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Bicartesian Squares of Nilpotent Groups[1)]

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0. Introduction

Our purpose in this paper is to draw attention to certain commutative squares of homomorphisms which arise naturally in the study of nilpotent groups and, in particular, of their localization theory. It turns out that these squares are *bicartesian* (that is, pullbacks and pushouts) in $\mathbb{N}$, the category of nilpotent groups. This is a somewhat remarkable fact, especially since, in general, $\mathbb{N}$ does not admit pushouts.

In the first of these squares (Section 2) we consider two families of primes, P, Q such that $P \cup Q = \Pi$, the family of all primes, and a commutative square of homomorphisms in $\mathbb{N}$,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ K & \xrightarrow[\sigma]{} & L \end{array} \qquad (*)$$

in which ϕ, σ are P-bijective and ψ, ϱ are Q-bijective (see [H, HMR]). Such squares turn out (Theorem 2.1) always to be bicartesian in $\mathbb{N}$. They arise whenever we have a nilpotent group G and its localizing maps,

$$\begin{array}{ccc} G & \rightarrow & G_P \\ \downarrow & & \downarrow \\ G_Q & \rightarrow & G_{P \cap Q}. \end{array}$$

Moreover every pair of homomorphisms $\phi: G \to H$, $\psi: G \to K$ in $\mathbb{N}$, with ϕ P-bijective and ψ Q-bijective, can be imbedded in a square $(*)$, as can every pair of homomorphisms $\varrho: H \to L$, $\sigma: K \to L$ in $\mathbb{N}$, with ϱ Q-bijective and σ P-bijective. The results of this section will be applied in a following paper [HM], in which we study an abelian group structure which may be imposed on the genus set of a finitely generated nilpotent group

[1)] Part of the content of this paper formed the subject matter of a talk given by the first author at a topology conference held at Ohio State University in August, 1974.

with finite commutator subgroup. We also draw attention to [K], in which (P, Q)-squares are studied in connection with the localization of manifolds.

In Section 3 we study a square that may be constructed out of any nilpotent group G. The p-localization maps $e_p: G \to G_p$, as p ranges over Π, determine an embedding [2])

$$\hat{e}: G \rightarrowtail \hat{G} = \prod G_p;$$

we call $\hat{G}$ the *local expansion* of G. Rationalizing $\hat{e}$, we obtain a commutative square

$$\begin{array}{ccc} G & \overset{\hat{e}}{\rightarrowtail} & \hat{G} \\ \downarrow r & & \downarrow r \\ G_0 & \overset{\hat{e}_0}{\rightarrowtail} & \hat{G}_0 \end{array}, \qquad (**)$$

and it turns out that $(**)$ is always bicartesian in $\mathbb{N}$; our method of proof is, in this case, homological. The square $(**)$ is relevant to the study of homotopical localization [HMR], and this connection is explained and elaborated in Section 3. In Section 4 we indicate further interesting properties of $(**)$, including the fact that the bicartesian property is preserved when we take the nth homology group H_n, $n \geqslant 0$.

The first section of the paper establishes some preliminary facts related to localization theory which are used in the present paper and in its sequel [HM].

1. Preliminaries on Nilpotent Groups and Localization

In this section we establish some results which will be used in the sequel. Our first result, crucial to the proof of Theorem 3.1, is certainly classical, so we give no proof.

PROPOSITION 1.1. *Let G be a nilpotent group, let T be its torsion subgroup and let T_p be the p-component of T. Then*

$$T = \dot{\Pi}\, T_p,$$

the restricted direct product of the groups T_p.

Our next result is used in Section 4 and will also be applied in [HM].

PROPOSITION 1.2. *Let G be a nilpotent group, H a subgroup of G and let $x \in G$ be such that $e_p x \in H_p$, for all p where $e_p: G \to G_p$ is the p-localizing map. Then $x \in H$.*

Proof. Assume first that H is normal in G, $H \lhd G$, with $K = G/H$. Then we have, for

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[2]) We may employ the notations $\hat{G}$, $\hat{e}$ in this paper, since we never discuss completions.

each p, a map of exact sequences

$$\begin{array}{ccccc} H & \rightarrowtail & G & \stackrel{\varepsilon}{\twoheadrightarrow} & K \\ \downarrow{\scriptstyle e_p} & & \downarrow{\scriptstyle e_p} & & \downarrow{\scriptstyle e_p} \\ H_p & \rightarrowtail & G_p & \stackrel{\varepsilon_p}{\twoheadrightarrow} & K_p . \end{array}$$

Then εx is an element of K such that $e_p \varepsilon x = 1$ for all primes p. This shows that $\varepsilon x = 1$ so $x \in H$.

In the general case we exploit the fact that H is *subnormal* in G, that is, we can find a normal series

$$H \lhd N_1 \lhd \cdots \lhd N_k \lhd G .$$

The argument above then shows successively that $x \in N_k$, $x \in N_{k-1}, \ldots, x \in N_1$, $x \in H$.

COROLLARY 1.3. *Let $\phi: G \to K$ be a homomorphism of nilpotent groups and let $H \subseteq G$, $L \subseteq K$. Then $\phi H \subseteq L$ if and only if $\phi_p H_p \subseteq L_p$ for all p.*

COROLLARY 1.4. *Let G be a nilpotent group, H a subgroup of G. Then $H \lhd G$ if and only if $H_p \lhd G_p$ for all p.*

PROPOSITION 1.5. *Let $\phi: G \to K$ be a homomorphism of nilpotent groups. Localizing at P yields*

$$\begin{array}{ccc} G & \stackrel{\phi}{\rightarrow} & K \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ G_P & \stackrel{\phi_P}{\rightarrow} & K_P . \end{array}$$

and hence an induced homomorphism[3] $\bar{e}: \operatorname{coker} \phi \to \operatorname{coker} \phi_P$. *Then $\bar{e}$ P-localizes.*

Proof. First, consider the map $\phi G \to \phi_P G_P$ induced by e. It is plain that $\phi_P G_P$, as a quotient of G_P, has qth roots for every $q \in P'$, and such roots are unique since $\phi_P G_P$ is a subgroup of K_P. Thus $\phi_P G_P$ is P-local. It is then easy to show that $\phi G \to \phi_P G_P$ is P-bijective so that (see [H]) it P-localizes.

Thus essentially we have the following situation. Let $H \subseteq K$ and let $\bar{H}$ be the normal closure of H in K. Similarly, let $\overline{(H_P)}$ be the normal closure of H_P in K_P. Then we claim

[3]) Note that the cokernel in $\mathbb{G}$, the category of groups, coincides with the cokernel in $\mathbb{N}$.

that $(\overline{H_P})$ is the P-localization of $\bar{H}$. Indeed, it is plain that

$$(\overline{H_P}) \subseteq \bar{H}_P$$

and we claim that the inclusion is an identity. For this it suffices to show that $(\overline{H_P})$ is P-local, since plainly e sends $\bar{H}$ to $(\overline{H_P})$. We must show that $(\overline{H_P})$ has unique qth roots for $q \in P'$. The uniqueness is evident since $(\overline{H_P})$ is a subgroup of K_P; and the existence follows from Blackburn's Theorem [B], since every element of $(\overline{H_P})$ is a product of conjugates of elements of H_P, and every element of H_P is a q^f-power for an arbitrary positive integer f.

The proof of the proposition is now completed by observing that localization is exact [H].

Now let $G, H, K \in \mathbb{N}_c$, the category of nilpotent groups N with $\operatorname{nil} N \leqslant c$, and let $\phi: G \to H$, $\psi: G \to K$. We may form the pushout in $\mathbb{N}_c$,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ K & \xrightarrow[\sigma]{} & L \end{array} \tag{1.3}$$

and we claim that, for any family of primes P,

PROPOSITION 1.6. *In the pushout diagram* (1.3),

(i) *ϕ is P-surjective if and only if σ is P-surjective*;

(ii) *if ϕ is P-bijective, σ is P-bijective.*

Proof. Let Loc: $\mathbb{N}_c \to \mathbb{N}_{cP}$ be the P-localization functor, where $\mathbb{N}_{cP}$ is the subcategory of $\mathbb{N}_c$ consisting of the P-local groups of $\mathbb{N}_c$. Then Loc is left adjoint to the embedding and so commutes with pushouts. Thus

$$\begin{array}{ccc} G_P & \xrightarrow{\phi_P} & H_P \\ {\scriptstyle\psi_P}\downarrow & & \downarrow{\scriptstyle\varrho_P} \\ K_P & \xrightarrow[\sigma_P]{} & L_P \end{array} \tag{1.4}$$

is a pushout diagram in $\mathbb{N}_{cP}$. Now it follows from Proposition 1.5 that $\mathbb{N}_{cP}$ has cokernels and indeed that the cokernel in $\mathbb{N}_{cP}$ coincides with the cokernel in $\mathbb{N}_c$, and hence with the cokernel in $\mathbb{N}$ (or $\mathbb{G}$). However, in the category $\mathbb{N}$, a homomorphism α is surjective if and only if $\operatorname{coker}\alpha$ is trivial. From (1.4) we infer that $\operatorname{coker}\phi_P \cong \operatorname{coker}\sigma_P$, and hence ϕ_P is surjective if and only if σ_P is surjective. By [H], this is equivalent to the first assertion of Proposition 1.6.

Now certainly we may infer from (1.4) that ϕ_P is invertible (in $\mathbb{N}_{cP}$) if σ_P is invertible. However an invertible morphism of $\mathbb{N}_{cP}$ is just an isomorphism so σ_P is an isomorphism if ϕ_P is an isomorphism and this is equivalent to the second assertion of Proposition 1.6.

Finally let G, H, $K \in \mathbb{N}$ and let $\phi: H \to G$, $\psi: K \to G$. We may form the pullback in $\mathbb{N}$ (or $\mathbb{G}$),

$$\begin{array}{ccc} M & \xrightarrow{\varrho} & H \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\phi} \\ K & \xrightarrow[\psi]{} & G \end{array} \tag{1.5}$$

and we claim that, for any family of primes P,

PROPOSITION 1.7. *In the pullback diagram* (1.5),

(i) *ϕ is P-injective if and only if σ is P-injective*;

(ii) *if ϕ is P-bijective, σ is P-bijective.*

Proof. It is shown in [HMR] that localization commutes with pullbacks, in $\mathbb{N}$. Now in the pullback diagram

$$\begin{array}{ccc} M_P & \xrightarrow{\varrho_P} & H_P \\ {\scriptstyle\sigma_P}\downarrow & & \downarrow{\scriptstyle\phi_P} \\ K_P & \xrightarrow[\psi_P]{} & G_P \end{array} \tag{1.6}$$

it is clear that ϕ_P is injective if and only if σ_P is injective (since ϕ_P, σ_P have isomorphic kernels) and that σ_P is invertible if ϕ_P is invertible. Again by [H] these statements are equivalent to the assertion of the proposition.

2. Commutative (P, Q)-squares

Let P, Q be families of primes such that $P \cup Q = \Pi$, the family of all primes and let

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ K & \xrightarrow[\sigma]{} & L \end{array} \tag{2.1}$$

be a commutative square in $\mathbb{N}$. Then we prove

THEOREM 2.1. *Suppose, in* (2.1), *that* ϕ, σ *are P-bijective and that* ψ, ϱ *are Q-bijective, with* $P \cup Q = \Pi$. *Then*

(i) (2.1) *is a pullback in* $\mathbb{N}$

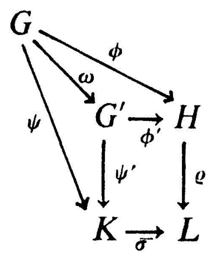
(ii) (2.1) *is a pushout in* $\mathbb{N}$

(iii) *if* $H, K \in \mathbb{N}_c$, *then* $G, L \in \mathbb{N}_c$

(iv) *every element x in L is expressible as*

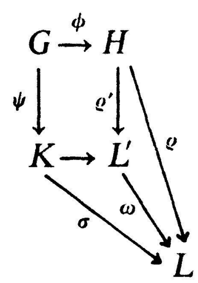
$$x = \varrho a \cdot \sigma b = \sigma b' \cdot \varrho a', \qquad a, a' \in H, \ b, b' \in K.$$

Proof. (i) Form the pullback of ϱ, σ in $\mathbb{N}$. We obtain a diagram



By Proposition 1.7 (ii) we infer that ϕ' is P-bijective and ψ' is Q-bijective. Since ϕ, ϕ' are P-bijective, so is ω; since ψ, ψ' are Q-bijective, so is ω. Thus ω is P-bijective and Q-bijective, and so, since $P \cup Q = \Pi$, ω is an isomorphism. Thus (2.1) is a pullback in $\mathbb{N}$.

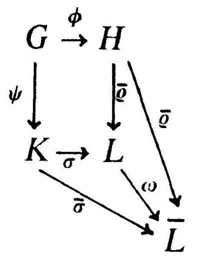
(ii) We first observe that $G \in \mathbb{N}_c$ if $H, K \in \mathbb{N}_c$ because G is embedded in $H \times K$. To prove (ii) we must show that (2.1) is a pushout in $\mathbb{N}_{c'}$, for every $c' \geqslant c$. First, let $L \in \mathbb{N}_d$, $d \geqslant c$. Form the pushout of ϕ, ψ in $\mathbb{N}_d$, and so obtain a diagram



By Proposition 1.6 (ii) we infer that σ' is P-bijective and ϱ' is Q-bijective. As above, we deduce that ω is both P-bijective and Q-bijective, hence an isomorphism, so that (2.1) is a pushout in $\mathbb{N}_d$.

Now form the pushout of ϕ, ψ in $\mathbb{N}_c$. Such a pushout is a commutative diagram in

$\mathbb{N}_d$, so we obtain



where $(\bar{\varrho}, \bar{\sigma})$ is the pushout of (ϕ, ψ) in $\mathbb{N}_c$. Again we infer that ω is an isomorphism, so that (2.1) is, in fact, a pushout in $\mathbb{N}_c$ and we have proved that $L \in \mathbb{N}_c$, completing the proof of (iii). It is moreover now clear that (2.1) is a pushout in $\mathbb{N}_{c'}$ for every $c' \geqslant c$, so that (ii) is also proved.

It remains to prove (iv). Now, given $x \in L$, there exist $m \in P'$, $n \in Q'$, $h \in H$, $k \in K$, with

$$x^m = \sigma k, \qquad x^n = \varrho h.$$

But $P \cup Q = \Pi$, so that m, n are mutually prime and there exist integers r, s with $rm + sn = 1$. Then

$$x = x^{rm+sn} = \sigma k^r \cdot \varrho h^s = \varrho h^s \cdot \sigma k^r.$$

COROLLARY 2.2. *Let $\varrho: H \to L$, $\sigma: K \to L$ in $\mathbb{N}$ with ϱ Q-bijective, σ P-bijective, $P \cup Q = \Pi$. Form the pullback of ϱ, σ* in $\mathbb{N}$,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ K & \xrightarrow[\sigma]{} & L \end{array} \tag{2.2}$$

Then (2.2) *is also a pushout in* $\mathbb{N}$.

COROLLARY 2.3. *Let $\phi: G \to H$, $\psi: G \to K$ in $\mathbb{N}_c$ with ϕ P-bijective, ψ Q-bijective, $P \cup Q = \Pi$. Form the pushout of ϕ, ψ in $\mathbb{N}_c$,*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ K & \xrightarrow[\sigma]{} & L \end{array} \tag{2.3}$$

Then (2.3) *is a pullback and pushout in* $\mathbb{N}$.

COROLLARY 2.4. *Under the hypotheses of Corollary* 2.3, *form the pushout of* ϕ, ψ *in* $\mathbb{G}$,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ K & \rightarrow & R \end{array}$$

Then $\Gamma^{c+1}R=\Gamma^{c+2}R$, *where* Γ^i *is the ith term of the lower central series.*

EXAMPLE. We note that, for any nilpotent group G, the square

$$\begin{array}{ccc} G & \xrightarrow{e_P} & G_P \\ {\scriptstyle e_Q}\downarrow & & \downarrow{\scriptstyle r_P} \\ G_Q & \xrightarrow{r_Q} & G_0 \end{array}, \tag{2.4}$$

made up of localization maps e_P, e_Q, and rationalization maps r_P, r_Q, with $P\cup Q=\Pi$, satisfies the conditions of Theorem 2.1, provided that $P\cap Q=\emptyset$, that is, provided that $\{P, Q\}$ is a partition of Π. Of course, even if $\{P, Q\}$ is not a partition, we obtain an example by replacing G_0 in (2.4) by $G_{P\cap Q}$.

Remark. If we apply the homology functor H_n (with integer coefficients) to the square (2.1) we again get a (P, Q)-square

$$\begin{array}{ccc} H_nG & \rightarrow & H_nH \\ \downarrow & & \downarrow \\ H_nK & \rightarrow & H_nL. \end{array} \tag{2.5}$$

Thus (2.5) is bicartesian. It follows that (2.1) gives rise to a Mayer-Vietoris sequence in integral homology (which, in fact, breaks up into short exact sequences). The existence of such a sequence is of interest since the homomorphisms ϕ, ψ of (2.1) are not assumed to be injective – and, in any case, (2.1) is a pushout in $\mathbb{N}$, not in $\mathbb{G}$.

3. A Commutative Square in N.

In this section we study a commutative square associated with a given nilpotent group G. Let

$$\hat{G}=\prod_p G_p; \tag{3.1}$$

we refer to $\hat{G}$ as the *local expansion* of G. Then the localizing maps $e_p: G \to G_p$ induce an embedding

$$\hat{e}: G \rightarrowtail \hat{G} \tag{3.2}$$

of G in its local expansion. The square in question is obtained by rationalizing (3.2), thus,

$$\begin{array}{ccc} G & \overset{\hat{e}}{\rightarrowtail} & \hat{G} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ G_0 & \underset{\hat{e}_0}{\rightarrowtail} & \hat{G}_0 . \end{array} \tag{3.3}$$

Then (3.3) was studied in [HMR] in the special case that G is a finitely-generated abelian group. This study was relevant to homotopical localization. We will point out, at the end of the section, that the general properties of (3.3) also have applications in homotopy theory. Note that $\hat{G}_0 = (\hat{G})_0$; we rationalize the local expansion of G.

Since $\operatorname{nil} G = \operatorname{nil} \hat{G}$, it is plain that (3.3) is a diagram in $\mathbb{N}_c$ if G is in $\mathbb{N}_c$. We will prove

THEOREM 3.1. *The diagram* (3.3) *is a pullback in* $\mathbb{G}$ *and a pushout in* $\mathbb{N}_c$. *Hence it is bicartesian in* $\mathbb{N}$.

Proof. We first prove that (3.3) is a pushout in $\mathbb{N}_c$. It follows from Proposition 1.1 that $\hat{e}$ maps the torsion subgroup T of G isomorphically onto the torsion subgroup of $\hat{G}$. To see this it suffices to recall that localization commutes with the torsion subgroup functor and then to remark that the torsion subgroup of $\hat{G}$ is obviously the restricted direct product $\dot{\prod}_p T_p$. We will therefore also write T for the torsion subgroup of $\hat{G}$, so that $\hat{e} \mid T = 1$. We may factor (3.3) as

$$\begin{array}{ccc} G & \overset{\hat{e}}{\rightarrowtail} & \hat{G} \\ \downarrow & & \downarrow \\ G/T & \overset{\bar{e}}{\rightarrowtail} & \hat{G}/T \\ \downarrow{\scriptstyle \bar{r}} & & \downarrow{\scriptstyle \bar{r}} \\ G_0 & \overset{\hat{e}_0}{\rightarrowtail} & \hat{G}_0 \end{array} \tag{3.4}$$

Moreover, $\bar{e}$ is injective since $\hat{e}$ is injective, and $\bar{r}: G/T \rightarrowtail G_0$ $\bar{r}: \hat{G}/T \rightarrowtail \hat{G}_0$ are again rationalization maps, so we will write r for $\bar{r}$. Further it is easy to see that (3.3) is a pushout in $\mathbb{N}_c$ if and only if the lower square of (3.4) is a pushout in $\mathbb{N}_c$. Note, however, that $\bar{e}$ is not the canonical embedding in the local expansion unless G has p-torsion for only finitely many primes p.

Our next observation is that $\hat{e}$ induces

$$\hat{e}_*: H_n(G; \mathbf{Z}/p) \cong H_n(\hat{G}; \mathbf{Z}/p) \quad \text{for all } n, p. \tag{3.5}$$

To see this, fix p and write $\hat{G} = G_p \times \tilde{G}$, where $\tilde{G} = \prod_{q \neq p} G_q$. Notice that each G_q is p'-local; so therefore is $\tilde{G}$, and so therefore too is $H_n\tilde{G}$, $n \geq 1$. It follows that $H_n(\tilde{G}; \mathbf{Z}/p) = 0$, $n \geq 1$. Thus the projection $\pi_p: G_p \times \tilde{G} \to G_p$ induces an isomorphism

$$\pi_*: H_n(G_p \times \tilde{G}; \mathbf{Z}/p) \cong H_n(G_p; \mathbf{Z}/p);$$

for, by the Künneth formula,

$$H_n(G_p \times \tilde{G}; \mathbf{Z}/p) = \bigoplus_{r+s=n} H_r(G_p; \mathbf{Z}/p) \otimes H_s(\tilde{G}; \mathbf{Z}/p),$$

the tensor product being taken over the field $\mathbf{Z}/p$. But, since $\mathbf{Z}/p$ is p-local,

$$e_{p*}: H_n(G; \mathbf{Z}/p) \cong H_n(G_p; \mathbf{Z}/p),$$

and (3.5) now follows from the fact that $e_p = \pi_p \hat{e}$.

We now complete the proof that (3.3) is a pushout in $\mathbb{N}_c$.

We first prove a result corresponding to (3.5), namely,

$$\bar{e}_*: H_n(G/T; \mathbf{Z}/p) \cong H_n(\hat{G}/T; \mathbf{Z}/p) \quad \text{for all } n, p. \tag{3.6}$$

The proof of (3.6) is along precisely the same lines as that of (3.5); namely, we fix p, so that

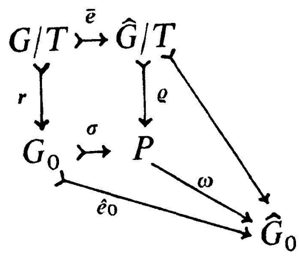
$$\hat{G}/T = G_p/T_p \times \tilde{G}/T_{p'},$$

where $\tilde{G}$, as in (3.5), is $\prod_{q \neq p} G_q$. Since $\tilde{G}$ and $T_{p'}$ are both p'-local, so is $\tilde{G}/T_{p'}$, and the rest of the argument proceeds as for (3.5), in view of the fact that $(G/T)_p = G_p/T_p$.

Now form a pushout diagram

$$\begin{array}{ccc} G/T & \overset{\bar{e}}{\rightarrowtail} & \hat{G}/T \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \varrho} \\ G_0 & \underset{\sigma}{\rightarrowtail} & P \end{array} \tag{3.7}$$

in $\mathbb{G}$; we then have



(3.8)

The pushout (3.7) gives rise to a Mayer-Vietoris sequence in homology; from this and (3.6) we immediately infer that

$$\sigma_*: H_n(G_0; \mathbf{Z}/p) \cong H_n(P; \mathbf{Z}/p).$$

But since $H_n G_0$ is rational, $n \geqslant 1$, this implies that

$$H_n(P; \mathbf{Z}/p) = 0, \quad n \geqslant 1, \quad \text{all } p. \tag{3.9}$$

Now plainly $r_*: H_n(G/T; \mathbf{Q}) \cong H_n(G_0; \mathbf{Q})$, $r_*: H_n(\hat{G}/T; \mathbf{Q}) \cong H_n(\hat{G}_0; \mathbf{Q})$. From the former isomorphism and the Mayer-Vietoris sequence we infer that

$$\varrho_*: H_n(\hat{G}; \mathbf{Q}) \cong H_n(P; \mathbf{Q}). \tag{3.10}$$

This with the latter isomorphism implies that

$$\omega_*: H_n(P; \mathbf{Q}) \cong H_n(\hat{G}_0; \mathbf{Q}). \tag{3.11}$$

Since $H_n\hat{G}_0$ is rational, $n \geqslant 1$, so that $H_n(\hat{G}_0; \mathbf{Z}/p) = 0$, $n \geqslant 1$, all p, we may now infer from (3.9) and (3.11) that

$$\omega_*: H_n P \cong H_n \hat{G}_0. \tag{3.12}$$

By the Stallings-Stammbach Theorem, (3.12) implies that ω induces an isomorphism

$$\omega_i: P/\Gamma^{i+1}P \cong \hat{G}_0/\Gamma^{i+1}\hat{G}_0 \tag{3.13}$$

for all $i \geqslant 0$. In particular, we may take $i = c$, so that ω induces

$$\omega_c: P/\Gamma^{c+1}P \cong \hat{G}_0. \tag{3.14}$$

But if τ projects P onto $P/\Gamma^{c+1}P$, then $\omega = \omega_c \tau$ and

$$\begin{array}{ccc} G/T & \overset{\bar{e}}{\rightarrowtail} & \hat{G}/T \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \tau\varrho} \\ G_0 & \overset{\tau\sigma}{\to} & P/\Gamma^{c+1}P \end{array}$$

is a pushout diagram in $\mathbb{N}_c$. This proves that the lower square of (3.4), and hence (3.3), is a pushout diagram in $\mathbb{N}_c$. Since $G \in \mathbb{N}_c$, it follows that $G \in \mathbb{N}_d$, for all $d \geqslant c$, so that (3.3) is a pushout diagram in $\mathbb{N}_d$ for all $d \geqslant c$, and hence a pushout diagram in $\mathbb{N}$.

We now prove that (3.3) is a pullback diagram in $\mathbb{G}$. Of course, we may abandon the categorical definition and simply prove that if $x \in \hat{G}$, $y \in G_0$ with $rx = \hat{e}_0 y$, then there exists $g \in G$ with $\hat{e}g = x$, $rg = y$ (the uniqueness of g follows because $\hat{e}$ is injective). We will invoke the following principle.

PROPOSITION 3.2. *Let S be an assertion about nilpotent groups such that*

(i) *S is true of all abelian groups*;

(ii) *if $N \rightarrowtail G \twoheadrightarrow Q$ is a central extension of nilpotent groups and if S is true of N and Q, then S is true of G.*

Then S is true of all nilpotent groups.

Proposition 3.2 follows from an easy induction on nilpotency class. Now consider the assertion S that (3.3) is a pullback (we need not specify the category). Then S is true of all abelian groups G since (3.3) is a pushout (in Ab) and $\hat{e}$ is injective. It thus remains to establish property (ii) of Proposition 3.2. But this is easy, noting the fact that local expansion and rationalization are exact functors (preserving centrality, though this is not important here). Thus Theorem 3.1 is completely proved.

We close this section by giving an application of Theorem 3.1 to homotopy theory. Given a nilpotent space X [HMR] we may form the square analogous to (3.3)

$$\begin{array}{ccc} X & \overset{\hat{e}}{\to} & \hat{X} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ X_0 & \overset{\hat{e}_0}{\to} & \hat{X}_0 . \end{array} \tag{3.15}$$

THEOREM 3.3. *The square* (3.15) *is a weak pullback in the homotopy category.*

The meaning of this theorem is the following. Replacing $r: \hat{X} \to \hat{X}_0$ by a fibre-map, we may form the pullback

$$\begin{array}{ccc} \bar{X} & \overset{u}{\to} & \hat{X} \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle r} \\ X_0 & \overset{\hat{e}_0}{\to} & \hat{X}_0 \end{array} \tag{3.16}$$

and hence we obtain a map $w: X \to \hat{X}$ such that $vw = r$, $uw = \hat{e}$. Then we claim that w is a homotopy equivalence. We remark that no finiteness assumptions are being imposed here.

Proof of Theorem 3.3. Applying the nth homotopy group functor to (3.15) we obtain a square

$$\begin{array}{ccc} \pi_n X & \xrightarrow{\hat{e}_*} & \pi_n \hat{X} \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ \pi_n X_0 & \xrightarrow{\hat{e}_{0*}} & \pi_n \hat{X}_0 \end{array} \tag{3.17}$$

of type (3.3) which is therefore a pullback and a pushout in $\mathbb{N}$. Applying the Mayer-Vietoris *homotopy* sequence to (3.16) we get an exact sequence

$$\cdots \pi_n X \xrightarrow{\{u_*, v_*\}} \pi_n \hat{X} \oplus \pi_n X_0 \xrightarrow{\langle r_*, -\hat{e}_{0*} \rangle} \pi_n \hat{X}_0 \to \cdots \to \pi_1 X \xrightarrow{\{u_*, v_*\}} \pi_1 \hat{X} \times \pi_1 X_0 \underset{\hat{e}_{0*}q'}{\overset{r_*q}{\rightrightarrows}} \pi_1 \hat{X}_0 \tag{3.18}$$

where the final two arrows are obtained by composing r_*, $\hat{e}_{0*}$ with the projections $q: \pi_1 \hat{X} \times \pi_1 X_0 \to \pi_1 \hat{X}$, $q': \pi_1 \hat{X} \times \pi_1 X_0 \to \pi_1 X_0$, respectively; and exactness at $\pi_1 \hat{X} \times \pi_1 X_0$ asserts that $r_* \xi = \hat{e}_{0*} \eta$ if and only if there exists $\zeta \in \pi_1 X$ with $u_* \zeta = \xi$, $v_* \zeta = \eta$. (Actually, to complete the argument we only need the easy "if" part of this last assertion).

It now follows easily from the fact that (3.17) is bicartesian, and from (3.18) that w induces isomorphisms

$$w_*: \pi_n X \cong \pi_n \hat{X}, \qquad n \geqslant 1,$$

so that w is a homotopy equivalence, provided we know that $\hat{X}$ is connected. This in turn follows from Proposition II.7.11 of [HMR] and Proposition 3.4 below.

PROPOSITION 3.4. *In the square* (3.3), *each element of* $\hat{G}_0$ *is expressible as*

$$\begin{aligned} z &= rx \cdot \hat{e}_0 y \\ &= \hat{e}_0 y' \cdot rx', \end{aligned}$$

$z \in \hat{G}_0$, $x, x' \in \hat{G}$, $y, y' \in G_0$.

Proof. We apply Proposition 3.2 to the assertion S of Proposition 3.4. Then (i) holds since (3.3) is a pushout square. An elementary computation establishes (ii) (here the centrality of N is essential), so that Proposition 3.4, and hence Theorem 3.3, follows.

Remark. A proof of Theorem 3.3, for spaces X of finite type, is given in [HMR], but the argument that $\hat{X}$ is connected is there omitted.

4. Further Properties of the Square (3.3)

In this section we make further deductions about the square (3.3)

$$\begin{array}{ccc} G & \overset{\hat{e}}{\rightarrowtail} & \hat{G} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ G_0 & \overset{\hat{e}_0}{\rightarrowtail} & \hat{G}_0 ; \end{array}$$

and a deduction from Theorem 3.1.

THEOREM 4.1. *The induced homology square*

$$\begin{array}{ccc} H_nG & \overset{H_n\hat{a}}{\rightarrowtail} & H_n\hat{G} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ H_nG_0 & \overset{H_n\hat{e}_0}{\rightarrowtail} & H_n\hat{G}_0 \end{array} \tag{4.1}$$

is bicartesian for all $n \geqslant 1$, *with* $H_n\hat{e}$, $H_n\hat{e}_0$ *injective.*

Proof. Let M be the mapping cone of the map $K(G, 1) \to K(\hat{G}, 1)$ associated with $\hat{e}$. It follows from (3.5) that $H_n(M; \mathbf{Z}/p) = 0$ for all $n \geqslant 1$ and all primes p. Thus H_nM is a rational vector space, $n \geqslant 1$.

Now the maps $\pi_p : \hat{G} \to G_p$ induce $H_n\pi_p : H_n\hat{G} \to H_nG_p$ and hence $\tilde{\pi} : H_n\hat{G} \to \hat{H}_nG$, such that

$$\tilde{\pi} \circ H_n\hat{e} = \hat{e} : H_nG \to \hat{H}_nG .$$

Since $\hat{e}$ is injective, so is $H_n\hat{e}$ – and hence also $H_n\hat{e}_0$. We thus obtain the diagram, with exact rows, and with $\bar{M}$ the mapping cone of $K(G_0, 1) \to K(\hat{G}_0, 1)$,

$$\begin{array}{ccccc} H_nG & \underset{H_n\hat{e}}{\rightarrowtail} & H_n\hat{G} & \twoheadrightarrow & H_nM \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ H_nG_0 & \overset{H_n\hat{e}_0}{\rightarrowtail} & H_n\hat{G}_0 & \twoheadrightarrow & H_n\bar{M} \end{array}$$

But since H_nM is rational, it follows that $r : H_nM \cong H_n\bar{M}$. Thus (4.1) is a pushout and, $H_n\hat{e}$ being injective, therefore bicartesian.

THEOREM 4.2. *In* (3.3) *coker* $\hat{e}$ *is a rational group. Further, if* $x \in \hat{G}$ *and* $x^n \in \hat{e}G$ *for some* $n \geqslant 1$, *then* $x \in \hat{e}G$.

Proof. Since (3.3) is a pushout in $\mathbb{N}$, the induced map $\operatorname{coker}\hat{e} \to \operatorname{coker}\hat{e}_0$ is an isomorphism, the cokernels being taken in $\mathbb{N}$. However, the cokernel in $\mathbb{N}$ coincides with the cokernel in $\mathbb{G}$. Moreover, it follows from Proposition 1.5 that the induced map is rationalization. Thus $\operatorname{coker}\hat{e}$ is rational.

To prove the second assertion of the theorem we invoke Proposition 3.2. For if G is abelian then the assertion is an immediate consequence of the fact that $\operatorname{coker}\hat{e}$ is rational. Now suppose given a central extension $N \rightarrowtail G \overset{\varepsilon}{\twoheadrightarrow} Q$ (where we think of N as a central subgroup of G) and assume our assertion true for N and Q. We write $\hat{\varepsilon}: \hat{G} \twoheadrightarrow \hat{Q}$, etc., for the surjections induced by ε. Let $x \in \hat{G}$ with $x^n = \hat{e}y, y \in G$. Then $(\hat{\varepsilon}x)^n = \hat{e}\varepsilon y$, so that $\hat{\varepsilon}x = \hat{e}z, z \in Q$. Let $z = \varepsilon g, g \in G$, so $\hat{\varepsilon}x = \hat{\varepsilon}\hat{e}g$. Thus $x = (\hat{e}g)\,h, h \in \hat{N}$. Then $\hat{e}y = x^n = (\hat{e}g^n)\,h^n$, so that $h^n \in \hat{e}G$, say

$$h^n = \hat{e}u, \qquad u \in G.$$

Now the element $u \in G$ is such that $e_p u \in N_p$ for all primes p. Thus by Proposition 1.2 it follows that $u \in N$, and our assertion for N implies that $h = \hat{e}v, v \in N$. Thus finally $x = \hat{e}(gv)$ and our assertion is proved for G.

THEOREM 4.3. *Let* $\operatorname{nil} G \leqslant c$ *and let*

$$\begin{array}{ccc} G & \overset{\hat{e}}{\rightarrowtail} & \hat{G} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \varrho} \\ G_0 & \overset{\sigma}{\to} & P \end{array}$$

be a pushout in $\mathbb{G}$. *Then* σ *is injective and* $\Gamma^{c+1}P = \Gamma^{c+2}P$. *Moreover,* P/Γ^{c+1} *is a rational group.*

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