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Poincaré Duality and Groups of Type (FP)

F. THOMAS FARRELL

0. Introduction

This paper continues our study of the groups $H^n(\Gamma, k\Gamma)$ begun in [3]. (Here Γ is a group and k is an arbitrary field.) There we generally restricted ourselves to the case n=2; here we allow n to be arbitrary, but usually require Γ to satisfy rather strong finiteness conditions.

In particular our main result (Theorem 1) applies only to groups of type (FP) over k. (See section 1 for the definition of this term.) It states that if the first non-vanishing $H^n(\Gamma, k\Gamma)$ contains a non-zero finite-dimensional (over k) sub- $k\Gamma$ -module, then $H^n(\Gamma, k\Gamma)$ has dimension 1 and the remaining $H^i(\Gamma, k\Gamma)$ vanish.

As a consequence we obtain the following extension of some results from [3].

THEOREM 2. If Γ is a finitely presented, torsion-free group, then any sub- $k\Gamma$ -module of $H^2(\Gamma, k\Gamma)$ has dimension $0, 1, or \infty$.

Our second application shows that Γ satisfies Poincaré duality under weaker assumptions than were previously known. Namely Theorem 3 states the following. If Γ is a finitely presented group of type (FP) and the first non-vanishing $H^n(\Gamma, \mathbb{Z}\Gamma)$ is finitely generated (as an abelian group), then Γ is a Poincaré duality group.

This paper is an extension of some observations of A. Borel and J-P. Serre. They had obtained, previous to my work, the following facts about groups Γ of type (FP) such that $H^i(\Gamma, k\Gamma) = 0$ for all $i \neq n$:

- (a) dim $H^n(\Gamma, k\Gamma) = 0$, 1, or ∞ ;
- (b) if $H^n(\Gamma, k\Gamma)$ has a proper $k\Gamma$ -subspace of finite codimension, then $H^n(\Gamma, k\Gamma)$ has no non-zero finite-dimensional $k\Gamma$ -subspace.

They had also obtained results in the case where k is replaced by \mathbb{Z} .

I wish to thank Professor Serre for communicating their results to me and for encouraging me in my own work.

1. Preliminaries

Notation. Throughout this paper k denotes an arbitrary field and Γ a group. Let V and W be two k-vector spaces, then the collection of linear transformations from V to

W is denoted by $\operatorname{Hom}(V, W)$, and $V \otimes W$ expresses the tensor product of V with W over k. If V and W are $k\Gamma$ -modules, then $\operatorname{Hom}(V, W)$ and $V \otimes W$ are also $k\Gamma$ -modules where the Γ -structures are defined by the equations

$$(\gamma \cdot f)(x) = \gamma f(\gamma^{-1}x)$$
, and $\gamma \cdot (x \otimes y) = \gamma x \otimes \gamma y$

for all $\gamma \in \Gamma$, $f \in \text{Hom}(V, W)$, $x \in V$ and $y \in W$. If V is a $k\Gamma$ -module (or k-vector space), then the dimension of V, abbreviated dim V, refers to the dimension of the underlying k-vector space.

LEMMA 1. If V and W are two $k\Gamma$ -modules with W free and $0 < \dim V < \infty$, then $\operatorname{Hom}(V, W)$ is free. In fact, $\operatorname{Hom}(V, W)$ is $k\Gamma$ -isomorphic to the direct sum of s-copies of W where $s = \dim V$.

Proof. Our argument is modeled after that of Proposition 1 on page 149 of [8]. Since W is free, it contains a k-subspace X such that W can be expressed as the following direct sum.

$$W = \sum_{\gamma \in \Gamma} \gamma \cdot X.$$

Because dim V is finite, $\operatorname{Hom}(V, W)$ is the direct sum of the k-subspaces $\operatorname{Hom}(V, \gamma \cdot X)$; but $\operatorname{Hom}(V, \gamma \cdot X) = \operatorname{Hom}(\gamma^{-1} \cdot V, \gamma \cdot X) = \gamma \cdot \operatorname{Hom}(V, X)$. Hence if Y denotes $\operatorname{Hom}(V, X)$ given the trivial Γ -structure, then $\operatorname{Hom}(V, W)$ is $k\Gamma$ -isomorphic to $k\Gamma \otimes Y$. If we also give X the trivial Γ -structure, then Y is isomorphic to S-copies of X. Therefore $\operatorname{Hom}(V, W)$ is $K\Gamma$ -isomorphic to S-copies of $K\Gamma \otimes X$. But this completes our proof since W is $K\Gamma$ -isomorphic to $K\Gamma \otimes X$.

LEMMA 2. If V and W are two $k\Gamma$ -modules, then

$$\operatorname{Ext}_{k\Gamma}^{n}(V, W) \cong H^{n}(\Gamma, \operatorname{Hom}(V, W))$$

for all $n \ge 0$.

Proof. This lemma is well-known. (Compare [7], page 272, exercises 4–6.) Hence we only sketch its proof.

Denote the functors $A \mapsto H^n(\Gamma, \text{Hom}(A, W))$ by $E^n(A)$. (Here A is a $k\Gamma$ -module and $n \ge 0$.) Then the E^n satisfy the axiomatic description ([7], Theorem 10.1) of the functors $A \mapsto \text{Ext}_{k\Gamma}^n(A, W)$.

The only axiom which is difficult to verify is that

$$E^{n}(F)=0$$
 for $n>0$ and all free modules F .

To do this one proves first, by an argument similar to that in the proof of Lemma 1, that Hom(F, W) is co-induced over k: that is, $k\Gamma$ -isomorphic to $Hom(k\Gamma, X)$ for

some k-vector space X with trivial Γ -structure. Then one shows that $H^n(\Gamma, A) = 0$ when A is co-induced over k and n > 0. (Compare [8], Proposition 1, page 120.)

We next recall some well-known facts about dual modules. The dual of a $k\Gamma$ -module M is the $k\Gamma$ -module $M^* = \operatorname{Hom}_{k\Gamma}(M, k\Gamma)$. If P is a finitely generated, projective, right $k\Gamma$ -module and A is a left $k\Gamma$ -module, then P^* is finitely generated and projective, and

$$P \otimes_{k\Gamma} A$$
 and $\operatorname{Hom}_{k\Gamma}(P^*, A)$

are naturally isomorphic.

Given a chain complex of $k\Gamma$ -modules of finite length $K: K_n \to K_{n-1} \to \cdots \to K_0$, where each K_i is finitely generated and projective, we can form its dual cochain complex $K^*: K_0^* \to K_1^* \to \cdots \to K_n^*$. Given, in addition, a $k\Gamma$ -module A, we can form chain complexes

$$K \otimes_{k\Gamma} A : K_n \otimes_{k\Gamma} A \to K_{n-1} \otimes_{k\Gamma} A \to \cdots \to K_0 \otimes_{k\Gamma} A$$
,

and

$$\operatorname{Hom}_{k\Gamma}(K^*, A) : \operatorname{Hom}_{k\Gamma}(K_n^*, A) \to \operatorname{Hom}_{k\Gamma}(K_{n-1}^*, A) \to \cdots \to \operatorname{Hom}_{k\Gamma}(K_0^*, A).$$

By the above remarks, $K \otimes_{k\Gamma} A$ and $\operatorname{Hom}_{k\Gamma}(K^*, A)$ are isomorphic chain complexes. Denote the *i*-th homology group of $K \otimes_{k\Gamma} A$ by C_i and the *i*-th cohomology group of K^* by C^i .

PROPOSITION 1. Under the above assumptions, there exists a spectral sequence with

$$E_2^{pq} \cong H^p(\Gamma, \operatorname{Hom}(C^{n-q}, A))$$

and converging to C_{n-p+q} .

Proof. Proposition 1 is a special case of the spectral universal coefficient theorem. (See [4], page 100, Theorem 5.4.1.) In order to fit with Godement's notation, let

$$L_i = K_{n-i}^*$$
, $M^0 = A$, and $M^i = 0$ for all $i \neq 0$.

Then Theorem 5.4.1 of [4] posits the existence of a spectral sequence with $E_2^{pq} = \operatorname{Ext}_{k\Gamma}^p$ (C^{n-q} , A) and converging to $H^{p+q}(\operatorname{Hom}_{k\Gamma}(L,A))$. But Lemma 2 states that $\operatorname{Ext}_{k\Gamma}^p$ (C^{n-q} , A) $\cong H^p(\Gamma, \operatorname{Hom}(C^{n-q}, A))$. On the other hand $H^{p+q}(\operatorname{Hom}_{k\Gamma}(L,A))$ and $H_{n-p+q}(\operatorname{Hom}_{k\Gamma}(K^*,A))$ are identical, and by the remarks preceding the statement of Proposition 1, $H_{n-p+q}(\operatorname{Hom}_{k\Gamma}(K^*,A))$ and C_{n-p+q} are isomorphic. Concatenating this information completes the proof of Proposition 1.

We say that Γ is a group of type (n-FP) over k if k with the trivial Γ -structure has a resolution of finite length $0 \to P_s \to P_{s-1} \to \cdots \to P_0 \to k \to 0$ by projective $k\Gamma$ -modules such that P_i is finitely generated for all $i \le n$. When $n = \infty$ we say more simply that Γ

is a group of type (FP) over k. Moreover, if $n = \infty$ and k is replaced by Z in the above definition, then we say that Γ is a group of type (FP).

COROLLARY 1. If Γ is a group of type (FP) over k and A is a $k\Gamma$ -module, then there exists a spectral sequence (whose differentials d, have bidegree (r, r-1)) with

$$\mathscr{E}_2^{pq} \cong H^p(\Gamma, \operatorname{Hom}(H^q(\Gamma, k\Gamma), A))$$

and converging to $H_{q-p}(\Gamma, A)$.

Proof. Consider a resolution of k $0 \to K_n \to K_{n-1} \to \cdots \to K_0 \to k \to 0$ by finitely generated, projective modules K_i , and let K denote the chain complex $K_n \to K_{n-1} \to \cdots \to K_0$. Applying Proposition 1 to the complex K and the $k\Gamma$ -module A, we obtain a spectral sequence with $E_2^{pq} \cong H^p(\Gamma, \text{Hom}(H^{n-p}(\Gamma, k\Gamma), A))$ and converging to $H_{n-p+q}(\Gamma, A)$. Then let \mathscr{E}_s^{pq} be $E_s^{p,n-q}$ and we are done.

The next corollary partially recovers the "inverse duality" discovered by Bieri. (See [1], Remark following Proposition 5.3.)

COROLLARY 2. Let Γ be a group of type (FP) over k such that $H^i(\Gamma, k\Gamma) = 0$ for all $i \neq n$. If C denotes $H^n(\Gamma, k\Gamma)$, then

$$H_s(\Gamma, A) \cong H^{n-s}(\Gamma, \operatorname{Hom}(C, A))$$

for every integer s and every $k\Gamma$ -module A.

Proof. Under the above assumptions, the spectral sequence of Corollary 1 collapses and yields that $H_{n-p}(\Gamma, A)$ is isomorphic to $H^p(\Gamma, \text{Hom}(C, A))$. The result now follows by substituting n-s for p in this isomorphism.

Remark. Prior to my work, Borel and Serre had observed (private communication) that Bieri-Eckmann duality [2] could be recovered from a spectral sequence (constructed under the same hypotheses as Corollary 1) with $E_{pq}^2 \cong H_p(\Gamma, H^q(\Gamma, k\Gamma) \otimes A)$ and converging to $H^{q-p}(\Gamma, A)$. This spectral sequence is obtainable in a manner analogous to the one from Proposition 1 by making use of the spectral Künneth formula ([4], page 102, Theorem 5.5.1) together with the natural isomorphism between $P^* \otimes_{k\Gamma} A$ and $\operatorname{Hom}_{k\Gamma}(P, A)$ valid for any pair of left $k\Gamma$ -modules, provided that P is finitely generated and projective.

2. The Main Theorem

We now come to the main result of this paper.

THEOREM 1. Suppose that $H^i(\Gamma, k\Gamma) = 0$ for all i < n and that $H^n(\Gamma, k\Gamma)$ contains a non-zero finite-dimensional sub- $k\Gamma$ -module. If Γ is of type (n-FP) over k, then we conclude the following:

- (a) Γ is of type (FP) over k;
- (b) $H^{i}(\Gamma, k\Gamma) = 0$ for all $i \neq n$;
- (c) dim $H^n(\Gamma, k\Gamma) = 1$.

Proof. For n=0 this result is well-known. Hence we may assume that n>0. Consider a projective resolution of k with minimal length m

$$0 \to K_m \stackrel{d_m}{\to} K_{m-1} \to \cdots \to K_0 \to k \to 0,$$

where K_i is finitely generated for all $i \le n$. Clearly $m \ge n$, and we intend to show that m=n. Let K be the chain complex $K_n \to K_{n-1} \to \cdots \to K_0$, and A be a $k\Gamma$ -module. Applying Proposition 1 to this pair and noting that the resulting spectral sequence collapses, we obtain an isomorphism between $H^p(\Gamma, \operatorname{Hom}(C^n, A))$ and C_{n-p} for all p. Recall that C^i is the i-th cohomology group of K^* and that C_i is the i-th homology group of $K \otimes_{k\Gamma} A$. In particular, C_i and $H_i(\Gamma, A)$ are isomorphic for all i < n; consequently,

- (i) $H^m(\Gamma, \text{Hom}(\mathbb{C}^n, A)) = 0$ if m > n, and
- (ii) $H^n(\Gamma, \operatorname{Hom}(C^n, A)) \cong H_0(\Gamma, A)$.

By the hypotheses of Theorem 1, $H^n(\Gamma, k\Gamma)$ contains a sub- $k\Gamma$ -module V such that $0 < \dim V < \infty$. Since $H^n(\Gamma, k\Gamma)$ is a sub- $k\Gamma$ -module of C^n , we see that V is also a sub- $k\Gamma$ -module of C^n . Applying the functor $Hom(\cdot, A)$ to the short exact sequence $0 \to V \to C^n \to C^n/V \to 0$, we obtain a new short exact sequence of $k\Gamma$ -modules

$$0 \to \operatorname{Hom}(C^n/V, A) \to \operatorname{Hom}(C^n, A) \to \operatorname{Hom}(V, A) \to 0$$
.

Now, applying the functor $H^*(\Gamma, \cdot)$ to this sequence, we obtain the exact sequence

$$H^m(\Gamma, \operatorname{Hom}(C^n, A)) \to H^m(\Gamma, \operatorname{Hom}(V, A)) \to H^{m+1}(\Gamma, \operatorname{Hom}(C^n/V, A)).$$

Since k has a projective resolution of length m, $H^{m+1}(\Gamma, \text{Hom}(\mathbb{C}^n/V, A))$ must vanish, and hence the above sequence degenerates into the following epimorphism:

(iii)
$$H^m(\Gamma, \operatorname{Hom}(C^n, A)) \to H^m(\Gamma, \operatorname{Hom}(V, A)) \to 0$$
.

Suppose that m > n. (We intend to show that this assumption leads to a contradiction.) Then, by (i) and (iii), $H^m(\Gamma, \text{Hom}(V, A)) = 0$ for every $k\Gamma$ -module A. This fact, in conjunction with Lemma 1, yields that $H^m(\Gamma, W) = 0$ for every free (hence, also every projective) module W. In particular $H^m(\Gamma, K_m)$ vanishes, which implies that $d_m: K_m \to K_{m-1}$ is a split- $k\Gamma$ -monomorphism. Therefore $K_{m-1}/d_m K_m$ is projective (and finitely generated if m-1=n), and

$$0 \to K_{m-1}/d_m K_m \to K_{m-2} \to \cdots \to K_0 \to k \to 0$$

is a projective resolution of k with length m-1 whose first n+1-terms (starting with K_0) are finitely generated. But this is a contradiction. Hence m=n, which proves assertions (a) and (b) of Theorem 1.

Since $H_0(\Gamma, k\Gamma) = k$ we obtain, using (ii) and (iii), the following inequality:

(iv) $\dim H^n(\Gamma, \operatorname{Hom}(V, k\Gamma)) \leq 1$. But Lemma 1 states that $\operatorname{Hom}(V, k\Gamma)$ is the direct sum of s-copies of $k\Gamma$ where $s = \dim V$. This fact, together with the inequality (iv), implies that $\dim H^n(\Gamma, k\Gamma) = 1$, which completes the proof of Theorem 1.

One says that Γ is a group of type (VFP) over k if Γ contains a subgroup of finite index of type (FP) over k.

ADDENDUM. If we replace in the hypotheses of Theorem 1 (n-FP) by (VFP), then conclusions (b) and (c) remain true.

Proof. This is a consequence of the following well-known fact: If Γ' is a subgroup of finite index in Γ , then $H^i(\Gamma, k\Gamma)$ and $H^i(\Gamma', k\Gamma')$ are isomorphic $k\Gamma'$ -modules for all integers i.

3. Applications

Our first application of Theorem 1 is to extend some results from [3].

THEOREM 2. If Γ is a finitely presented, torsion-free group, then any sub- $k\Gamma$ -module of $H^2(\Gamma, k\Gamma)$ has dimension $0, 1, \text{ or } \infty$.

The proof of Theorem 2 depends on the following elementary lemma.

LEMMA 3. Let l be a subfield of k, and A a $l\Gamma$ -module. If $A \otimes_l k$ contains a sub- $k\Gamma$ -module V such that

$$0 < \dim_k V < \infty$$
,

then A contains a sub- $l\Gamma$ -module W such that

$$\dim_k V \leq \dim_l W < \infty$$
.

Proof. Regarding k as a vector space over l, let $f:k \to l$ be a non-zero linear functional. Then define a $l\Gamma$ -homomorphism $g:A \otimes_l k \to A$ by composing $\mathrm{id} \otimes f:A \otimes_l k \to A \otimes_l l$ with the natural isomorphism from $A \otimes_l l$ to A. Let W = g(V), then one easily checks that W satisfies the conclusion of Lemma 3.

Proof of Theorem 2. Because of Theorem 5.1 of [3], it suffices to consider the case where k has characteristic 0. Since Γ is finitely presented, $H^2(\Gamma, k\Gamma)$ and $H^2(\Gamma, \mathbf{Q}\Gamma) \otimes_{\mathbf{Q}} k$ are isomorphic $k\Gamma$ -modules. (Here \mathbf{Q} denotes the rational numbers.) Let V be a sub- $k\Gamma$ -module of $H^2(\Gamma, k\Gamma)$ such that $0 < \dim_k V < \infty$. By Lemma 3, $H^2(\Gamma, \mathbf{Q}\Gamma)$ contains a sub- $\mathbf{Q}\Gamma$ -module W such that $\dim_k V \leq \dim_{\mathbf{Q}} W < \infty$; hence to prove Theorem 2, we need only show that $\dim_{\mathbf{Q}} W = 1$.

But because of Theorem 5.3 of [3], we may assume that Γ is a group of type (2-FP) over \mathbb{Q} . Since $H^0(\Gamma, \mathbb{Q}\Gamma)=0$, Theorem 1 implies Theorem 2 provided we can show that $H^1(\Gamma, \mathbb{Q}\Gamma)$ vanishes.

To do this we assume its opposite, i.e. $H^1(\Gamma, \mathbf{Q}\Gamma) \neq 0$, and show that this assumption leads to a contradiction. As a consequence of Lemma 3.5 of [10] and section 5.1 of [9], Γ has infinitely many ends. Hence by the Main Theorem of [9], Γ is a non-trivial free product of subgroups Γ_1 and Γ_2 ; both of which are finitely presented by a result of Stallings ([11], Lemma 1.3). By the "Mayer-Vietoris" sequence ([6] or [10], Theorem 2.3), $H^2(\Gamma, \mathbf{Q}\Gamma)$ is $\mathbf{Q}\Gamma$ -isomorphic to the direct sum of $H^2(\Gamma_1, \mathbf{Q}\Gamma)$ and $H^2(\Gamma_2, \mathbf{Q}\Gamma)$. Therefore one of these modules, say $H^2(\Gamma_1, \mathbf{Q}\Gamma)$ to be specific, contains a non-zero finite-dimensional sub- $\mathbf{Q}\Gamma$ -module. But this is impossible, since

$$H^2(\Gamma_1, \mathbf{Q}\Gamma) \cong H^2(\Gamma_1, \mathbf{Q}\Gamma_1) \otimes_{\mathbf{Q}\Gamma_1} \mathbf{Q}\Gamma$$

as $Q\Gamma$ -modules. This completes the proof of Theorem 2.

One says that Γ is virtually torsion-free if Γ contains a torsion-free subgroup of finite index. Then the following extension of Theorem 2 is easily proven.

ADDENDUM. If Γ is finitely presented and virtually torsion-free, then any sub- $k\Gamma$ -module of $H^2(\Gamma, k\Gamma)$ has dimension $0, 1, or \infty$.

Our second application is the following result.

THEOREM 3. Suppose that Γ is a finitely presented group of type (FP), and let n be the smallest integer such that $H^n(\Gamma, \mathbf{Z}\Gamma) \neq 0$. If $H^n(\Gamma, \mathbf{Z}\Gamma)$ is a finitely generated abelian group, then Γ is an n-dimensional Poincaré duality group.

Remark. Such an integer n exists, since for groups of type (FP) $H^i(\Gamma, \mathbf{Z}\Gamma)$ cannot vanish for all i.

Proof. Since Γ is a group of type (FP), it is also of type (FP) over k. Furthermore $H^i(\Gamma, k\Gamma)$ is k-isomorphic to the direct sum of $H^i(\Gamma, \mathbf{Z}\Gamma) \otimes k$ and $\operatorname{Tor}(H^{i+1}(\Gamma, \mathbf{Z}\Gamma), k)$, and $H^i(\Gamma, \mathbf{Z}\Gamma) \otimes k$ is embedded as a sub- $k\Gamma$ -module of $H^i(\Gamma, k\Gamma)$ via this isomorphism. (Here, and for the rest of this paper, \otimes and Tor are over \mathbf{Z} .)

Suppose $H^n(\Gamma, \mathbf{Z}\Gamma)$ has p-torsion for some prime p. Then by the above discussion, we have the following facts:

- (a) $H^{i}(\Gamma, \mathbb{Z}_{p}\Gamma) = 0$ for all i < n-1;
- (b) $H^n(\Gamma, \mathbf{Z}_p\Gamma) \neq 0$; and
- (c) $0 < \dim_{\mathbb{Z}_p} H^{n-1}(\Gamma, \mathbb{Z}_p \Gamma) < \infty$.

(Here \mathbb{Z}_p denotes the field with *p*-elements.) But these facts contradict Theorem 1. Thus $H^n(\Gamma, \mathbb{Z}\Gamma)$ is a free abelian group of rank s where $0 < s < \infty$.

Therefore $H^i(\Gamma, k\Gamma) = 0$ for all i < n; and $H^n(\Gamma, k\Gamma)$ contains a sub- $k\Gamma$ -module of

dimension s. Now by a second application of Theorem 1, we have $\dim H^n(\Gamma, k\Gamma) = 1$ and $H^i(\Gamma, k\Gamma) = 0$ for all $i \neq n$. Consequently we have s = 1 and both $H^i(\Gamma, \mathbf{Z}\Gamma) \otimes k$ and $\operatorname{Tor}(H^i(\Gamma, \mathbf{Z}\Gamma), k)$ vanish for all $i \neq n$. By setting k equal to \mathbf{Q} and \mathbf{Z}_p respectively, we see that $H^i(\Gamma, \mathbf{Z}\Gamma) = 0$ for all $i \neq n$. And since s = 1, $H^n(\Gamma, \mathbf{Z}\Gamma)$ is infinite cyclic. Hence Γ satisfies the conditions of [5] to be an n-dimensional Poincaré duality group.

ADDENDUM. The conclusion of Theorem 3 remains true when the hypothesis

" $H^n(\Gamma, \mathbf{Z}\Gamma)$ is a finitely generated abelian group"

is replaced by the following two assumptions:

- (a) $H^n(\Gamma, \mathbf{Z}\Gamma)$ contains a non-zero finitely generated (as an abelian group) sub- Γ -module, and
 - (b) $H^n(\Gamma, \mathbb{Z}\Gamma)$ is a free abelian group.

Proof. Let A be a non-zero sub- Γ -module of $H^n(\Gamma, \mathbf{Z}\Gamma)$ such that A is finitely generated as an abelian group. By assumption (b), $A \otimes \mathbf{Q}$ is a non-zero finite-dimensional sub- $\mathbf{Q}\Gamma$ -module of $H^n(\Gamma, \mathbf{Z}\Gamma) \otimes \mathbf{Q}$. But $H^i(\Gamma, \mathbf{Z}\Gamma) \otimes \mathbf{Q}$ and $H^i(\Gamma, \mathbf{Q}\Gamma)$ are isomorphic $\mathbf{Q}\Gamma$ -modules for all $i \ge 0$. Therefore Theorem 1 implies $\dim_{\mathbf{Q}} H^n(\Gamma, \mathbf{Z}\Gamma) \otimes \mathbf{Q} = 1$. This fact, together with (b), yields that $H^n(\Gamma, \mathbf{Z}\Gamma)$ is infinite cyclic. Now apply Theorem 3 to complete the proof.

4. Appendix

We mention a consequence of Theorem 2.

COROLLARY 3. If Γ is finitely presented and virtually torsion-free, then any sub- Γ -module of $H^2(\Gamma, \mathbb{Z}\Gamma)$ is either

- (a) zero,
- (b) an infinite cyclic abelian group, or
- (c) not finitely generated as an abelian group.

(This result extends Corollary 5.2 of [3].)

Proof. Corollary 3.7 of [10] implies that $H^2(\Gamma, \mathbb{Z}\Gamma)$ is a torsion-free abelian group. Thus it suffices to show that $\dim_{\mathbb{Q}} A \otimes \mathbb{Q} = 1$ when A is a non-zero finitely generated (as an abelian group) sub- Γ -module of $H^2(\Gamma, \mathbb{Z}\Gamma)$. But this follows from the addendum to Theorem 2 where we specify k to be \mathbb{Q} .

Note added in proof: 1) There are analogues to our results in the theory of homology manifolds, namely in the work of P. E. Conner and E. E. Floyd (Michigan Math. J. 6 (1959), 33-43).

2) K. Brown has recently found an elegant new proof for Theorem 1 which avoids the use of spectral sequences.

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