Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 51 (1976)

Artikel: Aspherical Manifolds and Higher-Dimensional Knots

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DOI: https://doi.org/10.5169/seals-39430

Comment. Math. Helvetici 51 (1976) 93–98 Birkhäuser Verlag, Basel

Aspherical Manifolds and Higher-Dimensional Knots

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E. Dyer and R. Vasquez [3] proved that the complement of a higher-dimensional knot $S^{n-2} \subset S^n$, $n \geq 4$, is never aspherical unless the knot group is infinite cyclic (and hence, for $n \geq 5$, the imbedding is unknotted[1]). In the present note we give a simple proof of this fact based on some remarks concerning compact ∂-manifolds. By the same method we show that the complement of a link in S^n, $n \geq 4$, is never aspherical.

Let X be a compact n-dimensional ∂-manifold, $G = \pi_1(X)$ its fundamental group. If ∂X is connected, let G_0 be the image of $\pi_1(\partial X)$ in G. Using the connection between G_0 and the boundary $\partial \tilde{X}$ of the universal cover of X we first note that $H^{n-1}(X; \mathbf{Z}G) = 0$ if and only if $G_0 = G$. If, moreover, X is aspherical, we show that $\operatorname{cd} G_0 < n-1$ implies $G_0 = G$ (and vice-versa). Since in the case of a knot-complement in S^n, $n \geq 4$, the image G_0 is infinite cyclic, the Dyer-Vasquez result follows. Actually the asphericity is used here in a weak form only, cf. 3.2 below. – In the case where ∂X is not connected, and if X is aspherical, then for at least one of the components of ∂X one has $\operatorname{cd} G_0 = n-1$. This immediately implies the sphericity of higher-dimensional links.

1. The Fundamental Group of a ∂-Manifold

1.1. Let X be a ∂-manifold; by this we mean a connected cellular manifold with non-empty boundary ∂X. We write i for the inclusion map $\partial X \to X$, and G for the fundamental group $\pi_1(X)$.

We will always assume X to be compact. The universal cover $\tilde{X}$ of X is a ∂-manifold which may be compact or not; its boundary $\partial \tilde{X}$ is the inverse image $p^{-1}(\partial X)$ under the covering map $p: \tilde{X} \to X$. We want to get information on the number of connected components of $\partial \tilde{X}$; i.e., on the integral homology group $H_0(\partial \tilde{X})$. The exact homology sequence

$$\cdots \to H_1(\tilde{X}) \to H_1(\tilde{X} \bmod \partial \tilde{X}) \to H_0(\partial \tilde{X}) \to H_0(\tilde{X}) \to 0$$

yields

$$H_0^{\text{red}}(\partial \tilde{X}) = H_1(\tilde{X} \bmod \partial \tilde{X}),$$

[1]) In [3], only $n \geq 6$ is mentioned (Levine, Stallings), but the result holds for $n = 5$ as well (C. T. C. Wall, Shaneson).

where H_0^{red} is the reduced homology group. Poincaré duality in $\tilde{X}$ further yields

$$H_0^{\mathrm{red}}(\partial\tilde{X})=\bar{H}^{n-1}(\tilde{X}),$$

n being the dimension of X, and $\bar{H}$ denoting cohomology with compact supports (i.e., if we use a cell decomposition, cohomology based on finite cellular cochains). If $C(\tilde{X})$ denotes the chain complex of $\tilde{X}$ corresponding to a finite cell decomposition of X, one may replace (cf. [2], p. 359) the finite cochain group $\mathrm{Hom}_{\mathrm{fin}}(C(\tilde{X}), \mathbf{Z})$ by the equivariant group $\mathrm{Hom}_G(C(\tilde{X}), \mathbf{Z}G)$. It follows that

$$\bar{H}^{n-1}(X)=H^{n-1}(X; \mathbf{Z}G),$$

the last group being cohomology with local coefficients given by the left G-module $\mathbf{Z}G$. We thus obtain

$$H_0^{\mathrm{red}}(\partial\tilde{X})=H^{n-1}(X; \mathbf{Z}G). \tag{1}$$

This yields the following results:

PROPOSITION 1.1. *$\partial\tilde{X}$ is connected if and only if $H^{n-1}(X; \mathbf{Z}G)=0$.*

PROPOSITION 1.2. *If ∂X is not connected, then $H^{n-1}(X; \mathbf{Z}G)\neq 0$.*

1.2. We now assume the boundary manifold ∂X to be connected and write G_0 for $i_*\pi_1(\partial X)$, the image of $\pi_1(\partial X)$ under the inclusion map $i:\partial X\to X$. The connected components of $\partial\tilde{X}=p^{-1}(\partial X)$ correspond bijectively to the cosets of G modulo G_0. Proposition 1.1 can therefore be reformulated as follows.

PROPOSITION 1.3. *Let X be a compact manifold of dimension n with connected boundary ∂X. Then $H^{n-1}(X; \mathbf{Z}G)=0$ if and only if $G_0=G$; i.e., if $\pi_1(\partial X)\to\pi_1(X)$ is surjective.*

Let $K(G_0, 1)$ denote an Eilenberg-MacLane complex of the group G_0. There is a map $j:\partial X\to K(G_0, 1)$, determined up to homotopy, which induces the surjection $\pi_1(\partial X)\to G_0$. We now further assume that the inclusion $i:\partial X\to X$ can be factored up to homotopy through j:

$$i=hj:\partial X\xrightarrow{j} K(G_0, 1)\xrightarrow{h} X. \tag{2}$$

Then $i^*:H^{n-1}(X; \mathbf{Z}G)\to H^{n-1}(\partial X; \mathbf{Z}G)$ is factored as $i^*=j^*h^*$ through the cohomology group $H^{n-1}(G_0; \mathbf{Z}G)$ and will thus be 0 if we assume this group to be 0 (in particular, if the cohomology dimension $\mathrm{cd}\, G_0$ is $<n-1$). The homomorphism i^* appears in the exact sequence with local coefficients

$$\cdots \to H^{n-1}(X \bmod \partial X; \mathbf{Z}G) \to H^{n-1}(X; \mathbf{Z}G) \xrightarrow{i^*} H^{n-1}(\partial X; \mathbf{Z}G) \to \cdots. \tag{3}$$

By Poincaré duality $H^{n-1}(X \bmod \partial X; \mathbf{Z}G) = H_1(X; \mathbf{Z}G)$; the latter group is computed from $\mathbf{Z}G \otimes_G C(\tilde{X}) = C(\tilde{X})$, i.e., it is equal to $H_1(\tilde{X})$ and hence 0.

Note that the argument is valid both in the orientable and non-orientable case: in the non-orientable case the duality yields $H^{n-1}(X \bmod \partial X; \mathbf{Z}G) = H_1(X; \check{\mathbf{Z}} \otimes \mathbf{Z}G)$, where $\check{\mathbf{Z}}$ is the group of twisted integers. But $\check{\mathbf{Z}} \otimes \mathbf{Z}G$ (with diagonal action) is easily seen to be isomorphic to $\mathbf{Z}G$.

Thus by (3) i^* is always injective. Under the factorization assumption (2), and if $H^{n-1}(G_0; \mathbf{Z}G) = 0$, we have seen that $i^* = 0$, and therefore $H^{n-1}(X; \mathbf{Z}G) = 0$. Combining this with Prop. 1.3 we get

THEOREM 1.4 *Let X be a compact manifold of dimension n with connected boundary ∂X, and let $i: \partial X \to X$ be the inclusion, $G = \pi_1(X)$, $G_0 = i_* \pi_1(\partial X)$. If i can be factored up to homotopy as $i = hj: \partial X \to K(G_0, 1) \to X$ and if $H^{n-1}(G_0; \mathbf{Z}G) = 0$, then $G_0 = G$.*

1.3. In Theorem 1.4 the condition $H^{n-1}(G_0; \mathbf{Z}G) = 0$ can be replaced by $H_{n-1}(G_0) = 0$.

To prove this, let $e \in H_{n-1}(\partial X)$ be the fundamental class of ∂X [$e \in H_{n-1}(\partial X; \check{\mathbf{Z}})$ in the non-orientable case]. For any $z \in H^{n-1}(G_0; \mathbf{Z}G)$, the cap-product formula

$$j_*(e \cap j^* z) = j_* e \cap z$$

together with $j_* e = 0$ yields $j^* z = 0$, since $j_*: H_0(\partial X; \mathbf{Z}G) \to H_0(G_0; \mathbf{Z}G)$ and $e \cap -$ are both isomorphisms. Now $j^* = 0$ implies $i^* = 0$.

1.4. If we do not assume that ∂X is connected, Theorem 1.4 has to be restated in a slightly different form.

Let $\partial_\nu X$, $\nu = 0, 1, \ldots, k$ be the connected components of ∂X, and $G_\nu = i_{\nu *} \pi_1(\partial_\nu X)$ the images in G of their fundamental groups under the inclusions $i_\nu: \partial_\nu X \to X$ (determined up to conjugacy only). Let K be the disjoint union of the $K(G_\nu, 1)$ and $j: \partial X \to K$ the union of the maps $j_\nu: \partial_\nu X \to K(G_\nu, 1)$ inducing $i_{\nu *}$. If i can be factored up to homotopy as $i = hj: \partial X \to K \to X$ and if $H^{n-1}(G_\nu; \mathbf{Z}G) = 0$ for all ν (or: if $H_{n-1}(G_\nu) = 0$ for all ν) then it follows as above that $H^{n-1}(X; \mathbf{Z}G) = 0$; i.e., ∂X must be connected, $k = 0$, $G = G_0$.

THEOREM 1.5. *Let X be a compact ∂-manifold, $G=\pi_1(X)$ and $G_\nu=i_{\nu *}\pi_1(\partial_\nu X)$, $\nu=0,\ldots,k$. If $i:\partial X\to X$ can be factored as $i=hj:\partial X\to K\to X$ and if $H^{n-1}(G_\nu;\mathbf{Z}G)=0$ (or: $H_{n-1}(G_\nu)=0$) for $\nu=0,\ldots,k$, then ∂X is connected and $G_0=G$.*

2. Aspherical Manifolds

2.1. The notations being as in 1.1, we now assume the manifold X to be aspherical; in other words, an Eilenberg-MacLane complex $K(G,1)$ for its fundamental group $G=\pi_1(X)$. Since cohomology of X with local coefficients vanishes in dimensions $k\geqslant n$, the cohomology dimension $\mathrm{cd}\,G$ is $\leqslant n-1$. Note that the chain complex $C(\tilde{X})$ constitutes a finitely generated free resolution for G; therefore $H^{n-1}(G;\mathbf{Z}G)=H^{n-1}(X;\mathbf{Z}G)=0$ implies $H^{n-1}(G;A)=0$ for all free G-modules A and hence (cf. [1] p. 105) for all G-modules A, and thus is equivalent to $\mathrm{cd}\,G<n-1$.

The results of Section 1 can now be applied as follows.

PROPOSITION 2.1. *Let G be a group admitting a $K(G,1)=X$ which is a compact manifold of dimension n with non-empty boundary ∂X. Then* $\mathrm{cd}\,G<n-1$ *if and only if $\partial\tilde{X}$ is connected; in particular, if ∂X is not connected then* $\mathrm{cd}\,G=n-1$.

Note that any group admitting a $K(G,1)$ which is a *finite* cell-complex admits a compact manifold with non-empty boundary as Eilenberg-MacLane space (imbed $K(G,1)$ in some $\mathbf{R}^N$ and take a regular neighborhood).

2.2. For aspherical X, assuming ∂X connected, the factorization (2) of $i:\partial X\to X$ is always possible. Hence Theorem 1.4 yields

THEOREM 2.2. *Let G be a group admitting a $K(G,1)=X$ which is a compact manifold of dimension n with connected boundary ∂X, and $G_0=i_*\pi_1(\partial X)$. Then $G_0=G$ if and only if* $\mathrm{cd}\,G_0<n-1$. *In other words, one always has* $\mathrm{cd}\,G=\mathrm{cd}\,G_0$; *namely, $<n-1$ if $G_0=G$ and $=n-1$ if $G_0\neq G$.*

From Theorem 1.5 we immediately get

THEOREM 2.3. *Let G be a group admitting a $K(G,1)=X$ which is a compact ∂-manifold of dimension n. If ∂X is not connected, then* $\mathrm{cd}\,G_\nu=n-1$ *for at least one component $\partial_\nu X$ of ∂X, G_ν being the image of $\pi_1(\partial_\nu X)$ under the inclusion.*

3. Higher-dimensional Knots

3.1. Let throughout this section $S^{n-2}\subset S^n$, $n\geqslant 4$, be a knot, i.e., a differentiable imbedding of S^{n-2} in S^n, $C=S^n-S^{n-2}$ its complement, and X its closed complement $S^n - V^n$ where V^n is an open tubular neighborhood of S^{n-2} in S^n. Then X and C have the same homotopy type, and $G = \pi_1(X)$ is the corresponding knot group. ∂X is a product $S^1 \times S^{n-2}$, and $\pi_1(\partial X) \cong \mathbf{Z}$ imbeds injectively into G.

THEOREM 3.1. *If the knot complement is aspherical, then* $G \cong \mathbf{Z}$.

Proof. Since $i_* \pi_1(\partial X) = G_0 \cong \mathbf{Z}$, we have $\operatorname{cd} G_0 = 1$, and hence Theorem 2.2 applies: $G_0 = G \cong \mathbf{Z}$.

3.2. Note that the asphericity of X is not used in full here. The factorization (2) of i can be obtained under weaker assumptions, as follows.

Let $j : \partial X \to S^1 = K(G_0, 1)$ be the projection of $\partial X = S^1 \times S^{n-2}$ onto $S^1 \times pt$, and h the imbedding $S^1 \times pt \to \partial X \to X$. If we assume

(a) a sphere $pt \times S^{n-2} \subset \partial X$ is nullhomotopic in X

then hj and i can be made, by a homotopy, to agree on $S^1 \times pt \vee pt \times S^{n-2}$. If we further assume

(b) $\pi_{n-1}(X) = 0$

then i and hj are homotopic, and thus, by Theorem 1.4, $G = G_0 \cong \mathbf{Z}$.

THEOREM 3.1′. *If* $pt \times S^{n-2}$ *is nullhomotopic in the knot complement* C *and if* $\pi_{n-1}(C) = 0$ *then* $G \cong \mathbf{Z}$.

3.3. G. A. Swarup [4] has proved that Theorem 3.1′ holds without the assumption that $\pi_{n-1}(C) = 0$ provided G is *accessible*. Since it is conjectured that all finitely generated groups are accessible, it is possible that the nullhomotopy of S^{n-2} in C alone is sufficient to conclude that $G \cong \mathbf{Z}$.

3.4. HIGHER-DIMENSIONAL LINKS. If X is the closed complement of a link

$$\bigcup_{\nu = 0, \dots, k} S_\nu^{n-2} \subset S^n, \quad n \geqslant 4, \quad k > 0,$$

then ∂X is not connected. The images G_ν of $\pi_1(\partial_\nu X)$ are all $\cong \mathbf{Z}$. By Theorem 2.3 X can not be aspherical.

THEOREM 3.2. *The complement of a link in* S^n, $n \geqslant 4$, *is never aspherical.*

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Received September, 1975