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Autor: Frank, David
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Irregular Primes and Integrality Theorems for Manifolds

DAVID FRANK¹⁾

1. Introduction

Number theory has long played an important part in topology. We examine here a topological problem whose solution depends on the distinction between regular and irregular primes. (Recall that a prime p is said to be irregular if p is odd and p divides the ideal class number of the p -th cyclotomic field; the first irregular prime is 37.)

Let M be a closed oriented topological manifold of dimension $4k$, smooth on the complement of a point. A necessary condition for M itself to be smoothable is that the Pontrjagin number $p_k[M]$ be an integer. If, moreover, M is a spin manifold ($w_2M=0$), the integrality theorem of Atiyah-Borel-Hirzebruch-Singer [2], [3], [4] says an additional necessary condition is that $(1/a_k)\hat{A}[M]$ be an integer, where \hat{A} is the \hat{A} -genus and $a_k=1$ (k even) or 2 (k odd).

We wish to know to what extent these necessary conditions for the smoothability of M are sufficient. If $k > 1$, M is triangulable [11]; then the obstruction to extending the smoothing of $(M\text{-point})$ to M is an element Σ_M of the group of exotic spheres Γ^{4k-1} . There is a splitting [6], [9] $\Gamma^{4k-1} = bP(4k) \oplus \pi'_{4k-1}$, where $bP(4k)$ is the subgroup of exotic spheres bounding parallelizable manifolds and π'_{4k-1} is a certain complementary summand. The integrality of $p_k[M]$ and $\hat{A}[M]$ can provide no information on the component of Σ_M in π'_{4k-1} (Lemma 3.2 below). Hence we let Σ_M^b denote the component of Σ_M in $bP(4k)$, and pose the

Question. Let M be a closed oriented spin manifold of dimension $4k$, $k > 1$, smooth on the complement of a point. Suppose M satisfies the integrality condition: $p_k[M]$ and $(1/a_k)\hat{A}[M]$ are integers. Is $\Sigma_M^b = 0$ in $bP(4k)$?

THEOREM 1.1. *Suppose $\dim M \leq 200$, $\dim M \neq 136$. If M satisfies the integrality condition, then $\Sigma_M^b = 0$.*

THEOREM 1.2. *There is a manifold N of dimension 136 satisfying the integrality condition with Σ_N of order 37 in $bP(136)$.*

Since 37 is the first irregular prime, this suggests

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THEOREM 1.3. *Let p be a prime. The following are equivalent:*

- i) p is irregular
- ii) for some $k > 1$, there is a manifold M of dimension $4k$ satisfying the integrality condition with order Σ_M^b divisible by p .

In each dimension $4k$, it is possible to determine which primes p (if any) can divide the order of a Σ_M^b (for some M satisfying the integrality condition). Table 1.4 lists all such values of k and p which can occur in the range $k < 109$. If a value of k does not appear in the table, then for a manifold M of dimension $4k$, the integrality condition implies $\Sigma_M^b = 0$.

Table 1.4

$k = \frac{1}{4} \dim M$	34	51	52	62	63	70	76	80	84	88	95	106
$p = \text{order } \Sigma_M^b$	37	59	37	67	103	37	131	59	101	37	67	37

A reference for the number theory we use is [5]. However, we follow the notation of [10], which is standard in topology and differs from the notation of [5].

2. Proof of the Main Theorems

Let B_i be the i -th Bernoulli number [10]. For convenience, define B_0 to be 1. Kummer gave a numerical criterion for regularity.

ASSERTION 2.1. *A prime p is irregular if and only if p divides the numerator of B_n , for some $n < \frac{1}{2}(p - 1)$.*

We give a table of all irregular primes less than 233. For each p , we give those $n < \frac{1}{2}(p - 1)$ such that p divides $\text{num } B_n$. This information is in fact available for all $p \leq 4001$.

Table 2.2

p	37	59	67	101	103	131	149	157
n	16	22	29	34	12	11	65	31, 55

If k is an integer and p is a prime, let $c (= c(k, p))$ denote the residue class of k modulo $\frac{1}{2}(p - 1)$. Thus $0 \leq c < \frac{1}{2}(p - 1)$. If $p = 2$, set $c(k, 2) = 0$.

THEOREM 2.3. *Let $k > 1$ be an integer and p a prime. The following are equivalent:*

- i) $p \leq 2k - 1$ and p divides $\text{num } B_c$, where c is the residue class of k modulo $\frac{1}{2}(p - 1)$,
- ii) there is a manifold M of dimension $4k$ satisfying the integrality condition with order Σ_M^b divisible by p .

We show that Theorem 2.3, whose proof we defer to the next section, implies the results of the introduction. Note first that (2.3) implies Theorem 1.3. For if p is irregular, then p divides $\text{num} B_c$ for some $c < \frac{1}{2}(p-1)$. Let $k = a(\frac{1}{2})(p-1) + c$, for any integer $a \geq 1$. Then $p \leq 2k-1$ and c is the residue class of $k \pmod{\frac{1}{2}(p-1)}$, so by (2.3) there is a manifold M with the desired properties. Conversely, if M exists, then p divides $\text{num} B_c$, so p is irregular.

We now prove Theorems 1.1 and 1.2. Let $\dim M = 4k$, $k \leq 50$. Suppose M satisfies the integrality condition and p divides the order of Σ_M^b . Then $p \leq 99$ and p is irregular, so $p = 37, 59$, or 67 . Suppose $p = 59$. Then $k \geq 30$. Let c be the residue class of $k \pmod{29}$. Then 59 divides $\text{num} B_c$, and from Table 2.2 we find that $c = 22$. Thus $k \equiv 22 \pmod{29}$ and $30 \leq k \leq 50$. Since no such k exists, we conclude that $p \neq 59$. Similarly, we may exclude $p = 67$.

If $p = 37$, then $k \geq 19$ and, from Table 2.2, $k \equiv 16 \pmod{18}$. Thus $k = 34$ is the only possibility, proving Theorems 1.1 and 1.2.

We leave the further verification of (1.4) to the reader.

3. Proof of Theorem 2.3

We will need

THEOREM (von Staudt). *Let p be a prime. Then p divides $\text{denom} B_k$ if and only if $p-1$ divides $2k$.*

THEOREM (Kummer's Congruence). *Suppose $p-1$ does not divide $2k$, p a prime. Let $r = \frac{1}{2}(p-1)$. Then B_k/k is a p -integer (p does not divide its denominator), and in the ring of p -integers*

$$B_k/k \equiv \pm B_{k+r}/(k+r) \pmod{p}.$$

LEMMA 3.1. *The prime p divides $\text{num}(B_k/k)$ if and only if p divides $\text{num} B_c$, where c is the residue class of $k \pmod{\frac{1}{2}(p-1)}$.*

Proof. Suppose p divides $\text{num}(B_k/k)$. Then by von Staudt's Theorem, $p-1$ does not divide $2k$. (In particular, p is odd.) Write $k = a(\frac{1}{2})(p-1) + c$, where $0 \leq c < \frac{1}{2}(p-1)$. Then Kummer's Congruence shows p divides $\text{num}(B_c/c)$. Hence p divides $\text{num} B_c$.

Conversely, suppose p divides $\text{num} B_c$, where $0 \leq c < \frac{1}{2}(p-1)$. Since $2c < p-1$, we see that p divides $\text{num}(B_c/c)$ and $p-1$ does not divide $2c$. Hence by Kummer, p divides $\text{num}(B_k/k)$.

LEMMA 3.2. *Let M and N be manifolds of dimension $4k$, $k > 1$. If $\Sigma_M^b = \Sigma_N^b$, then $p_k[M] = p_k[N] \pmod{1}$ and $\hat{A}[M] = \hat{A}[N] \pmod{a_k}$.*

Proof. Let $\Sigma = \Sigma_N - \Sigma_M$. Then the $bP(4k)$ -component of Σ is zero. It follows from [6] or [9] that there is a spin manifold V with $\Sigma_V = \Sigma$ such that all Pontrjagin numbers of V are zero. Let X be the connected sum of M , $-N$, and V . Then $\Sigma_X = \Sigma_M - \Sigma_N + \Sigma_V = 0$, so X is smoothable. Thus $p_k[M] - p_k[N] = p_k[M] - p_k[N] + p_k[V] = p_k[X]$ is an integer.

A similar argument applies to the \hat{A} -genus.

Proof of Theorem 2.3. Let W be the closed Milnor manifold of dimension $4k$ and signature 8. Thus W -(point) is smooth and parallelizable. Then $\Sigma_W \in bP(4k)$. In fact, Σ_W generates $bP(4k)$, which is a cyclic group of order $N_k = a_k 2^{2k-2} (2^{2k-1} - 1) \text{ num}(B_k/k)$.

Now let qW be the connected sum of q copies of W . Then qW is smoothable on the complement of a point and $\Sigma_{qW} = q\Sigma_W$. Note that $\Sigma_{qW} = 0$ if and only if N_k divides q . Suppose M is a manifold of dimension $4k$ which satisfies the integrality condition but $\Sigma_M^b \neq 0$. Then $\Sigma_M^b = \Sigma_{qW}$ for some q . Therefore by Lemma 3.2, the manifold qW satisfies the integrality condition. Using the Hirzebruch signature theorem for the L -genus and the definition of the \hat{A} -genus [10], we have

$$8 = \text{signature } W = L_k[W] = \frac{2^{2k} (2^{2k-1} - 1) B_k}{(2k)!} p_k[W]$$

$$\hat{A}[W] = \frac{-B_k}{2(2k)!} p_k[W].$$

We easily compute

$$p_k[qW] = q \frac{(2k)!}{2^{2k-3} (2^{2k-1} - 1) B_k}$$

$$\hat{A}[qW] = \frac{-q}{2^{2k-2} (2^{2k-1} - 1)}.$$

Now suppose the order of Σ_{qW} is ps , where p is prime and s is an integer. Then we can write

$$q = \frac{rN_k}{ps}, \quad r \text{ prime to } p.$$

Thus

$$p_k[qW] = \frac{4a_k r (2k-1)! \text{ denom}(B_k/k)}{ps}$$

$$\hat{A}[qW] = \frac{(-a_k r) \text{ num}(B_k/k)}{ps}.$$

Then the integrality condition for qW says that p divides $\text{num}(B_k/k)$ and $p \leq 2k - 1$. But if p divides $\text{num}(B_k/k)$, then by Lemma 3.1, p divides $\text{num} B_c$, which proves half of Theorem 2.3.

Conversely, if p divides $\text{num} B_c$, then p divides $\text{num}(B_k/k)$. Let $q = N_k/p$. Thus $\text{order } \Sigma_{qW} = p$. If $p \leq 2k - 1$, then qW satisfies the integrality condition, concluding the proof of Theorem 2.3.

Remarks. 1. The manifold $N = qW$ ($q = N_k/p$, $p \leq 2k - 1$) in fact satisfies the stronger integrality theorem of [2], [3], [4]: $(1/a_k) \langle \hat{A}(N) ph(\gamma), [N] \rangle$ is an integer for every real vector bundle γ on N . Here ph is the Pontrjagin character. Nevertheless qW is not oriented cobordant to a smooth manifold. This does not contradict [12].

2. Note that $(1/a_k) \hat{A}[M] \bmod 1$ is the Eells-Kuiper invariant μ of Σ_M [7]. Similarly $p_k[M] \bmod 1$ is the Milnor invariant $\lambda(\Sigma_M)$ [13]. Thus our results say, for example, that if Σ is an exotic sphere with $\mu(\Sigma) = \lambda(\Sigma) = 0$, then Σ^b has order a product of irregular primes and $\dim \Sigma \geq 135$, or else $\Sigma^b = 0$.

3. There is a manifold N of dimension 436 satisfying the integrality condition with $\text{order } \Sigma_N^b$ equal to the product of distinct primes (59) (157). This is the smallest dimension of such an example.

4. Information on the component of Σ_M in π'_{4k-1} can be obtained using the invariant of [8].

5. For the 136-dimensional manifold of Theorem 1.2 we may take the connected sum of $2^{66}(2^{67} - 1) (1/37) \text{num}(B_{34}/34)$ copies of W . Knowing B_{34} (Adams [1]) we find $(1/37) \text{num}(B_{34}/34) =$

125, 235, 502, 160, 125, 163, 977, 598, 011, 460, 214, 000, 388, 469.

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*State University of New York
at Stony Brook*

Current address:

University of Minnesota

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