$  be a  $\partial$ -manifold; by this we mean a connected cellular manifold with non-empty boundary  $\partial X$ . We write  $i$  for the inclusion map  $\partial X \rightarrow X$ , and  $G$  for the fundamental group  $\pi_1(X)$ .

We will always assume  $X$  to be compact. The universal cover  $\tilde{X}$  of  $X$  is a  $\partial$ -manifold which may be compact or not; its boundary  $\partial \tilde{X}$  is the inverse image  $p^{-1}(\partial X)$  under the covering map  $p: \tilde{X} \rightarrow X$ . We want to get information on the number of connected components of  $\partial \tilde{X}$ ; i.e., on the integral homology group  $H_0(\partial \tilde{X})$ . The exact homology sequence

$$\cdots \rightarrow H_1(\tilde{X}) \rightarrow H_1(\tilde{X} \text{ mod } \partial \tilde{X}) \rightarrow H_0(\partial \tilde{X}) \rightarrow H_0(\tilde{X}) \rightarrow 0$$

yields

$$H_0^{\text{red}}(\partial \tilde{X}) = H_1(\tilde{X} \text{ mod } \partial \tilde{X}),$$

<sup>1)</sup> In [3], only  $n \geq 6$  is mentioned (Levine, Stallings), but the result holds for  $n = 5$  as well (C. T. C. Wall, Shaneson).

where  $H_0^{\text{red}}$  is the reduced homology group. Poincaré duality in  $\tilde{X}$  further yields

$$H_0^{\text{red}}(\partial\tilde{X}) = \bar{H}^{n-1}(\tilde{X}),$$

$n$  being the dimension of  $X$ , and  $\bar{H}$  denoting cohomology with compact supports (i.e., if we use a cell decomposition, cohomology based on finite cellular cochains). If  $C(\tilde{X})$  denotes the chain complex of  $\tilde{X}$  corresponding to a finite cell decomposition of  $X$ , one may replace (cf. [2], p. 359) the finite cochain group  $\text{Hom}_{\text{fin}}(C(\tilde{X}), \mathbf{Z})$  by the equivariant group  $\text{Hom}_G(C(\tilde{X}), \mathbf{Z}G)$ . It follows that

$$\bar{H}^{n-1}(X) = H^{n-1}(X; \mathbf{Z}G),$$

the last group being cohomology with local coefficients given by the left  $G$ -module  $\mathbf{Z}G$ . We thus obtain

$$H_0^{\text{red}}(\partial\tilde{X}) = H^{n-1}(X; \mathbf{Z}G). \quad (1)$$

This yields the following results:

**PROPOSITION 1.1.**  *$\partial\tilde{X}$  is connected if and only if  $H^{n-1}(X; \mathbf{Z}G) = 0$ .*

**PROPOSITION 1.2.** *If  $\partial X$  is not connected, then  $H^{n-1}(X; \mathbf{Z}G) \neq 0$ .*

1.2. We now assume the boundary manifold  $\partial X$  to be connected and write  $G_0$  for  $i_*\pi_1(\partial X)$ , the image of  $\pi_1(\partial X)$  under the inclusion map  $i: \partial X \rightarrow X$ . The connected components of  $\partial\tilde{X} = p^{-1}(\partial X)$  correspond bijectively to the cosets of  $G$  modulo  $G_0$ . Proposition 1.1 can therefore be reformulated as follows.

**PROPOSITION 1.3.** *Let  $X$  be a compact manifold of dimension  $n$  with connected boundary  $\partial X$ . Then  $H^{n-1}(X; \mathbf{Z}G) = 0$  if and only if  $G_0 = G$ ; i.e., if  $\pi_1(\partial X) \rightarrow \pi_1(X)$  is surjective.*

Let  $K(G_0, 1)$  denote an Eilenberg-MacLane complex of the group  $G_0$ . There is a map  $j: \partial X \rightarrow K(G_0, 1)$ , determined up to homotopy, which induces the surjection  $\pi_1(\partial X) \rightarrow G_0$ . We now further assume that the inclusion  $i: \partial X \rightarrow X$  can be factored up to homotopy through  $j$ :

$$i = hj: \partial X \xrightarrow{j} K(G_0, 1) \xrightarrow{h} X. \quad (2)$$

Then  $i^*: H^{n-1}(X; \mathbf{Z}G) \rightarrow H^{n-1}(\partial X; \mathbf{Z}G)$  is factored as  $i^* = j^*h^*$  through the cohomology group  $H^{n-1}(G_0; \mathbf{Z}G)$  and will thus be 0 if we assume this group to be 0

(in particular, if the cohomology dimension  $\text{cd } G_0$  is  $< n-1$ ). The homomorphism  $i^*$  appears in the exact sequence with local coefficients

$$\cdots \rightarrow H^{n-1}(X \bmod \partial X; \mathbf{Z}G) \rightarrow H^{n-1}(X; \mathbf{Z}G) \xrightarrow{i^*} H^{n-1}(\partial X; \mathbf{Z}G) \rightarrow \cdots. \quad (3)$$

By Poincaré duality  $H^{n-1}(X \bmod \partial X; \mathbf{Z}G) = H_1(X; \mathbf{Z}G)$ ; the latter group is computed from  $\mathbf{Z}G \otimes_G C(\tilde{X}) = C(\tilde{X})$ , i.e., it is equal to  $H_1(\tilde{X})$  and hence 0.

Note that the argument is valid both in the orientable and non-orientable case: in the non-orientable case the duality yields  $H^{n-1}(X \bmod \partial X; \mathbf{Z}G) = H_1(X; \check{\mathbf{Z}} \otimes \mathbf{Z}G)$ , where  $\check{\mathbf{Z}}$  is the group of twisted integers. But  $\check{\mathbf{Z}} \otimes \mathbf{Z}G$  (with diagonal action) is easily seen to be isomorphic to  $\mathbf{Z}G$ .

Thus by (3)  $i^*$  is always injective. Under the factorization assumption (2), and if  $H^{n-1}(G_0; \mathbf{Z}G) = 0$ , we have seen that  $i^* = 0$ , and therefore  $H^{n-1}(X; \mathbf{Z}G) = 0$ . Combining this with Prop. 1.3 we get

**THEOREM 1.4** *Let  $X$  be a compact manifold of dimension  $n$  with connected boundary  $\partial X$ , and let  $i: \partial X \rightarrow X$  be the inclusion,  $G = \pi_1(X)$ ,  $G_0 = i_*\pi_1(\partial X)$ . If  $i$  can be factored up to homotopy as  $i = hj: \partial X \rightarrow K(G_0, 1) \rightarrow X$  and if  $H^{n-1}(G_0; \mathbf{Z}G) = 0$ , then  $G_0 = G$ .*

1.3. In Theorem 1.4 the condition  $H^{n-1}(G_0; \mathbf{Z}G) = 0$  can be replaced by  $H_{n-1}(G_0) = 0$ .

To prove this, let  $e \in H_{n-1}(\partial X)$  be the fundamental class of  $\partial X$  [ $e \in H_{n-1}(\partial X; \check{\mathbf{Z}})$  in the non-orientable case]. For any  $z \in H^{n-1}(G_0; \mathbf{Z}G)$ , the cap-product formula

$$j_*(e \cap j^*z) = j_*e \cap z$$

together with  $j_*e = 0$  yields  $j^*z = 0$ , since  $j_*: H_0(\partial X; \mathbf{Z}G) \rightarrow H_0(G_0; \mathbf{Z}G)$  and  $e \cap -$  are both isomorphisms. Now  $j^* = 0$  implies  $i^* = 0$ .

1.4. If we do not assume that  $\partial X$  is connected, Theorem 1.4 has to be restated in a slightly different form.

Let  $\partial_v X$ ,  $v = 0, 1, \dots, k$  be the connected components of  $\partial X$ , and  $G_v = i_{v*}\pi_1(\partial_v X)$  the images in  $G$  of their fundamental groups under the inclusions  $i_v: \partial_v X \rightarrow X$  (determined up to conjugacy only). Let  $K$  be the disjoint union of the  $K(G_v, 1)$  and  $j: \partial X \rightarrow K$  the union of the maps  $j_v: \partial_v X \rightarrow K(G_v, 1)$  inducing  $i_{v*}$ . If  $i$  can be factored up to homotopy as  $i = hj: \partial X \rightarrow K \rightarrow X$  and if  $H^{n-1}(G_v; \mathbf{Z}G) = 0$  for all  $v$  (or: if  $H_{n-1}(G_v) = 0$  for all  $v$ ) then it follows as above that  $H^{n-1}(X; \mathbf{Z}G) = 0$ ; i.e.,  $\partial X$  must be connected,  $k = 0$ ,  $G = G_0$ .

**THEOREM 1.5.** *Let  $X$  be a compact  $\partial$ -manifold,  $G = \pi_1(X)$  and  $G_v = i_{v*}\pi_1(\partial_v X)$ ,  $v = 0, \dots, k$ . If  $i: \partial X \rightarrow X$  can be factored as  $i = hj: \partial X \rightarrow K \rightarrow X$  and if  $H^{n-1}(G_v; \mathbb{Z}G) = 0$  (or:  $H_{n-1}(G_v) = 0$ ) for  $v = 0, \dots, k$ , then  $\partial X$  is connected and  $G_0 = G$ .*

## 2. Aspherical Manifolds

2.1. The notations being as in 1.1, we now assume the manifold  $X$  to be aspherical; in other words, an Eilenberg-MacLane complex  $K(G, 1)$  for its fundamental group  $G = \pi_1(X)$ . Since cohomology of  $X$  with local coefficients vanishes in dimensions  $k \geq n$ , the cohomology dimension  $\text{cd } G$  is  $\leq n - 1$ . Note that the chain complex  $C(\tilde{X})$  constitutes a finitely generated free resolution for  $G$ ; therefore  $H^{n-1}(G; \mathbb{Z}G) = H^{n-1}(X; \mathbb{Z}G) = 0$  implies  $H^{n-1}(G; A) = 0$  for all free  $G$ -modules  $A$  and hence (cf. [1] p. 105) for all  $G$ -modules  $A$ , and thus is equivalent to  $\text{cd } G < n - 1$ .

The results of Section 1 can now be applied as follows.

**PROPOSITION 2.1.** *Let  $G$  be a group admitting a  $K(G, 1) = X$  which is a compact manifold of dimension  $n$  with non-empty boundary  $\partial X$ . Then  $\text{cd } G < n - 1$  if and only if  $\partial \tilde{X}$  is connected; in particular, if  $\partial X$  is not connected then  $\text{cd } G = n - 1$ .*

Note that any group admitting a  $K(G, 1)$  which is a finite cell-complex admits a compact manifold with non-empty boundary as Eilenberg-MacLane space (imbed  $K(G, 1)$  in some  $\mathbb{R}^N$  and take a regular neighborhood).

2.2. For aspherical  $X$ , assuming  $\partial X$  connected, the factorization (2) of  $i: \partial X \rightarrow X$  is always possible. Hence Theorem 1.4 yields

**THEOREM 2.2.** *Let  $G$  be a group admitting a  $K(G, 1) = X$  which is a compact manifold of dimension  $n$  with connected boundary  $\partial X$ , and  $G_0 = i_*\pi_1(\partial X)$ . Then  $G_0 = G$  if and only if  $\text{cd } G_0 < n - 1$ . In other words, one always has  $\text{cd } G = \text{cd } G_0$ ; namely,  $< n - 1$  if  $G_0 = G$  and  $= n - 1$  if  $G_0 \neq G$ .*

From Theorem 1.5 we immediately get

**THEOREM 2.3.** *Let  $G$  be a group admitting a  $K(G, 1) = X$  which is a compact  $\partial$ -manifold of dimension  $n$ . If  $\partial X$  is not connected, then  $\text{cd } G_v = n - 1$  for at least one component  $\partial_v X$  of  $\partial X$ ,  $G_v$  being the image of  $\pi_1(\partial_v X)$  under the inclusion.*

## 3. Higher-dimensional Knots

3.1. Let throughout this section  $S^{n-2} \subset S^n$ ,  $n \geq 4$ , be a knot, i.e., a differentiable imbedding of  $S^{n-2}$  in  $S^n$ ,  $C = S^n - S^{n-2}$  its complement, and  $X$  its closed comple-

ment  $S^n - V^n$  where  $V^n$  is an open tubular neighborhood of  $S^{n-2}$  in  $S^n$ . Then  $X$  and  $C$  have the same homotopy type, and  $G = \pi_1(X)$  is the corresponding knot group.  $\partial X$  is a product  $S^1 \times S^{n-2}$ , and  $\pi_1(\partial X) \cong \mathbf{Z}$  imbeds injectively into  $G$ .

**THEOREM 3.1.** *If the knot complement is aspherical, then  $G \cong \mathbf{Z}$ .*

*Proof.* Since  $i_*\pi_1(\partial X) = G_0 \cong \mathbf{Z}$ , we have  $\text{cd } G_0 = 1$ , and hence Theorem 2.2 applies:  $G_0 = G \cong \mathbf{Z}$ .

3.2. Note that the asphericity of  $X$  is not used in full here. The factorization (2) of  $i$  can be obtained under weaker assumptions, as follows.

Let  $j: \partial X \rightarrow S^1 = K(G_0, 1)$  be the projection of  $\partial X = S^1 \times S^{n-2}$  onto  $S^1 \times pt$ , and  $h$  the imbedding  $S^1 \times pt \rightarrow \partial X \rightarrow X$ . If we assume

(a) a sphere  $pt \times S^{n-2} \subset \partial X$  is nullhomotopic in  $X$

then  $hj$  and  $i$  can be made, by a homotopy, to agree on  $S^1 \times pt \vee pt \times S^{n-2}$ . If we further assume

(b)  $\pi_{n-1}(X) = 0$

then  $i$  and  $hj$  are homotopic, and thus, by Theorem 1.4,  $G = G_0 \cong \mathbf{Z}$ .

**THEOREM 3.1'.** *If  $pt \times S^{n-2}$  is nullhomotopic in the knot complement  $C$  and if  $\pi_{n-1}(C) = 0$  then  $G \cong \mathbf{Z}$ .*

3.3. G. A. Swarup [4] has proved that Theorem 3.1' holds without the assumption that  $\pi_{n-1}(C) = 0$  provided  $G$  is *accessible*. Since it is conjectured that all finitely generated groups are accessible, it is possible that the nullhomotopy of  $S^{n-2}$  in  $C$  alone is sufficient to conclude that  $G \cong \mathbf{Z}$ .

3.4. HIGHER-DIMENSIONAL LINKS. If  $X$  is the closed complement of a link

$$\bigcup_{v=0, \dots, k} S_v^{n-2} \subset S^n, \quad n \geq 4, \quad k > 0,$$

then  $\partial X$  is not connected. The images  $G_v$  of  $\pi_1(\partial_v X)$  are all  $\cong \mathbf{Z}$ . By Theorem 2.3  $X$  can not be aspherical.

**THEOREM 3.2.** *The complement of a link in  $S^n$ ,  $n \geq 4$ , is never aspherical.*

## REFERENCES

- [1] R. BIERI and B. ECKMANN, *Groups with homological duality generalizing Poincaré duality*, Inv. Math. 20 (1973), 103–124.
- [2] H. CARTAN and S. EILENBERG, *Homological Algebra*, Princeton University Press, 1956.
- [3] E. DYER and A. T. VASQUEZ, *The sphericity of higher dimensional knots*, Can. J. Math. 25 (1973), 1132–1136.
- [4] G. A. SWARUP, *Accessible groups and an unknotting criterion* (to appear in J. of Pure and Applied Algebra).

*E.T.H. Zürich*

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