

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 50 (1975)

Artikel: Abelian p-adic Group Rings.
Autor: Spiegel, Eugene
DOI: <https://doi.org/10.5169/seals-38789>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 03.05.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Abelian p -adic Group Rings

EUGENE SPIEGEL

Introduction

In the following we investigate the question of when two finite abelian groups G and H have isomorphic group-algebras over the p -adic integers and p -adic field.

We use the following notation: N denotes the positive integers, p and q are distinct primes in N , Z_n is the ring of integers modulo n ($n \in N$), P is the ring of p -adic integers, Q_p is the field of p -adic numbers and C_n denotes a cyclic group of order n ($n \in N$).

If G is a finite group and R is a principal ideal domain, $RG = R(G)$ will be the group algebra of G over R .

§1

Throughout this section, G and H denote finite abelian groups of order q^n . If F is a field of characteristic $k \neq q$, by the theorem of Perlis-Walker [3], $F(G) \simeq \sum_{d=1}^n (n_d/v_d) F(\zeta_d)$, where ζ_d is a primitive q^d th root of unity over F , $v_d = \text{degree}(F(\zeta_d)/F)$ and n_d is the number of elements of order q^d in G . Also, n_d/v_d is a non-negative integer.

As $(q^d, p) = 1$, the splitting field of the polynomial $g(x) = x^{q^d} - 1$ over Q_p , is a totally unramified extension of Q_p . This says that $\text{degree}(Q_p(\zeta_d)/Q_p) = \text{degree}(Z_p(\zeta_d)/Z_p)$, where ζ_d is a primitive q^d th root of unity over Z_p . Hence we can write

$$Q_p G \simeq \sum_{d=1}^n a_d Q_p(\zeta_d) \tag{*}$$

and

$$Z_p G \simeq \sum_{d=1}^n a_d Z_p(\zeta_d)$$

for a common collection of integers $a_d = n_d/v_d$, $d = 1, 2, \dots, n$.

The following generalization of the Perlis-Walker result is due to Raggi Cárdenas.

PROPOSITION 1.1. *Let A be a local ring, with maximal ideal M , and residue field K of characteristic p . Suppose M is finite, and $\bigcap_{n=0}^{\infty} M^n = \{0\}$. Then $AG \simeq \sum_{d=1}^n (n_d/v_d) A(\zeta_d)$ where ζ_d is a primitive q^d th root of unity, $v_d = \text{degree}(K(\zeta_d)/K)$ and n_d is the number of elements of order q^d in G .*

Proof. See [4].

LEMMA 1.2. *If a_d , $d=1, 2, \dots, n$, are defined as in (*), then $PG \simeq \sum_{d=1}^n a_d P(\zeta_d)$.*

Proof. If $m \in N$, the residue field of the local ring Z_{p^m} is Z_p . Due to Proposition 1.1 we must have

$$Z_{p^m}G \simeq \sum_{d=1}^n a_d Z_{p^m}(\zeta_d).$$

By taking injective limits, (the injective limit of Z_{p^m} is P), the above isomorphism leads to an isomorphism

$$PG \simeq \sum_{d=1}^n a_d P(\zeta_d).$$

COROLLARY 1.3. *If a_d , $d=1, 2, \dots, n$, are defined as in (*), and R is either Z_p , P or Q_p then*

$$RG \simeq \sum_{d=1}^n a_d R(\zeta_d)$$

where ζ_d is a primitive q^d th root of unity.

We define the function $\gamma: N \rightarrow N$ by $\gamma_q(m)$ is the least positive integer such that $p^{\gamma_q(m)} \equiv 1 \pmod{q^m}$. Then $\gamma_q(1) \leq \gamma_q(2) \leq \dots$ and $\gamma_q(d) \rightarrow \infty$. If $d \in N$, and ζ_d is a primitive q^d th root of unity over Z_p , $1 = \zeta^{q^d} = \zeta^{p^{v_d} a - 1}$, so that $\gamma_q(d) = v_d$, and $\gamma_q(d) \mid \gamma_q(d+1)$, $d=1, 2, \dots$

Define the sequence $\{\alpha_m\}$, $m=1, 2, \dots$, by $\alpha_1=1$, and if α_m is defined, α_{m+1} is the smallest positive integer such that $\gamma_q(\alpha_{m+1}) > \gamma_q(\alpha_m)$. Thus $\gamma_q(\alpha_1) < \gamma_q(\alpha_2) < \dots$ and

$$\{\gamma_q(\alpha_i) \mid i \in N\} = \{\gamma_q(i) \mid i \in N\}.$$

If R is Z_p , P or Q_p , then $R(\zeta_d) = R(\zeta_{d+1}) \Leftrightarrow \gamma_q(d) = \gamma_q(d+1)$ so that we define the sequence b_1, b_2, \dots, b_n by

$$b_i = \sum_{j=\alpha_i}^{\alpha_{i+1}-1} a_j$$

where $a_r = 0$ if $r > n$ and a_1, a_2, \dots, a_n are as in (*).

In terms of this notation, we can restate Corollary 1.3 as

COROLLARY 1.3'. *If $R = Z_p$, P or Q_p , then there exist non-negative integers b_1, b_2, \dots, b_n such that*

$$RG \simeq \sum_{i=1}^n b_i R(\zeta_{\alpha_i})$$

where ζ_{α_i} is a primitive q^{α_i} th root of unity.

We now show that the sequence b_1, b_2, \dots, b_n are invariants of RG .

If $m \in \mathbb{N}$, and r is a non-negative integer, by the symbol rC_m we mean the direct sum of r copies of C_m . Let s_1, s_2, \dots, s_n be a sequences of non-negative integers such that $s_i \mid s_{i+1}$, $i=1, \dots, n-1$. Suppose that A and B are finite abelian groups of the the same order and $A \simeq \sum_{i=1}^n r_i C_{p^{s_i-1}}$ $B \simeq \sum_{i=1}^n \bar{r}_i C_{p^{s_i-1}}$ where r_i, \bar{r}_i are non-negative integers for $i=1, \dots, n$. Since $(p^{s_i}-1) \mid (p^{s_{i+1}}-1)$, the fundamental theorem of abelian groups tells us that $A \simeq B$ if and only if $r_i = \bar{r}_i$, $i=1, \dots, n$.

If S is a commutative ring with identity, let $S^* = \{\delta \in S \mid \delta \text{ is of finite multiplicative order}\}$.

THEOREM 1.4. *Let R be Z_p, P , or Q_p . If G and H are finite abelian groups of order q^n , then*

$$RG \simeq RH \Leftrightarrow (RG)^* \simeq (RH)^*.$$

Proof. Since adjoining ζ_{α_i} to Q_p gives a totally unramified extension of Q_p , $(Q_p(\zeta_{\alpha_i}))^* \simeq (Z_p(\zeta_{\alpha_i}))^* \simeq C_{p^{\nu_q(\alpha_i)}-1}$. But any element of finite order in $Q_p(\zeta_{\alpha_i})$ is in fact in $P(\zeta_{\alpha_i})$ so that $(P(\zeta_{\alpha_i}))^* \simeq C_{p^{\nu_q(\alpha_i)}-1}$.

By Corollary 1.3', there exist non-negative integers b_1, b_2, \dots, b_n such that $RG \simeq \sum_{i=1}^n b_i R(\zeta_{\alpha_i})$. Thus $(RG)^* \simeq \sum_{i=1}^n b_i C_{p^{\nu_q(\alpha_i)}-1}$. If $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n$ is another sequence of non-negative integers such that $RG \simeq \sum_{i=1}^n \bar{b}_i R(\zeta_{\alpha_i})$, $(RG)^* \simeq \sum \bar{b}_i C_{p^{\nu_q(\alpha_i)}-1}$. By the fundamental theorem of abelian groups, we must have $\bar{b}_i = b_i$, $i=1, \dots, n$. Thus the sequence b_1, b_2, \dots, b_n is determined by RG and in turn determines, via a one-one correspondence, the group $(RG)^*$. Therefore $RG \simeq RH \Leftrightarrow (RG)^* \simeq (RH)^*$.

COROLLARY 1.5. *Let R be Z_p, P or Q_p . If G is a finite abelian group of order q^n , and $RG \simeq \sum_{i=1}^n b_i R(\zeta_{\alpha_i})$, $RG \simeq \sum_{i=1}^n \bar{b}_i R(\zeta_{\alpha_i})$, where b_i, \bar{b}_i , $i=1, \dots, n$, are non-negative integers then $b_i = \bar{b}_i$, $i=1, 2, \dots, n$.*

COROLLARY 1.6. *Suppose G and H are finite abelian groups of order q^n . The following are equivalent.*

- (i) $Z_p G \simeq Z_p H$, (ii) $P G \simeq P H$, (iii) $Q_p G \simeq Q_p H$.

Proof. Note that $(Z_p G)^* \simeq (P G)^* \simeq (Q_p G)^*$ and use Theorem 1.4

§2

If A is a finite abelian group of order $n = p^e q_1^{e_1} q_2^{e_2} \dots q_r^{e_r}$, we let A_p be the p -Sylow subgroup of A and A_{q_i} the q_i -Sylow subgroup of A . Again we start from a result of Perlis-Walker.

PROPOSITION 2.1. *If A and B are finite abelian groups of order $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ where $p_i, i = 1, \dots, r$, are distinct primes, and F is a field of characteristic k , where $k = 0$ or $(k, n) = 1$, then $FA \simeq FB \Leftrightarrow FA_{p_i} \simeq FB_{p_i}, i = 1, \dots, r$.*

Proof. See [3] and [1].

PROPOSITION 2.2. *Let A and B be finite abelian groups of order p^n . Then $Q_p A \simeq Q_p B \Leftrightarrow A \simeq B$.*

Proof. By the result of Perlis-Walker (of the last section), $Q_p A \simeq \sum_{d=1}^n a_d Q_p(\zeta_d)$ where ζ_d is a primitive p^d th root of unity, $a_d = n_d/v_d$, n_d is the number of elements in A of order p^d , and $v_d = \text{degree}(Q_p(\zeta_d)/Q_p)$. Similarly, $Q_p B \simeq \sum_{d=1}^n \bar{a}_d Q_p(\zeta_d)$, where $\bar{a}_d = \bar{n}_d/v_d$, and \bar{n}_d is the number of elements in B of order p^d . $Q_p A \simeq Q_p B$ implies

$$(Q_p A)^* \simeq \sum_{d=1}^n a_d (Q_p(\zeta_d))^* \simeq \sum_{d=1}^n \bar{a}_d (Q_p(\zeta_d))^* \simeq (Q_p B)^*.$$

Q_p contains only the $(p-1)$ st roots of unity, and $Q_p(\zeta_d)$ is a totally ramified extension of Q_p , so that $(Q_p(\zeta_d))^* \simeq C_{p^d(p-1)}$. By the fundamental theorem of abelian groups we have $a_d = \bar{a}_d, d = 1, \dots, n$, and so $n_d = \bar{n}_d, d = 1, \dots, n$. Thus $A \simeq B$.

COROLLARY 2.3. *If A and B are finite abelian groups of order n , and Q is the field of rational numbers, then $QA \simeq QB \Leftrightarrow A \simeq B$.*

Proof. By Proposition 2.1, it is sufficient to suppose that $n = p^m$. If $QA \simeq QB$, then $QA \otimes Q_p \simeq QB \otimes Q_p$; i.e., $Q_p A \simeq Q_p B$. But by Proposition 2.2, $A \simeq B$.

COROLLARY 2.4. *If A and B are finite abelian groups of order p^n , then $PA \simeq PB \Leftrightarrow A \simeq B$.*

Proof. $PA \simeq PB$ implies $PA \otimes Q_p \simeq PB \otimes Q_p$. By Proposition 2.2 $A \simeq B$.

Note that if A and B are finite abelian groups of order p^n , then $Z_p A \simeq Z_p B \Leftrightarrow A \simeq B$. See [2], e.g.

THEOREM 2.5. *Suppose R is Z_p, P or Q_p . If G and H are finite abelian groups of order $n = p^e q_1^{e_1} \cdots q_r^{e_r}$ where p, q_1, \dots, q_r are distinct primes, then*

$$RG \simeq RH \Leftrightarrow RG_p \simeq RH_p \quad \text{and} \quad RG_{q_i} \simeq RH_{q_i} \quad i = 1, \dots, r.$$

Proof. If $R = Q_p$, the result follows immediately by Proposition 2.1.

Suppose $R = Z_p$, and $Z_p G \simeq Z_p H$. By the results in May [2], $G_p \simeq H_p$. Suppose $|G_p| = p^{c_1}$, $|H_p| = p^{c_2}$, and $c = \max(c_1, c_2)$. Let $(Z_p G)^{p^c} = \{\delta^{p^c} \mid \delta \in Z_p G\}$. $(Z_p G)^{p^c} \simeq (Z_p H)^{p^c}$. As $Z_p^p \simeq Z_p$, we have $(Z_p G)^{p^c} \simeq Z_p(G/G_p) \simeq (Z_p H)^{p^c} \simeq Z_p(H/H_p)$. By Proposition 2.1, $Z_p(G/G_p) \simeq Z_p(H/H_p) \Rightarrow Z_p(G_{q_i}) \simeq Z_p(H_{q_i}), i = 1, \dots, r$. Thus $Z_p(G) \simeq Z_p(H) \Rightarrow Z_p G_p \simeq Z_p H_p$ and $Z_p(G_{q_i}) \simeq Z_p(H_{q_i}), i = 1, \dots, r$. The opposite implication follows from the tensor product.

If $R=P$, and $PG \simeq PH$, we let $J=(p)$ be the ideal of P generated by p . The epimorphism $P \rightarrow P/J \simeq Z_p$, induces an isomorphism $Z_p G \simeq Z_p H$. By the previous case $Z_p G \simeq Z_p H \Rightarrow G_p \simeq H_p$ and $Z_p G_{q_i} \simeq Z_p H_{q_i}$, $i=1, \dots, r$. But by Corollary 1.6 $Z_p G_{q_i} \simeq Z_p H_{q_i} \Leftrightarrow PG_{q_i} \simeq PH_{q_i}$, so that $PG \simeq PH \Rightarrow PG_p = PH_p$ and $PG_{q_i} \simeq PH_{q_i}$, $i=1, \dots, r$. The opposite implication follows from the tensor product.

COROLLARY 2.6. *Let G and H be finite abelian groups of order n . The following are equivalent.*

- (i) $Z_p G \simeq Z_p H$, (ii) $PG \simeq PH$, (iii) $Q_p G \simeq Q_p H$.

§3

In this section we investigate for which integers $m \in N$, is it true that two abelian groups G and H of order m have isomorphic p -adic group rings if and only if G and H are isomorphic. To study this question, it is sufficient, by Theorem 2.5, to again suppose that G and H are abelian groups of order q^n . We will use the notation of Section 1.

DEFINITION. We let $\mathcal{J}(q) = r$, if there is an $r \in N$ such that $\alpha_r = r$ and $\alpha_{r+1} \neq (r+1)$. Otherwise, we define $\mathcal{J}(q) = \infty$. We will call $\mathcal{J}(q)$, the index of q , (relative to p).

THEOREM 3.1. *Let R be Z_p, P or Q_p . Let $n \in N$.*

- (i) *If $n < 2\mathcal{J}(q)$, and G and H are abelian groups of order q^n , then*

$$RG \simeq RH \Leftrightarrow G \simeq H.$$

- (ii) *If $n \geq 2\mathcal{J}(q)$, there exist non-isomorphic abelian groups G and H , of order q^n such that $RG \simeq RH$.*

Proof. (i) Suppose $G \simeq C_{q^{y_1}} \times C_{q^{y_2}} \times \dots \times C_{q^{y_r}}$ and $H \simeq C_{q^{z_1}} \times C_{q^{z_2}} \times \dots \times C_{q^{z_r}}$ where $y_1 \geq y_2 \geq \dots \geq y_r \geq 0$, $z_1 \geq z_2 \geq \dots \geq z_r \geq 0$ and $y_1 + y_2 + \dots + y_r = z_1 + z_2 + \dots + z_r = n$. By Corollary 1.3', there exist non-negative integers b_1, b_2, \dots, b_n and $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n$ such that $RG \simeq \sum_{i=1}^n b_i R(\zeta_{\alpha_i})$ and $RH \simeq \sum_{i=1}^n \bar{b}_i R(\zeta_{\alpha_i})$. By Corollary 1.5, $RG \simeq RH \Leftrightarrow b_i = \bar{b}_i$, $i=1, \dots, n$. We suppose $RG \simeq RH$. $b_i = n_i / \gamma_q(i)$ for $i=1, \dots, \mathcal{J}(q)-1$, where n_i denotes the number of elements of G of order q^i . Similarly, $\bar{b}_i = m_i / \gamma_q(i)$, where m_i denotes the number of elements of H of order q^i . Thus if $y_1 < \mathcal{J}(q)$ or $z_1 < \mathcal{J}(q)$ we must have $n_i = m_i$, $i=1, \dots, n$, and $G \simeq H$.

We will assume now $y_1 \geq \mathcal{J}(q)$ and $z_1 \geq \mathcal{J}(q)$.

Suppose $y_1 > z_1 \geq \mathcal{J}(q)$. Let $\bar{G} \simeq C_{q^{y_2}} \times C_{q^{y_3}} \times \dots \times C_{q^{y_r}}$ and $\bar{H} \simeq C_{q^{z_2}} \times C_{q^{z_3}} \times \dots \times C_{q^{z_r}}$. Then $G \simeq C_{q^{y_1}} \times \bar{G}$, $H \simeq C_{q^{z_1}} \times \bar{H}$, $|\bar{H}| > |\bar{G}|$, $|\bar{G}| \leq q^{n - \mathcal{J}(q) - 1} \leq q^{\mathcal{J}(q) - 2}$ and

$$|\bar{H}| \leq q^{n-\mathcal{J}(q)} \leq q^{\mathcal{J}(q)-1}$$

$$b_{\mathcal{J}(q)-1} = \frac{n_{\mathcal{J}(q)-1}}{\gamma_{q(\mathcal{J}(q)-1)}} = \bar{b}_{\mathcal{J}(q)-1} = \frac{m_{\mathcal{J}(q)-1}}{\gamma_{(q^{\mathcal{J}(q)-1})}}$$

Clearly, $n_{\mathcal{J}(q)-1} = \phi(q^{\mathcal{J}(q)-1})|\bar{G}|$ (ϕ = Euler “phi” functions) while $m_{\mathcal{J}(q)-1} \geq \phi(q^{\mathcal{J}(q)-1})|\bar{H}| > \phi(q^{\mathcal{J}(q)-1})|\bar{G}| = n_{\mathcal{J}(q)-1}$. This contradiction shows $y_1 = z_1$.

Finally, we assume $y_i = z_i, i = 1, \dots, t, t \geq 1$, and $z_{t+1} < y_{t+1} < \mathcal{J}(q)$. Let

$$G' = C_{q^{y_1}} \times C_{q^{y_2}} \times \dots \times C_{q^{y_r}}$$

$$G'' = C_{q^{y_{t+1}}} \times \dots \times C_{q^{y_r}}$$

$$H'' = C_{q^{z_{y+1}}} \times \dots \times C_{q^{z_r}}$$

then $G \simeq G' \times G'', H \simeq G' \times H''$ and $|G''| = |H''| < q^{\mathcal{J}(q)}$.

Let $\mu = y_{t+1}$. Then

$$b_\mu = \frac{n_\mu}{\gamma_{q(\mu)}} = \frac{m_\mu}{\gamma_{q(\mu)}} = \bar{b}_\mu.$$

If λ is the number of elements of G' of order q^μ , then $m_\mu = \lambda|H''|$. But $n_\mu \geq \lambda|G''| + |G'|(\phi(q^\mu)) > \lambda|G''| = m_\mu$. This is the desired contradiction, and we conclude $y_i = z_i, i = 1, \dots, r$, and $GW = H$.

(ii) If $n = 2\mathcal{J}(q)$, let

$$G \simeq C_{q^{\mathcal{J}(q)}} \times C_{q^{\mathcal{J}(q)}}$$

and

$$H \simeq C_{q^{\mathcal{J}(q)+1}} \times C_{q^{\mathcal{J}(q)-1}}.$$

A straightforward verification shows that $b_i = \bar{b}_i, i = 1, \dots, \mathcal{J}(q) - 1$ and $b_i = \bar{b}_i = 0, i > \mathcal{J}(q)$. Since $|G| = |H|$, we must have $b_{\mathcal{J}(q)} = \bar{b}_{\mathcal{J}(q)}$ and $RG \simeq RH$.

If $n > 2\mathcal{J}(q)$, we merely tack on a sufficient number of copies of C_q to each of the above examples.

REFERENCES

[1] COHEN, D. E., *On abelian group algebra*, Math. Z. 105 (1968), 267–268.
 [2] MAY, W., *Commutative group algebras*, Trans. Am. Math. Soc. 136 (1969), 139–149.
 [3] PERLIS, S. and WALKER, G., *Abelian group algebras of finite order*, Trans. Am. Math. Soc. 68 (1950), 420–426.
 [4] RAGGI CÁRDENAS, *Units in group rings with coefficients in K_{p^n}, Z_{p^n} and \hat{Z}_p* . (Spanish) An. Inst. Mat. Univ. Nac. Autonoma Mexico 10 (1970), 29–65.

École Polytechnique Fédérale – Lausanne
The University of Connecticut
Storrs, Connecticut, 06268, U.S.A.

Received March 9, 1974