Fixed-Point Sets of Group Actions on Finite Acyclic Complexes.

Comment. Math. Helvetici 50 (1975) 155–177 Birkhäuser Verlag, Basel

Fixed-Point Sets of Group Actions on Finite Acyclic Complexes

ROBERT OLIVER[1])

This paper is an outgrowth of the author's thesis [9], in which he attempted to classify which compact Lie groups have smooth fixed-point free actions on disks (or equivalently, simplicial fixed-point free actions on finite contractible complexes). Here, the general problem is studied, by completely different methods, of classifying which finite complexes can be fixed-point sets of simplicial actions of a given group on finite contractible or $\mathbf{Z}_p$-acyclic complexes.

P. A. Smith has shown [12] that an action of a p-group on a $\mathbf{Z}_p$-acyclic complex must have a $\mathbf{Z}_p$-acyclic fixed point set. The converse was proven by Lowell Jones [8]: Any finite $\mathbf{Z}_p$-acyclic complex may be the fixed-point set of an action of $\mathbf{Z}_p$ (and thus of any p-group) on some finite contractible complex. Thus, in the case of p-group actions on contractible or $\mathbf{Z}_p$-acyclic complexes, the answer to these questions is already known.

In the other cases of actions of finite groups, it is shown here that the Euler characteristic is the only obstruction to a finite complex being a fixed-point set. More specifically:

For any prime p, and any finite group G not of p-power order, there is an integer $m_p(G)$ such that a finite complex K is the fixed-point set of an action of G on some finite $\mathbf{Z}_p$-acyclic complex if and only if $\chi(K)\equiv 1 \pmod{m_p(G)}$. For any group G not of prime power order, there is an integer n_G such that a finite complex K is the fixed-point set of an action of G on some finite contractible complex if and only if $\chi(K)\equiv 1 \pmod{n_G}$.

The following notation for classes of finite groups is used for the calculation of these constants. For p and q primes, let $\mathscr{G}_p^q$ be the class of finite groups G with normal subgroups $P\lhd H\lhd G$, such that P is of p-power order, G/H is of q-power order, and H/P is cyclic. Let $\mathscr{G}_p^1$ and $\mathscr{G}_1^q$ be the classes of such G where $H=G$ and $|P|=1$, respectively. Let $\mathscr{G}_p=\bigcup_q \mathscr{G}_p^q$, $\mathscr{G}^q=\bigcup_p \mathscr{G}_p^q$, $\mathscr{G}^1=\bigcup_p \mathscr{G}_p^1$, etc. The following will be proven:

The integer $m_p(G)$ is zero if and only if $G\in\mathscr{G}_p^1$. If $G\notin\mathscr{G}_p^1$, then $m_p(G)$ is a product of distinct primes (or 1), and $q \mid m_p(G)$ if and only if $G\in\mathscr{G}_p^q$.

The calculation of n_G is less complete, but the following is shown:

---

[1]) This work was done while the author was supported by an NSF graduate fellowship.

The integer n_G is zero if and only if $G \in \mathscr{G}^1$. If $G \notin \mathscr{G}^1$, then n_G is divisible by at most the square of any prime, and $q \mid n_G$ if and only if $G \in \mathscr{G}^q$.

The question of the existence of fixed-point free actions on contractible manifolds goes back to Smith. In the case of smooth actions on disks, the only previously known example of a fixed-point free action was the action of A_5 constructed by Floyd and Richardson [3]. Greever [4] described certain restrictions which must occur for a group to have a fixed-point free action on a contractible space where certain conditions hold, which include the case of smooth actions on disks. The results described above immediately yield:

A finite group G has a fixed-point free action on a finite $\mathbf{Z}_p$-acyclic complex if and only if $G \notin \mathscr{G}_p$. The group G has a fixed-point free action on a finite contractible complex (and then on a disk, via regular neighborhoods) if and only if $G \notin \mathscr{G} = \bigcup_p \mathscr{G}_p$. In particular, a finite abelian group has a smooth fixed-point free action on a disk if and only if it has three or more noncyclic Sylow subgroups.

The problem of constructing fixed-point free actions of connected positive-dimensional groups is also briefly discussed, and an example of SO(3) acting on a disk is constructed.

The above results have been described as applying to the category of simplicial actions on finite simplicial complexes. In obtaining the results of Section 2–4, it will be more convenient to work in a slightly broader category; that of cellular actions of finite groups on finite CW-complexes. In this category, the action of any group element is required to take the interior of any n-cell homeomorphically to the interior of some other n-cell, and via the identity whenever a cell is mapped to itself. Note that any simplicial action on a finite complex becomes a cellular action upon taking the first barycentric subdivision. Conversely any CW-complex with a cellular action can be made into a simplicial complex with simplicial action (of the same equivariant homotopy type) by subdividing cells and taking simplicial approximations to the attaching maps; if the fixed point set is already a simplicial complex then this can be done without changing it (except for subdividing). Thus, in terms of the question of which simplicial complexes can be fixed-point sets, these two categories are equivalent. Furthermore, the category of smooth actions of finite groups on compact manifolds is included in both of these categories (see, e.g., Illman [7]).

I would like to thank Wu-Chung Hsiang and William Browder for their many helpful suggestions while I was working on my thesis and on this further work.

1. Restrictions on Fixed-Point Sets

In this section, the methods used by Greever in [4] will be extended to prove some Euler characteristic conditions on the fixed-point sets of certain group actions. The results obtained will be basically the same as those proven in Chapter III of the author's thesis [9], but the proofs are simplified by restricting to the category of cellular actions on finite CW-complexes, as described above.

Two of the standard theorems from the homological theory of $\mathbf{Z}_p$-actions (p prime) will be used (see, e.g., Bredon [1], Chapter III, §7). If $\mathbf{Z}_p$ acts on the finite complex X, one has the relation

$$\chi(X)+(p-1)\cdot\chi(X^{\mathbf{Z}_p})=p\cdot\chi(X/\mathbf{Z}_p),$$

and from that, $\chi(X^{\mathbf{Z}_p})\equiv\chi(X)\pmod{p}$. If X is $\mathbf{Z}_p$-acyclic, then so is $X^{\mathbf{Z}_p}$.

If P is any group of p-power order, then P has a normal series $O=P_0\lhd P_1\lhd\cdots\lhd R_k=P$, such that $P_i/P_{i-1}\cong\mathbf{Z}_p$. In general, for an action of G on X and some $H\lhd G$, there is an induced action of G/H on X^H, with $(X^H)^{G/H}=X^G$. Thus, the above theorems for $\mathbf{Z}_p$-actions carry over to any p-group P: if P acts on the finite complex X, then $\chi(X^P)\equiv\chi(X)\pmod{p}$, and if X is $\mathbf{Z}_p$-acyclic, then so is X^P.

The following lemma is a special case of a more general proposition proven in [9]: if a cyclic group acts on a compact or finite dimensional space, such that the fixed-point set of any subgroup has finitely generated Čech cohomology, then the Euler characteristic of the fixed-point set of the whole group is equal to the Lefshetz number of the action of a generator.

LEMMA 1. *Assume $\mathbf{Z}_n$ acts on the finite $\mathbf{Q}$-acyclic complex X. Then* $\chi(X^{\mathbf{Z}_n})=1$.

Proof. First assume that $\mathbf{Z}_{p^k}$ acts on any finite complex X. In this case, $\mathbf{Z}_p$ acts on $X/\mathbf{Z}_{p^{k-1}}$ with fixed-point set $(X/\mathbf{Z}_{p^{k-1}})^{\mathbf{Z}_p}=X^{\mathbf{Z}_{p^k}}$, and application of the Euler characteristic formula gives the relation

$$\chi(X^{\mathbf{Z}_{p^k}})=\frac{1}{p-1}\left[p\cdot\chi(X/\mathbf{Z}_{p^k})-\chi(X/\mathbf{Z}_{p^{k-1}})\right].$$

Now assume that $n=m\cdot p^k$, where $p\nmid m$, and that the lemma has been proven for m. Then, if $\mathbf{Z}_n$ acts on the $\mathbf{Q}$-acyclic space X, one has $X^{\mathbf{Z}_n}=(X^{\mathbf{Z}_m})^{\mathbf{Z}_{p^k}}$, and

$$\chi(X^{\mathbf{Z}_n})=\frac{1}{p-1}\left[p\cdot\chi(X^{\mathbf{Z}_m}/\mathbf{Z}_{p^k})-\chi(X^{\mathbf{Z}_m}/\mathbf{Z}_{p^{k-1}})\right].$$

Since $\mathbf{Z}_m$ and $\mathbf{Z}_{p^k}$ have relatively prime order, one has $X^{\mathbf{Z}_m}/\mathbf{Z}_{p^k}=(X/\mathbf{Z}_{p^k})^{\mathbf{Z}_m}$ and $X^{\mathbf{Z}_m}/\mathbf{Z}_{p^{k-1}}=(X/\mathbf{Z}_{p^{k-1}})^{\mathbf{Z}_m}$. The existence of the transfer map shows that the orbit space of any finite group action on a $\mathbf{Q}$-acyclic space is $\mathbf{Q}$-acyclic, and so by the induction hypothesis, $\chi((X/\mathbf{Z}_{p^k})^{\mathbf{Z}_m})=\chi((X/\mathbf{Z}_{p^{k-1}})^{\mathbf{Z}_m})=1$. It follows that $\chi(X^{\mathbf{Z}_n})=1$. □

Now the desired restrictions on fixed-point sets follow quickly:

PROPOSITION 1. *Assume G acts on the finite $\mathbf{Z}_p$-acyclic complex X. If $G\in\mathscr{G}_p^1$, then $\chi(X^G)=1$. If $G\in\mathscr{G}_p^q$, then $\chi(X^G)\equiv 1\pmod{q}$.*

Proof. If $G\in\mathscr{G}_p^1$, then G has a normal subgroup P of p-power order, such that G/P is cyclic. Then X^P is $\mathbf{Z}_p$-acyclic, thus $\mathbf{Q}$-acyclic, and $\chi(X^G)=\chi((X^P)^{G/P})=1$ by Lemma 1. If $G\in\mathscr{G}_p^q$, then G has a normal subgroup $H\in\mathscr{G}_p^1$ of q-power index. It follows that $\chi(X^H)=1$, and $\chi(X^G)=\chi((X^H)^{G/H})\equiv 1 \pmod q$. □

PROPOSITION 2. *Assume G acts on the finite contractible complex X. If* $G\in\mathscr{G}^1$, *then* $\chi(X^G)=1$; *if* $G\in\mathscr{G}^q$, *then* $\chi(X^G)\equiv 1 \pmod q$.

Proof. If $G\in\mathscr{G}^1$, then $G\in\mathscr{G}_p^1$ for some prime p, and Proposition 1 applies. If $G\in\mathscr{G}^q$, then $G\in\mathscr{G}_p^q$ for some p, and again Proposition 1 applies. □

In the case of actions of $\mathbf{Z}_p$-acyclic complexes, it turns out that Proposition 1 gives the only restrictions on the fixed-point set when G is not of p-power order.

2. Resolving Functions

In this section, a bookkeeping method will be introduced for studying actions of a group on acyclic spaces terms of the Euler characteristics of the fixed point sets of subgroups. The following lemma will motivate the definitions. For a finite group G, $\mathscr{S}(G)$ will denote the set of subgroups of G.

LEMMA 2. *Assume G acts on the finite complex X; then there is a unique function* $\varphi:\mathscr{S}(G)\to\mathbf{Z}$ *such that*

$$\chi(X^H)=1+\sum_{K\supset H}\varphi(K) \tag{1}$$

for all $H\subseteq G$. *Furthermore, if* $\mathscr{H}\subseteq\mathscr{S}(G)$ *is any non-empty subset with the maximality condition:* $H\in\mathscr{H}$ *and* $H\subseteq K\subseteq G$ *imply* $K\in\mathscr{H}$, *then*

$$\chi\Big(\bigcup_{H\in\mathscr{H}}X^H\Big)=1+\sum_{H\in\mathscr{H}}\varphi(H). \tag{2}$$

In particular, $\varphi(H)=\chi(X^H,\bigcup_{K\supsetneq H}X^K)$ *for* $H\subsetneq G$.

Moreover, φ *is constant on conjugacy classes of subgroups, and* $[N(H):H]\mid\varphi(H)$ *for any* $H\subseteq G$.

Proof. Choose an ordering for $\mathscr{S}(G)=\{H_0, H_1,\dots, H_s\}$ such that $H_i\subseteq H_j$ implies $i\geqslant j$. Then

$$\varphi(H)=\chi(X^{H_i})-\sum_{H_j\supsetneq H_i}\varphi(H_j)-1$$

for all H_i is a necessary and sufficient condition for (1) to hold, and can be applied successively for $i=0, 1,\dots, s$ to define φ uniquely.

Now let $\mathscr{H}$ be any non-empty set of subgroups with the maximality condition, and assume (2) holds for all appropriate proper subsets of $\mathscr{H}$. Let $H_0 \in \mathscr{H}$ be minimal in $\mathscr{H}$. Let $\mathscr{H}_1 = \mathscr{H} - \{H_0\}$, $\mathscr{H}_2 = \{K: H_0 \subseteq K \subseteq G\}$.

If $\mathscr{H}_2 = \mathscr{H}$, then

$$\chi(\bigcup_{H \in \mathscr{H}} X^H) = \chi(X^{H_0}) = 1 + \sum_{H \supseteq H_0} \varphi(H) = 1 + \sum_{H \in \mathscr{H}} \varphi(H).$$

Otherwise, (2) holds for the sets $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_1 \cap \mathscr{H}_2$ by assumption,

$$(\bigcup_{H \in \mathscr{H}_1 \cap \mathscr{H}_2} X^H) = (\bigcup_{H \in \mathscr{H}_1} X^H) \cap (\bigcup_{H \in \mathscr{H}_2} X^H),$$

and so

$$\begin{aligned}
\chi(\bigcup_{H \in \mathscr{H}} X^H) &= \chi(\bigcup_{H \in \mathscr{H}_1} X^H) + \chi(\bigcup_{H \in \mathscr{H}_2} X^H) - \chi(\bigcup_{H \in \mathscr{H}_1 \cap \mathscr{H}_2} X^H) \\
&= 1 + \sum_{H \in \mathscr{H}_1} \varphi(H) + \sum_{H \in \mathscr{H}_2} \varphi(H) - \sum_{H \in \mathscr{H}_1 \cap \mathscr{H}_2} \varphi(H) \\
&= 1 + \sum_{H \in \mathscr{H}} \varphi(H).
\end{aligned}$$

Then, for $H \subsetneq G$,

$$\begin{aligned}
\chi(X^H, \bigcup_{K \supsetneq H} X^K) &= \chi(X^H) - \chi(\bigcup_{K \supsetneq H} X^K) \\
&= \sum_{K \supseteq H} \varphi(K) - \sum_{K \supsetneq H} \varphi(K) = \varphi(H).
\end{aligned}$$

The action of an element $a \in G$ takes the pair $(X^H, \bigcup_{K \supsetneq H} X^K)$ homeomorphically to the pair $(X^{aHa^{-1}}, \bigcup_{K \supsetneq aHa^{-1}} X^K)$, and so $\varphi(H) = \varphi(aHa^{-1})$. The group $N(H)/H$ acts semi-freely on $X^H / \bigcup_{K \supsetneq H} X^K$ with one fixed point, and so

$$\varphi(H) = \chi(X^H, \bigcup_{K \supsetneq H} X^K) = \chi(X^H / \bigcup_{K \supsetneq H} X^K) - 1 \equiv 0 \ (\mathrm{mod}\, |N(H)/H|). \quad \square$$

Now define a *modp resolving function* for G to be any function $\varphi: \mathscr{S}(G) \to \mathbf{Z}$ such that:

1) φ is constant on conjugacy classes of subgroups.
2) $[N(H):H] \mid \varphi(H)$ for all $H \subseteq G$.
3) For any $H \subseteq G$ such that $H \in \mathscr{G}_p^1$, $\sum_{K \supseteq H} \varphi(K) = 0$.

Define an *integral resolving function* for G to be a function $\varphi: \mathscr{S}(G) \to \mathbf{Z}$ which is a mod$p$ resolving function for all primes p; this amounts to replacing condition (3) by the condition: $\sum_{K \supseteq H} \varphi(K) = 0$ for all $H \in \mathscr{G}^1$.

If G acts on any finite complex X, and φ is defined as in Lemma 2, then conditions (1) and (2) have been shown to hold for φ. If in addition X is $\mathbf{Z}_p$-acyclic, then Proposition 1 shows that φ is a mod$p$ resolving function for G. If X is $\mathbf{Z}$-acyclic, then by Proposition 2, φ is an integral resolving function for G.

The sets of all mod$p$ or integral resolving functions of G form groups under pointwise addition, and so one may define $m_p(G)$ and $m(G)$ as the unique non-negative integers such that

$$m_p(G)\cdot\mathbf{Z}=\{\varphi(G): \varphi \text{ is a mod}p \text{ resolving function for } G\}$$
$$m(G)\cdot\mathbf{Z}=\{\varphi(G): \varphi \text{ is an integral resolving function for } G\}.$$

Thus, if F is the fixed-point set of an action of G on a finite $\mathbf{Z}_p$-acyclic complex, then $\chi(F)\equiv 1\ (\mathrm{mod}\, m_p(G))$. If F is the fixed-point set of an action on a finite $\mathbf{Z}$-acyclic complex, then $\chi(F)\equiv 1\ (\mathrm{mod}\, m(G))$.

It will be shown that for G not of p-power order, a finite complex F is the fixed point set of an action of G on some finite $\mathbf{Z}_p$-acyclic complex if and only if $\chi(F)\equiv 1$ $(\mathrm{mod}\, m_p(G))$. The result for action on contractible complexes will be similar, but not as complete. The basic idea will be to use the function φ as a pattern for building up a space X with G-action, such that $\chi(X^H)=1+\sum_{K\supseteq H}\varphi(K)$ for all $H\subseteq G$. Two lemmas will first be needed.

LEMMA 3. *Assume that P is a p-group acting on the n-dimensional $(n-1)$-connected complex X. If X^H is $\mathbf{Z}_p$-acyclic and of dimension less than n for all $0\neq H\subseteq P$, then $H_n(X)$ is a projective $\mathbf{Z}[P]$-module.*

Proof. The set $\{X^H: 0\neq H\subseteq P\}$ is a set of $\mathbf{Z}_p$-acyclic subcomplexes of X closed under intersections. It follows from the Meyer-Vietoris sequence (and inducting) that $\hat{X}=\bigcup_{0\neq H\subseteq P} X^H$ is $\mathbf{Z}_p$-acyclic. Thus, $H_n(X)=H_n(X/\hat{X})$ and $\bar{H}_i(X/\hat{X})$ consists of torsion prime to p for $i<n$. P acts semi-freely on $X/\hat{X}$ with one fixed point.

Since for any surjection $f: M\to T$ of $\mathbf{Z}[P]$-modules, where M is free and T consists of torsion prime to p, the kernel is projective, one may add free orbits of cells to $X/\hat{X}$, killing all homology in dimensions below n, and ending up with the n-dimensional $(n-1)$-connected complex Y, with a semi-free P-action with one fixed point, and with $H_n(Y)=H_n(X)\oplus N$, N a projective $\mathbf{Z}[P]$-module. The sequence

$$0\to H_n(Y)\to C_n(Y, Y^P)\to\cdots\to C_0(Y, Y^P)\to 0$$

is exact (where $C_*(Y, Y^P)$ is the cellular chain complex), all but the first group is free, and so $H_n(Y)$ is stably free. Thus, $H_n(X)$ is projective. $\square$

LEMMA 4. *If X is an n-dimensional $(n-1)$-connected finite complex with an action of some cyclic group $\mathbf{Z}_m$, such that $\chi(X^H)=1$ for all $0\neq H\subseteq \mathbf{Z}_m$, then $H_n(X; \Lambda)$ is a free $\Lambda[\mathbf{Z}_m]$-module for $\Lambda=\mathbf{Z}_p (p\nmid m)$ or $\Lambda=\mathbf{Q}$.*

Proof. For any finite complex Y, one may define an Euler characteristic for Y with values in the group $R_A(\mathbf{Z}_m)$ of virtual representations:

$$\chi_{\mathbf{Z}_m}(Y)=\sum_{i=0}^{\infty}(-1)^i[H_i(Y;A)]=\sum_{i=0}^{\infty}(-1)^i[C_i(Y,A)]\in R_A(\mathbf{Z}_m)$$

(where $C_*(Y;A)$ is again the cellular chain complex).

Let $\varphi:\mathscr{S}(\mathbf{Z}_m)\to\mathbf{Z}$ be defined by $\varphi(0)=\chi(X)-1$, $\varphi(H)=0$ for all subgroups $H\neq 0$. Then φ satisfies the conaition $\chi(X^H)=1+\sum_{K\supseteq H}\varphi(K)$ for all $H\subseteq\mathbf{Z}_m$, and so by Lemma 2, $\chi(X^H,\bigcup_{K\supsetneq H}X^K)=0$ for all $0\neq H\subsetneq G$. In other words, for any proper subgroup H, the numbers of even and odd dimensional cells in orbits of type G/H are the same.

It follows that $\chi_{\mathbf{Z}_m}(X^H,\bigcup_{K\supsetneq H}X^K)=0$ foi all $0\neq H\subsetneq\mathbf{Z}_m$, and so

$$\chi_{\mathbf{Z}_m}(\bigcup_{0\neq H\subseteq\mathbf{Z}_m}X^H)=\chi_{\mathbf{Z}_m}(X^{\mathbf{Z}_m})=[A].$$

Then

$$(-1)^n[H_n(X;A)]=\chi_{\mathbf{Z}_m}(X)-[A]\equiv\chi_{\mathbf{Z}_m}(\bigcup_{0\neq H\subseteq\mathbf{Z}_m}X^H)-[A]=0\,(\mathrm{mod}\,[A[\mathbf{Z}_m]]).$$

Since $A[\mathbf{Z}_m]$ is semi-simple, it follows that $H_n(X;A)$ is free. □

Now, actions on $\mathbf{Z}_p$-acyclic complexes can be constructed from mod$p$ resolving functions.

THEOREM 1. *Let G be a finite group not of p-power order, and φ a* mod$p$ *resolving function for G. Then for any finite complex F with $\chi(F)=1+\varphi(G)$, F is the fixed-point set of an action of G on some finite $\mathbf{Z}_p$-acyclic complex.*

Proof. The goal is to embed F as the fixed-point set of a G-space X, such that $\chi(X^H)=1+\sum_{K\supseteq H}\varphi(K)$ for all $H\subseteq G$, and such that X^H is $\mathbf{Z}_p$-acyclic whenever H is of p-power order. This has been done for X^G; assume the complex X_0 has been constructed, with G-action, such that the above properties hold for all $H\supsetneq H_0$, for some fixed $H_0\subseteq G$.

Cells must now be added in orbits of type G/H_0, until the fixed-point set of H_0 meets the conditions described above. This amounts to adding cells to $X_0^{H_0}$ in orbits of type $N(H_0)/H_0$; when these are extended equivariantly to the full orbits of type G/H_0, the remaining cells will be added to the fixed-point sets of other subgroups conjugate to H_0. Thus, the fixed point sets of these conjugate subgroups will be built up at the same time to meet the desired conditions.

From the above assumptions on X_0, one has $\chi(X_0^H)=1+\sum_{K\supseteq H}\varphi(K)$ for all $H\supsetneq H_0$; thus, using Lemma 2,

$$\chi(X_0^{H_0})\equiv\chi(\bigcup_{H\supsetneq H_0}X_0^H)=1+\sum_{H\supsetneq H_0}\varphi(H)\equiv 1+\sum_{H\subseteq H_0}\varphi(H)\quad(\mathrm{mod}|N(H_0)/H_0|).$$

If H_0 is not of p-power order, orbits of cells of type $N(H_0)/H_0$ may now be added to $X_0^{H_0}$ to produce a complex with the desired Euler characteristic.

If H_0 is of p-power order, add orbits of cells of type $N(H_0)/H_0$ to $X_0^{H_0}$ to produce a space Y which is n-dimensional and $(n-1)$-connected for some n larger than the dimension of X_0^H for any $H \supsetneqq H_0$. The fixed-point set of any non-zero p-group $\hat{P}$ in $N(H_0)/H_0$ under the action on Y is $\mathbf{Z}_p$-acyclic, since $\hat{P} = P_0/H_0$ for some p-group $P_0 \subseteq G$, and $Y^{\hat{P}} = X_0^{P_0}$. Lemma 3 applies: $H_n(Y)$ is projective as a $\mathbf{Z}[P]$-module, where P is a p-Sylow subgroup of $N(H_0)/H_0$. It follows that $H_n(Y; \mathbf{Z}_p)$ is a projective $\mathbf{Z}_p[P]$-module, and thus (from Rim [10], Proposition 4.8 and Corollary 2.4) a projective $\mathbf{Z}_p[N(H_0)/H_0]$-module.

Let $\hat{K} = K/H_0 \subseteq N(H_0)/H_0$ be any non-zero cyclic subgroup. Then $K \in \mathscr{G}_p^1$, and it follows from the definition of a mod p resolving function that

$$\chi(Y^{\hat{K}}) = \chi(X_0^K) = 1 + \sum_{H \supseteq K} \varphi(H) = 1.$$

Thus, by Lemma 4, $H_n(Y; \mathbf{Z}_p)$ is a free $\mathbf{Z}_p[\hat{K}]$-module for all cyclic subgroups $\hat{K} \subseteq N(H_0)/H_0$ of order prime to p. Since

$$\chi(Y) \equiv \chi(X_0^{H_0}) \equiv 1 + \sum_{H \supseteq H_0} \varphi(H) = 1 \qquad (\operatorname{mod} |N(H_0)/H_0|),$$

the dimension of $H_n(Y; \mathbf{Z}_p)$ as a $\mathbf{Z}_p$-vector space must be a multiple of $|N(H_0)/H_0|$. Let M be the free $\mathbf{Z}_p[N(H_0)/H_0]$-module of the same dimension.

Both M and $H_n(Y; \mathbf{Z}_p)$ are projective. They are isomorphic upon restricting the action to any cyclic subgroup of order prime to p, thus have the same Brauer character, and so are isomorphic as $\mathbf{Z}_p[N(H_0)/H_0]$-modules (see, e.g., [11], §§14, 16 and 18). Thus, $H_n(Y; \mathbf{Z}_p)$ is free, and so orbits of $(n+1)$-cells of type $N(H_0)/H_0$ may be added to Y to make it $\mathbf{Z}_p$-acyclic. □

COROLLARY. *For a fixed finite group G not of p-power order, a finite complex F is the fixed-point set of an action of G on some finite $\mathbf{Z}_p$-acyclic complex if and only if* $\chi(F) \equiv 1 \pmod{m_p(G)}$. □

In the case of actions on contractible complexes, things are less simple. Define a G-resolution of a complex F to be a finite complex X with a cellular G-action such that $F = X^G$, X is n-dimensional, $(n-1)$-connected (for some $n \geqslant 2$), and $H_n(X)$ is a projective $\mathbf{Z}[G]$-module.

THEOREM 2. *Assume G is not of prime power order, and acts on the finite complex Y with fixed-point set F, such that* $\chi(F) \equiv 1 \pmod{m(G)}$. *Then Y can be embedded equivariantly into a G-resolution X of F.*

Proof. Let φ be an integral resolving function for G with $\varphi(G)=\chi(F)-1$. By the techniques used in proving Theorem 1, Y may be built up, without changing the fixed point set, to produce a G-complex X_0, such that $\chi(X_0^H)=1+\sum_{K\supset H}\varphi(K)$ for all $0\neq H\subseteq G$, and such that X_0^H is $\mathbf{Z}_p$-acyclic for any $H\neq 0$ of p-power order, for any prime p.

Now add free G-orbits of cells to X_0 to produce an n-dimensional, $(n-1)$-connected space X, where $n\geqslant 2$ and $n>\dim(X^H)$ for any $0\neq H\subseteq G$. By Lemma 3, $H_n(X)$ is a projective $\mathbf{Z}[P]$-module for any Sylow-subgroup $P\subseteq G$. It follows (Rim [10], Proposition 4.9) that $H_n(X)$ is a projective $\mathbf{Z}[G]$-module. □

The problem remains to determine when a G-resolution may be modified to produce a contractible complex with the same fixed-point set. For convenience, let $\mathscr{P}(G)$ be the set of finite complexes F with $\chi(F)\equiv 1\ (\operatorname{mod} m(G))$: the finite complexes which have G-resolutions. For any n-dimensional G-resolution X of F, set $\gamma_G(F,X)=(-1)^n\,[H_n(X)]\in\tilde{K}_0(\mathbf{Z}[G])$, an obstruction lying in the projective class group. If $\gamma_G(F,X)=0$, then $H_n(X)$ is stably free, and n- and $(n+1)$-dimensional cells may be added to produce a contractible complex with fixed-point set F.

3. The Obstruction γ_G

The constructions in Section 2 now make it possible to define an obstruction $\gamma_G(F)$ for $F\in\mathscr{P}(G)$, such that F is the fixed-point set of an action of G on a finite contractible complex if and only if $\gamma_G(F)=0$. It will turn out that $\gamma_G(F)$ depends only on $\chi(F)$. The following lemma will be used in the constructions throughout this section.

LEMMA 5. *Assume G acts on the* 1*-connected space X with fixed-point set F, and assume that $H_i(X)$ is a projective* $\mathbf{Z}[G]$*-module for all $n\geqslant 2$. Then X can be embedded in a G-resolution Y of F, such that*

$$\gamma_G(F,Y)=\sum_{i=2}^{\infty}(-1)^i\,[H_i(X)].$$

Proof. Let $n=\dim X$, and let j be the smallest positive integer such that $H_j(X)\neq 0$. If $j=n$, then X is a G-resolution, and we are done.

If $j<n$, then choose some surjection $f\colon(\mathbf{Z}[G])^k\to H_j(X)$, and let $M=\operatorname{Ker}(f)$. Then M is projective, and

$$[M]=-[H_j(X)]\in\tilde{K}_0(\mathbf{Z}[G]).$$

Adding k free orbits of $(j+1)$-cells to X realizing f produces a new space X_0 with homology

$$\begin{aligned}\bar{H}_i(X_0) &= 0 && \text{for } i \leq j \\ &= H_{j+1}(X) \oplus M && \text{for } i = j+1 \\ &= H_i(X) && \text{for } i > j+1\end{aligned}$$

and so $\sum_{i=2}^{n} (-1)^i [H_i(X_0)] = \sum_{i=2}^{n} (-1)^i [H_i(X)]$. Repetition of this procedure will eventually produce a G-resolution Y of F, with

$$\gamma_G(F, Y) = \sum_{i=2}^{n} (-1)^i [H_i(X)]. \quad \square$$

Now set

$$\mathscr{B}(G) = \{\gamma_G(pt, X) : X \text{ is a } G\text{-resolution of a point}\}.$$

The set $\mathscr{B}(G)$ is actually a subgroup of $\tilde{K}_0(\mathbf{Z}[G])$. If X_1 and X_2 are G-resolutions of a point, then $X_1 \vee X_2$ (the one-point union at the fixed points) fulfills the hypotheses of Lemma 5 and there is a G-resolution X of a point with $\gamma_G(pt, X) = \gamma_G(pt, X_1) + {}+\gamma_G(pt, X_2)$. If X is any G-resolution of a point, them so is its reduced suspension ΣX, and $\gamma_G(pt, \Sigma X) = -\gamma_G(pt, X)$.

PROPOSITION 3. *For any $F \in \mathscr{P}(G)$, if X_1 and X_2 are two G-resolutions of F, then* $\gamma_G(F, X_1) - \gamma_G(F, X_2) \in \mathscr{B}(G)$.

Proof. Let X be the union of X_1 and X_2 identified at F. By Theorem 2, there is an embedding of X into a G-resolution Y of F; thus both X_1 and X_2 are embedded in Y. Let $n_i = \dim(X_i)$, $m = \dim(Y)$, and assume $m \geq n_i + 2$.

For $i = 1$ or 2, the space Y/X_i has reduced homology in only two dimensions: $H_m(Y/X_i) \cong H_m(Y)$, and $H_{n_i+1}(Y/X_i) \cong H_{n_i}(X_i)$. Thus, by Lemma 5, there is a G-resolution Y_i of $(Y/X_i)^G = pt$, with

$$\gamma_G(F, Y) - \gamma_G(F, X_i) = \gamma_G(pt, Y_i) \in \mathscr{B}(G).$$

It follows that $\gamma_G(F, X_1) - \gamma_G(F, X_2) \in \mathscr{B}(G)$. $\square$

Now, for $F \in \mathscr{P}(G)$, define $\gamma_G(F)$ to be the image in $\tilde{K}_0(\mathbf{Z}[G])/\mathscr{B}(G)$ of $\gamma_G(F, X)$ for any G-resolution X of F. If F is the fixed-point set of an action of G on a finite contractible complex X, then X is a G-resolution of F, and so $F \in \mathscr{P}(G)$ and $\gamma_G(F) = 0$. The converse is proven in the following proposition:

PROPOSITION 4. *If $F \in \mathscr{P}(G)$ and $\gamma_G(F) = 0$, then G has an action on some finite contractible complex with fixed-point set F.*

Proof. Choose a G-resolution X of F. If $X \neq \emptyset$, then let Y be a G-resolution of a point, such that $\gamma_G(pt, Y) = -\gamma_G(F, X)$. Let $X \vee Y$ be the space obtained by identifying the fixed-point of Y with any point of $F \subseteq X$. Then $(X \vee Y)^G = F$, and by Lemma

5, there is a G-resolution $\hat{X}$ of F with

$$\gamma_G(F, \hat{X}) = \gamma_G(F, X) + \gamma_G(pt, Y) = 0.$$

It follows that the top dimensional homology of $\hat{X}$ is stably free, and so free orbits of cells may be added to produce a contractible complex.

If $\emptyset \in \mathscr{P}(G)$, then $m(G)=1$, and so $m_p(G)=1$ for all primes p. It follows from the corollary to Theorem 1 that G has a fixed-point free action on a finite $\mathbf{Z}_p$-acyclic complex for every prime p. A contractible complex with a fixed-point free action may now be obtained by taking a finite join of these spaces. (Thus, $\gamma_G(\emptyset)=0$ whenever $\emptyset \in \mathscr{P}(G)$). $\square$

The following proposition relates the obstruction $\gamma_G(F)$ for different complexes.

PROPOSITION 5. *Let $F_1, F_2 \in \mathscr{P}(G)$ be non-empty finite complexes. Then:*

1) $\gamma_G(F_1) = \gamma_G(F_2)$ *if F_1 and F_2 are homotopically equivalent.*

2) $F_1 \vee F_2 \in \mathscr{P}(G)$ *for any choice of base points, and*

$$\gamma_G(F_1 \vee F_2) = \gamma_G(F_1) + \gamma_G(F_2).$$

3) *If $f: F_1 \to F_2$ is any (skeletal preserving) map with mapping cone C_f, then $C_f \in \mathscr{P}(G)$ and*

$$\gamma_G(C_f) = \gamma_G(F_2) - \gamma_G(F_1).$$

4) $\Sigma F_1 \in \mathscr{P}(G)$ *and* $\gamma_G(\Sigma F_1) = -\gamma_G(F_1)$

5) *If $\emptyset \in \mathscr{P}(G)$, then so is S^0, and* $\gamma_G(S^0) = -\gamma_G(\emptyset)$ $(=0)$.

Proof. To prove (1), let $f: F_1 \to F_2$ be a homotopy equivalence, and X_1 a G-resolution of F_1. Let $X_1^{(n)}$ denote the n-skeleton of X_1, and set $Y^{(-1)} = F_2$. Starting with the homotopy equivalence

$$f^{(-1)} = f: F_1 \cup X_1^{(-1)} \to Y^{(-1)}$$

inductively construct spaces $Y^{(n)}$ and homotopy equivalences $f^{(n)}: F_1 \cup X_1^{(n)} \to Y^{(n)}$: for every n-cell in $X_1 - F_1$ with attaching map $\alpha: S^{n-1} \to F_1 \cup X_1^{(n-1)}$, attach an n-cell to $Y^{(n-1)}$ via the map $f^{(n-1)} \circ \alpha$. Now, for $n = \dim(X_1)$, $Y^{(n)}$ will be a G-resolution of F_2, and $\gamma_G(F_2, Y^{(n)}) = \gamma_G(F_1, X_1)$.

In (2), let $X_i \supseteq F_i$ be G-resolutions. Then $X_1 \vee X_2$ has an action of G with fixed-point set $F_1 \vee F_2$, and by Lemma 5 can be embedded in a G-resolution X of $F_1 \vee F_2$ with $\gamma_G(F_1 \vee F_2, X) = \gamma_G(F_1, X_1) + \gamma_G(F_2, X_2)$.

In (3), let Z_f be the mapping cylinder of $f: F_1 \subseteq Z_f$ and $Z_f/F_1 = C_f$. Let X be a n-dimensional G-resolution of F_1 and set $\hat{X}$ to be the union of X and Z_f, joined at F_1. Let Y be an m-dimensional G-resolution of Z_f containing $\hat{X}$, for $m \geqslant n+2$. Then Y/X

has a G-action with fixed-point set C_f, and its reduced homology is zero except in two dimensions: $H_m(Y/X)=H_m(Y)$ and $H_{n+1}(Y/X)=H_n(X)$. By Lemma 5, Y may be embedded in a G-resolution $\hat{Y}$ of C_f, with

$$\gamma_G(C_f, \hat{Y})=(-1)^m\,[H_m(Y)]+(-1)^{n+1}\,[H_n(X)]=\gamma_G(Z_f, Y)-\gamma_G(F_1, X).$$

Thus, $\gamma_G(C_f)=\gamma_G(Z_f)-\gamma_G(F_1)=\gamma_G(F_2)-\gamma_G(F_1)$ (by (1)).

If X is a G-resolution of F_1, then ΣX is a G-resolution of ΣF_1, and $\gamma_G(\Sigma F_1, \Sigma X)=-\gamma_G(F_1, X)$. If $\emptyset\in\mathscr{P}(G)$ and X is a G-resolution of $\emptyset$, then the unreduced suspension ΣX is a G-resolution of S^0, with $\gamma_G(S^0, \Sigma X)=-\gamma_G\emptyset(, X)$. □

Now, the basic property of the obstruction can be proven:

THEOREM 3. *If G is not of prime power order, and $F_1, F_2\in\mathscr{P}(G)$, then $\chi(F_1)=\chi(F_2)$ implies $\gamma_G(F_1)=\gamma_G(F_2)$.*

Proof. It suffices to prove that $\gamma_G(F)=0$ for all F with $\chi(F)=1$. Then, whenever $\chi(F_1)=\chi(F_2)$, if neither is empty one may take any map $f:F_1\to F_2$; $\chi(C_f)=1$ and so $\gamma_G(F_2)-\gamma_G(F_1)=\gamma_G(C_f)=0$. If either is empty, then $\chi(\Sigma F_1)=\chi(\Sigma F_2)=2$ (setting $\Sigma\emptyset=S^0$), and

$$\gamma_G(F_1)=-\gamma_G(\Sigma F_1)=-\gamma_G(\Sigma F_2)=\gamma_G(F_2).$$

By Theorem 2, any complex of Euler characteristic 1 is in $\mathscr{P}(G)$; in particular $S^1\vee S^0$ is. Let $f:S^1\vee S^0\to S^1\vee S^0$ be the map which takes $S^1\to S^1$ via the identity, and takes S^0 to the base point. Then C_f has the homotopy type of $S^1\vee S^0$, and so

$$\gamma_G(S^1\vee S^0)=\gamma_G(C_f)=\gamma_G(S^1\vee S^0)-\gamma_G(S^1\vee S^0)=0.$$

Suspending, one gets $\gamma_G(S^n\vee S^{n-1})=0$ for all $n\geqslant 1$.

Let F be any wedge of spheres with $\chi(F)=1$: F is a wedge of equal numbers of odd and even dimensional spheres. Then, one may choose integers n_i $(i=1,\ldots, s)$ and m_i $(i=1,\ldots, t)$ $(n_i, m_i\geqslant 1)$ such that

$$F\vee\left(\bigvee_{i=1}^{s}(S^{n_i}\vee S^{n_i-1})\right)=\bigvee_{i=1}^{t}(S^{m_i}\vee S^{m_i-1}),$$

and so $\gamma_G(F)=0$.

Now, for $n\geqslant 1$, assume that $\gamma_G(F)=0$ for all finite complexes F of dimension less than n with $\chi(F)=1$. Let F_0 be an n-dimensional complex, with F_1 the $(n-1)$-skeleton. Let $\hat{F}_1$ be the space obtained by taking the wedge product of F_1 with enough 0- or 1-spheres so that $\chi(\hat{F}_1)=1$. Then $\gamma_G(\hat{F}_1)=0$: for $n\geqslant 2$, this follows by the induction hypothesis, and for $n=1$, $\hat{F}_1$ is a wedge of spheres.

Let $f: \hat{F}_1 \to F_0$ be the map which is the inclusion upon restriction to F_1, and which collapses the rest of $\hat{F}_1$ to a point. Then C_f has the homotopy type of a wedge of spheres, $\chi(C_f)=1$, and so

$$\gamma_G(F_0)=\gamma_G(F_0)-\gamma_G(\hat{F}_1)=\gamma_G(C_f)=0.$$

Thus, $\gamma_G(F)=0$ for all finite complexes F of Euler characteristic. 1. □

COROLLARY. *If G is not of prime power order, then there is an integer n_G such that a finite complex F is the fixed-point set of an action of G on a finite contractible complex if and only if $\chi(F)\equiv 1 \pmod{n_G}$.* □

Clearly $m(G) \mid n_G$. The following proposition puts some upper limit on n_G.

PROPOSITION 6. *For G not of prime power order, $n_G \mid m(G)^2$. In particular, $n_G=0$ if and only if $m(G)=0$, and $n_G=1$ if and only if $m(G)=1$.*

Proof. Let F be a finite complex with $\chi(F)=1+m(G)$. Let X be an n-dimensional G-resolution of F. Since $H_n(X)$ is projective, it follows from [2] (corollary to Theorem 78.3) That $H_n(X; \mathbf{Z}_p)=H_n(X)\otimes \mathbf{Z}_p$ is a free $\mathbf{Z}_p[G]$-module for any prime p. In particular, $(n+1)$-cells may be added to X to give a $\mathbf{Z}_2$-acyclic complex X_1, upon which G acts with fixed-point set F. Let $\{p_1, \dots, p_s\}$ be the finite set of primes for which torsion occurs in X.

Setting $k=p_1 \dots p_s$, one has $H_n(X; \mathbf{Z}_k)=\oplus_i H_n(X; \mathbf{Z}_{p_i})$, a free $\mathbf{Z}_k[G]$-module. Now, free orbits of cells may be added to X to produce the $\mathbf{Z}_k$-acyclic complex X_2, with $X_2^G=F$. Thus, G acts on the contractible complex $X_1 * X_2$ with fixed-point set $F*F$, $\chi(F*F)=1-m(G)^2$, and so $n_G \mid m(G)^2$. □

4. Computation of $m_p(G)$ and $m(G)$

In this section, the invariants $m_p(G)$ and $m(G)$ will be computed. Again, the following notation will be used for classes of finite groups. $\mathscr{G}_p^1$ is the collection of groups G with normal subgroups $P \lhd G$, such that P is of p-power order and G/P is cyclic. $\mathscr{G}_p^q$ denotes the collection of finite groups G with normal subgroups $H \in \mathscr{G}_p^1$ of q-power index. Furthermore, set $\mathscr{G}_p=\bigcup_q \mathscr{G}_p^q$, $\mathscr{G}^q=\bigcup_p \mathscr{G}_p^q$, $\mathscr{G}^1=\bigcup_p \mathscr{G}_p^1$, and $\mathscr{G}=\bigcup_p \mathscr{G}_p$. Note that from the definitions, $m_p(G)=0$ if $G\in\mathscr{G}_p^1$ and $m(G)=0$ if $G\in\mathscr{G}^1$.

LEMMA 6. *Assume $H\subseteq G$, a subgroup with index n. If H acts on X with fixed-point set F, then G has an action on X^n with fixed-point set F', the image of F under the diagonal map $\Delta: X\to X^n$.*

Proof. Let G/H be the finite set of right cosets; choose some splitting map $t: G/H \to G$ with $t(He)=e$. Define $p: G\to H$ by $p(g)=g\cdot t(Hg)^{-1}$. The function p is continuous, and

$$p(h)=h \quad \text{for} \quad h\in H$$
$$p(hg)=h\cdot p(g) \quad \text{for} \quad h\in H, g\in G.$$

The space X^n can be described as $X^{G/H}$: the space of functions from $G/H\to X$. Define the action $\pi: G\times X^{G/H}\to X^{G/H}$ by

$$\pi(g,\xi)(Ha)=p(a)^{-1}\,p(ag)\cdot\xi(Hag)$$

π is well-defined, since $p(ha)^{-1}\,p(hag)=(h\cdot p(a))^{-1}\,(h\cdot p(ag))$. It is an action of G, since

$$\begin{aligned}\pi(g_1,\pi(g_2,\xi))(Ha)&=p(a)^{-1}\,p(ag_1)\cdot\pi(g_2,\xi)(Hag_1)\\&=p(a)^{-1}\,p(ag_1)\cdot p(ag_1)^{-1}\,p(ag_1g_2)\,\xi(Hag_1g_2)\\&=\pi(g_1g_2,\xi)(Ha)\end{aligned}$$

The action is continuous; since for fixed g, the action on each coordinate is the action of some $h\in H$ (with the coordinates permuted).

Clearly, every point of $F'=\Delta(F)$ is fixed by π. For any $\xi\in X^n$, fixed by π: for any $a\in G$, $h\in H$,

$$\xi(Ha)=\pi(a^{-1}ha,\xi)(Ha)=p(a)^{-1}\,p(ha)\,\xi(Ha)=[p(a)^{-1}\cdot h\cdot p(a)]\,\xi(Ha)$$

so $\xi(Ha)\in F$ for all $Ha\in G/H$. Then $\xi(He)=\pi(a,\xi)(He)=\xi(Ha)$ for all $a\in G$, and $\xi\in F'$. $\square$

COROLLARY. *If $H\subseteq G$, then $m_p(G)\mid m_p(H)$ for all p, and $n_G\mid n_H$.*

Proof. Assume H acts simplicially on some finite simplicial complex X, and set $n=[G:H]$. The action of H on X induces an action of the wreath product $\Sigma_n\int H$ on X^n, which is simplicial under suitable choice of a simplicial structure for X^n, and the action of G constructed in Lemma 6 is actually a subaction of this, and thus simplicial.

For any prime p, it follows from the discussion in the introduction, together with Theorem 1, that H has a simplicial action on some finite $\mathbf{Z}_p$-acyclic complex X with $\chi(X^H)=1+m_p(H)$. Then G has a simplicial action on the finite $\mathbf{Z}_p$-acyclic complex X^n, with $(X^n)^G=X^H$, and so $m_p(G)\mid m_p(H)$. Similarly, H has a simplicial action on some finite contractible complex Y, with $\chi(Y^H)=1+n_H$, G has a simplicial action on Y^n with $\chi((Y^n)^G)=1+n_H$, and so $n_G\mid n_H$. $\square$

LEMMA 7. *Assume G has subgroups $H_0\lhd H_1\lhd\cdots\lhd H_n=G$, where $H_i\lhd H_{i+1}$ of index q for all i, and $H_0\in\mathcal{G}_p^1$. Then $G\in\mathcal{G}_p^q$.*

Proof. Let $P \lhd H_0$ be the p-Sylow subgroup of H_0; H_0/P is cyclic. One may assume $q \nmid |H_0/P|$.

If $p \neq q$, then P is the only p-Sylow subgroup in H_0, thus invariant under any automorphism of H_0, and so $P \lhd H_1$. Then P is the only p-Sylow subgroup of H_1, so $P \lhd H_2$, and continuing this procedure one gets $P \lhd G$. The same process shows that any Sylow subgroup of H_0/P is normal in G/P; thus $H_0/P \lhd G/P$, $H_0 \lhd G$, and so $G \in \mathscr{G}_p^q$.

Now assume $p=q$. If $H_0/P \cong \mathbf{Z}_m$, then $(|P|, m)=1$ implies, since H_0 is solvable, that there is a subgroup $\mathbf{Z}_m \subseteq H_0$ (see [5], p. 99). The group $\mathbf{Z}_m$ acts on the set of p-Sylow subgroups of G by conjugation; since G permutes them transitively so does $\mathbf{Z}_m$. Let $\hat{P}$ be the intersection of the p-Sylow subgroups of G; $\hat{P} \lhd G$, and $P \subseteq \hat{P}$ since P is invariant under the action of $\mathbf{Z}_m$.

Let $\pi: G \to G/\hat{P}$ be the projection; set $\tilde{H}_i = \pi(H_i)$. Then $\tilde{H}_0 \cong \mathbf{Z}_m (p \nmid m)$ and $\tilde{H}_i \lhd \tilde{H}_{i+1}$ of index 1 or p. Now $\tilde{H}_0 \lhd G/\hat{P}$, by the procedure described above (a normal Sylow subgroup is invariant under any automorphism), and so $G \in \mathscr{G}_p^p = \mathscr{G}_p^q$. □

The calculations of $m_p(G)$ and $m(G)$ will be based on the following technical lemma:

LEMMA 8. *For any group G, let k_G be the product of the distinct primes q such that G has a normal subgroup of index q. Let $\varphi: \mathscr{S}(G) \to \mathbf{Z}$ be the function defined uniquely by the conditions:*

$$\varphi(G) = k_G, \qquad \sum_{K \supseteq H} \varphi(K) = 0 \quad \textit{for} \quad H \subsetneq G.$$

Then $[N(H):H] \mid \varphi(H)$ for all $H \subseteq G$.

Proof. Assume that this has been proven for all groups of smaller order. For $0 \neq H \subseteq G$, assume that $[N(K):K] \mid \varphi(K)$ for all $K \supsetneq H$.

If $H \lhd G$, then setting $\hat{\varphi}(K/H) = \varphi(K)$ for all $K \supseteq H$ defines a function $\hat{\varphi}: \mathscr{S}(G/H) \to \mathbf{Z}$ with the properties: $\hat{\varphi}(G/H) = k_G$, and $\sum_{L \supseteq K} \hat{\varphi}(L) = 0$ for all $K \subsetneq G/H$. Since $k_{G/H} \mid k_G$, it follows from the induction hypothesis that $|G/H| \mid \hat{\varphi}(H/H) = \varphi(H)$.

Now assume $N(H) \neq G$. For all L such that $H \subseteq L \subseteq N(H)$, set $\mathscr{S}_L = \{K \supseteq H: K \cap N(H) = L\}$, partitioning $\{K : K \supseteq H\}$. Then $\sum_{K \in \mathscr{S}_{N(H)}} \varphi(K) = \sum_{K \supseteq N(H)} \varphi(K) = 0$; by inducting downward one gets $\sum_{K \in \mathscr{S}_L} \varphi(K) = 0$ for all L. In particular, $\sum_{K \in \mathscr{S}_H} \varphi(K) = 0$.

Partition $\mathscr{S}_H$ into the orbits of the action of $N(H)$. For $K \in \mathscr{S}_H$, $K \neq H$, let F_K be the orbit of K under this action; $|F_K| = |N(H)|/|N(H) \cap N(K)|$. Since $N(H) \cap K = H$, the map $(N(H) \cap N(K))/H \to N(K)/K$ is one-to-one, so

$$[N(H):H] = |F_K| \cdot |N(H) \cap N(K)|/|H| \mid |F_K| \cdot [N(K):K] \mid \sum_{L \in F_K} \varphi(L).$$

Thus,

$$[N(H):H] \mid \sum_{\substack{K \in \mathscr{S}_H \\ K \neq H}} \varphi(K), \quad \text{and so} \quad [N(H):H] \mid \varphi(H).$$

It has now been shown that $[N(H):H] \mid \varphi(H)$ for all $0 \neq H \subseteq G$, and it remains to show that $|G| \mid \varphi(0)$. If $\mathscr{H}$ is any conjugacy class of non-zero subgroups, and $H \in \mathscr{H}$, then $|\mathscr{H}| = [G:N(H)]$ and

$$|G| \mid [G:N(H)] \cdot |H| \cdot \varphi(H) = \sum_{K \in \mathscr{H}} |K| \cdot \varphi(K).$$

It follows that $|G| \mid \sum_{0 \neq H \subseteq G} |H| \cdot \varphi(H)$.

Thus, it will now suffice to show that $|G| \mid \sum_{H \subseteq G} |H| \cdot \varphi(H)$. This may be rewritten

$$\sum_{H \subseteq G} |H| \cdot \varphi(H) = \sum_{H \subseteq G} \sum_{a \in H} \varphi(H) = \sum_{a \in G} \sum_{H \supseteq \langle a \rangle} \varphi(H)$$
$$= k_G \cdot (\text{number of generators of } G).$$

If G is not cyclic, it follows that $\sum_{H \subseteq G} |H| \cdot \varphi(H) = 0$, and we are done. If $G = \mathbf{Z}_n$, where $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ is the primary decomposition, then $k_G = p_1 \dots p_s$ and

$$\sum_{H \subseteq G} |H| \cdot \varphi(H) = n \cdot (p_1 - 1) \dots (p_s - 1)$$

which again is a multiple of $n = |G|$. □

Now the numbers $m_p(G)$ and $m(G)$ may be calculated immediately.

THEOREM 4. *Assume $G \notin \mathscr{G}_p^1$. Then $m_p(G)$ is a product of distinct primes, and $q \mid m_p(G)$ if and only if $G \in \mathscr{G}_p^q$ (so $m_p(G) = 1$ if and only if $G \notin \mathscr{G}_p$).*

Proof. Since $G \notin \mathscr{G}_p^1$, the function φ defined for G in Lemma 8 is a mod$p$ resolving function for G, and $m_p(G) \mid k_G$ which is a product of distinct primes. Furthermore, if $q \mid m_p(G)$, then $q \mid k_G$ and G has a normal subgroup H_1 of index q. If $H_1 \notin \mathscr{G}_p^1$, then $q \mid m_p(G) \mid m_p(H_1)$ and so H_1 has $H_2 \lhd H_1$ of index q. This procedure can be repeated until one reaches a subgroup $H_n \in \mathscr{G}_p^1$. Thus, by Lemma 7, $G \in \mathscr{G}_p^q$.

If $G \in \mathscr{G}_p^q$, then the fixed-point set F of any action of G on a finite $\mathbf{Z}_p$-acyclic complex has $\chi(F) \equiv 1 \pmod{q}$ (Proposition1) and so $q \mid m_p(G)$. □

THEOREM 5. *Assume $G \notin \mathscr{G}^1$. Then $m(G)$ is a product of distinct primes, and $q \mid m(G)$ if and only if $G \in \mathscr{G}^q$ (so $m(G) = 1$ if and only if $G \notin \mathscr{G}$).*

Proof. Since $G \notin \mathscr{G}^1$, the function φ defined for G in Lemma 8 is an integral resolving function for G, and $m(G) \mid k_G$, a product of distinct primes. If $q \mid m(G)$, then $q \mid k_G$, and G has a normal subgroup H_1 of index q. Then $q \mid m(G) \mid n_G \mid n_{H_1}$, so by Proposition

6, $q \mid m(H_1)$, and either $H_1 \in \mathscr{G}^1$ or H_1 has a normal subgroup H_2 of index q. This gives a sequence $H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd G$ of subgroups, each normal of index q in the following one, and such that $H_n \in \mathscr{G}^1$. $H_n \in \mathscr{G}_p^1$ for some prime p, and so $G \in \mathscr{G}_p^q \subseteq \mathscr{G}^q$ by Lemma 7.

If $G \in \mathscr{G}^q$, then $G \in \mathscr{G}_p^q$ for some p, and so $q \mid m_p(G) \mid m(G)$. □

COROLLARY. *For G not of prime power order, $n_G = 0$ if and only if $G \in \mathscr{G}^1$. If $G \notin \mathscr{G}^1$, then n_G is divisible by at most the square of any prime, and $q \mid n_G$ if and only if $G \in \mathscr{G}^q$. In particular, $n_G = 1$ if and only if $G \notin \mathscr{G}$.*

Proof. This follows directly from Theorem 5 and Proposition 6. □

There is one more case where n_G can easily be computed. Consistent with the notation used previously, $\mathscr{G}_1$ will denote the set of finite groups which have a normal cyclic subgroup of prime power index.

PROPOSITION 7. *Assume $G \notin \mathscr{G}_1$. Then $n_G = m(G)$.*

Proof. If p is any prime not dividing the order of G, then $G \notin \mathscr{G}_1$ implies that $G \notin \mathscr{G}_p$. It follows that $m_p(G) = 1$, and so there exists a finite $\mathbf{Z}_p$-acyclic complex X_p upon which G acts with two fixed points.

Let Y be an n-dimensional G-resolution of a discrete set of $(1 + m(G))$ points. The group $H_n(Y)$ is a projective $\mathbf{Z}[G]$-module, and so $H_n(Y; \mathbf{Z}_q) = H_n(Y) \otimes \mathbf{Z}_q$ is a free $\mathbf{Z}_q[G]$-module for any prime q ([2], Corollary to Theorem 78.3). If $q_1, \ldots, q_t$ are the distinct primes dividing $|G|$, and $k = q_1 \ldots q_t$, then it follows that

$$H_n(Y; \mathbf{Z}_k) = \bigoplus_{i=1}^{t} H_n(Y; \mathbf{Z}_{q_i})$$

is a free $\mathbf{Z}_k[G]$-module. Thus, free orbits of $(n+1)$-cells may be added to Y to produce the $\mathbf{Z}_k$-acyclic space $\hat{Y}$, upon which G acts with $(1 + m(G))$ fixed points.

The space $\hat{Y}$ has torsion with respect to a finite number of primes $p_1, \ldots, p_s$, none of which divide the order of G. Now, taking the smash product $\hat{Y} \wedge (\bigwedge_{i=1}^{s} X_{p_i})$, one gets a finite contractible complex upon which G acts with $(1 + m(G))$ fixed points. Thus, $n_G = m(G)$. □

The only remaining groups G for which n_G has not been calculated are those groups in $\mathscr{G}_1$ not in $\mathscr{G}^1$. In these cases, $m(G)$ turns out to be prime, and so there are only two possible values which n_G could take, depending on whether γ_G is ever non-zero. The smallest example of such a group is D_6, the dihedral group of order 12.

5. Fixed-Point Free Actions

So far, all of the results have described fixed-point sets possible for simplicial actions on finite acyclic complexes. In the case of fixed-point free actions, however, there is a fairly simple procedure for constructing smooth actions on compact manifolds from the simplicial actions. This is actually possible for positive-dimensional compact Lie groups as well as finite groups, by means of the concept of equivariant CW-complexes (see [7]).

DEFINITION. Let G be a compact Lie group. A zero-dimensional G-equivariant CW-complex is a disjoint union of homogeneous spaces G/H_i. An n-dimensional G-equivariant CW-complex is a space X, obtained from an $(n-1)$-dimensional G-equivariant CW-complex by attaching spaces $G/H_i \times D_i^n$ (where G acts trivially on D_i^n), via equivariant maps $\varphi_i: G/H_i \times S_i^{n-1} \to Y$.

When G is a finite group, a G-equivariant CW-complex is the same as a CW-complex with a cellular action (as defined in the introduction). Note that for any compact Lie group G, any finite G-equivariant CW-complex has finitely generated homology (because $(G/H \times D^n, G/H \times S^{n-1})$ does). It has been proven by Illman [7] that any smooth manifold with a smooth G-action has the structure of an equivariant CW-complex; if the manifold is compact, it will be a finite complex. The following theorem will make it possible to go in the other direction, from finite complexes to smooth, compact manifolds.

THEOREM 6. *Let K be a finite G-equivariant* CW-*complex. Then there is a smooth, compact manifold M with a smooth G-action, and an equivariant embedding $i: K \to M$ into its interior, such that $\pi_1(i)$ and $H_*(i; \mathbf{Z})$ are isomorphisms. Furthermore, M can be chosen such that the isotropy subgroup of any point in M is contained in the isotropy subgroup of some point of K, and such that M^G is a regular neighborhood of K^G.*

Thus, if K is simply connected, M will have the same homotopy type, and if the action on K is fixed-point free, the same will be true of the action on M.

The equivariant tubular neighborhood theorem (see, e.g. Bredon [1, p. 306]) shows that the boundary of any smooth G-manifold M has an invariant neighborhood equivariantly diffeomorphic to $\partial M \times [0, 1)$ (with the fixed action on $[0, 1)$). Thus, the corners which occur when taking the product of two manifolds, a disk bundle over a manifold, or a manifold with a handle attached, all can be smoothed equivariantly.

The following lemma will be needed to prove the theorem:

LEMMA 9. *Given smooth manifolds M^n and N^p, where M is compact and $p \geq 2n+3$, and a continuous function $f_0: M \to N$, there is a homotopy $F: M \times I \to N$ of f_0, such that f_i is a smooth embedding for all $i > 0$.*

Proof. Define $\tilde{F}: M \times (0, 1] \to N$, by $\tilde{F}(m, t) = f_0(m)$. Define the positive function $\delta: M \times (0, 1] \to \mathbf{R}$ by $\delta(m, t) = t$. Fix a metric on N.

$\mathrm{Dim}(N) \geq 2 \cdot \dim(M \times (0, 1]) + 1$, so $\tilde{F}$ can be δ-approximated by a smooth one-to-one immersion $F': H \times (0, 1] \to N$. Set $f_t(m) = F'(m, t)$, then f_t is a smooth embedding ($t>0$) since M is compact. Set $F=f_0 \cup F': H \times [0, 1] \to N$. Then F is continuous and is the required homotopy. $\square$

Theorem 6 will be proven by induction, starting with the subcomplex $K^0 \cup K^G$, where K^0 is the union of the cells $(G/H \times D^0)$. The theorem is true for this subcomplex: embed K^G in some regular neighborhood, and leave alone the components of K not in K^G. Theorem 6 now follows from the following lemma:

LEMMA 10. *Assume K is a finite G-equivariant* CW *complex, with sub-complex L, where $K=L \bigcup_f (G/H \times D^n)$ for some equivariant $f: G/H \times S^{n-1} \to L$, some $H \subsetneq G$. Assume M_0 is a compact manifold with smooth G action, with the embedding $i: L \to M_0$ fulfilling the conclusion of the theorem. Then the theorem holds for K.*

Proof. Let $j: S^{n-1} \to G/H \times S^{n-1}$ be the inclusion map $j(x)=(eH, x)$. One may assume $\dim M_0^H \geqslant 2n+1$ (if not replace M_0 by $M_0 \times D^k \supseteq M_0$). Apply Lemma 1 to the map $ifj: S^{n-1} \to M_0^H$, obtaining the map

$$\alpha: S^{n-1} \times I \to \operatorname{int} M_0^H$$

where $\alpha_0 = ifj$ and α_t is a smooth embedding for $t>0$.

Choose D a disk with a linear action of G, such that some $x \in \partial D$ has isotropy subgroup H. Set $M_1 = M_0 \times D$, let $i_0: M_0 \to M_1$ be the embedding of the zero section, and define $\hat{\alpha}: G/H \times S^{n-1} \times I \to M_1$ by

$$\hat{\alpha}(gH, x, t) = (g \cdot \alpha(x, t), t \cdot gx).$$

Then $(\hat{\alpha} \mid G/H \times S^{n-1} \times 0) = i_0 if$, and the restriction of $\hat{\alpha}$ to $G/H \times S^{n-1} \times (0, 1]$ is embedded in $M_1 - M_0$, with $G/H \times S^{n-1} \times 1$ the inverse image of ∂M_1, and smoothly embedded. Denote that embedding by $\beta: G/H \times S^{n-1} \to \partial M_1$.

Let $i_1: M_1 \to W$ be a smooth, equivariant embedding of M_1 in a linear representation of G; let M_2 be an equivariant tubular neighborhood of M_1 in W. This induces a smooth embedding $\beta' = i_1 \beta: G/H \times S^{n-1} \to \partial M_2$, which restricts to $\beta' j: S^{n-1} \to \partial M_2$.

As H-bundles:

$$W \times S^{n-1} = \tau_W \mid S^{n-1} = \tau(S^{n-1}) \oplus \mathbf{R} \times S^{n-1} \oplus \tau_{eH}(G/H) \times S^{n-1} \\ \oplus \nu_{\partial M_2}(G/H \times S^{n-1}) \big|_{S^{n-1}}$$

Set $V = \mathbf{R}^n \oplus \tau_{eH}(G/H)$ (an H-representation where $\mathbf{R}^n$ has the trivial action); then $W \times S^{n-1} = V \times S^{n-1} \oplus \nu_{\partial M_2}(G/H \times S^{n-1}) \big|_{S^{n-1}}$.

Let $\tilde{V}$ be a real G-representation whose restriction to H contains V as a direct summand: $\tilde{V} = V \oplus V_1$ as H-representations. Set $M_3 = D(\tilde{V}) \times M$; $i_2: M_2 \to M_3$ the embedding $x \to (0, x)$.

As an H-bundle over S^{n-1},

$$\nu_{\partial M_3}(G/H\times S^{n-1})\,|_{S^{n-1}}=\nu_{\partial M_2}(G/H\times S^{n-1})\,|_{S^{n-1}}\oplus(V\oplus V_1)\times S^{n-1}=(W\oplus V_1)\times S^{n-1},$$

and so $\nu_{\partial M_3}(G/H\times S^{n-1})\cong(G\times_H(W\oplus V_1))\times S^{n-1}$. Let $D(W\oplus V_1)$ be the disk representation associated to $W\oplus V_1$, and attach $(G\times_H D(W\oplus V_1))\times D^n$ via this isomorphism to ∂M_3 to get the smooth G-manifold M_4. The embeddings

$$G/H\times D^n\to(G\underset{H}{\times}D(W\oplus V_1))\times D^n\qquad\text{(zero section)}$$
$$i_2i_1\hat{\alpha}\colon G/H\times S^{n-1}\times I\to M_3$$
$$i_2i_1i_0i\colon L\to M_3$$

define an embedding of K into M_4. By the Van Kampen and Meyer Vietoris theorems, this embedding still induces an isomorphism of fundamental groups and integral homology.

The manifold M_3 is a vector bundle over M_0, and therefore any isotropy subgroup of M_3 is contained in one of M_0. The handle $(G\times_H D(W\oplus V_1))\times D^n$ is a bundle over $G/H\times D^n$, and so any of its isotropy subgroups is contained in a conjugate of H.

Since $G\neq H$, $(M_4)^G$ is a disk bundle over $(M_0)^G$, which was assumed to be a regular neighborhood of $L^G=K^G$, so $(M_4)^G$ is a regular neighborhood of K^G. □

COROLLARY. *Assume K a contractible finite G-equivariant* CW *complex. Then G has a smooth action on a disk, any of whose isotropy subgroups is contained in an isotropy subgroup of K.*

Proof. By the theorem, G has a smooth action on some compact contractible manifold M_0, where all isotropy subgroups of M_0 are contained in isotropy subgroups of K. Embed M_0 smoothly in some linear representation of G; let M_1 be the disk bundle of an equivariant tubular neighborhood of M_0. By a theorem of Whitehead [13, p. 298], M_1 is a disk if M_0 was embedded with sufficiently high codimension. Isotropy subgroups of M_1 are contained inside those of M_0. □

Now, Theorem 6 and its corollary may be combined with the results of Section 2, 3, and 4 to classify finite groups having smooth fixed-point free actions on compact acyclic manifolds.

THEOREM 7. *A finite group G has a smooth fixed-point free action on some compact $\mathbf{Z}_p$-acyclic manifold if and only if $G\notin\mathcal{G}_p$. G has a smooth fixed-point free action on a disk if and only if $G\notin\mathcal{G}$. In particular, any non-solvable group has a smooth fixed-point free action on a disk, and an abelian group has such an action if and only if it has three or more non-cyclic Sylow subgroups.*

Proof. By the corollary to Theorem 1, G has a fixed-point free action on a finite $\mathbf{Z}_p$-acyclic complex if and only if $m_p(G)=1$; this occurs if and only if $G \notin \mathscr{G}_p$ by Theorem 4. By the corollary to Theorem 3, G has a fixed-point free action on a finite contractible complex if and only if $n_G=1$; if and only if $G \notin \mathscr{G}$ (by the corollary to Theorem 5). □

Thus, the smallest abelian group with a smooth fixed-point free action on a disk is $\mathbf{Z}_{30} \oplus \mathbf{Z}_{30}$, of order 900. The next theorem will show that the smallest solvable groups with such action have order 72: two such groups are $S_4 \oplus \mathbf{Z}_3$ and $A_4 \oplus S_3$. It thus follows that the smallest group with a smooth fixed-point free action on a disk is A_5 of order 60 (the one compact group previously known [3] to have such an action).

The following notation will be used to simplify the proof of Theorem 8. A finite group will be said to be of type $\langle F_1, \ldots, F_n \rangle$ if there is a normal series

$$0=H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G\,(H_i \lhd G)$$

such that $F_i \cong H_i/H_{i-1}$ for all i. Under this notation, $G \in \mathscr{G}_p^q$ if and only if G is of type $\langle P, \mathbf{Z}_n, Q \rangle$ for some p-group P and some q-group Q. Note that any solvable group is of type $\langle \mathbf{Z}_{p_n}^{\alpha_1}, \ldots, \mathbf{Z}_{p_n}^{\alpha_n} \rangle$ for some sequence of elementary abelian groups.

THEOREM 8. *If G is a finite solvable group of order less than 72, then $G \in \mathscr{G}$.*

Proof. Any group of order $p^\alpha q$ or pqr, for primes p, q, r, is in some $\mathscr{G}_p$ by examination of its composition series. This leaves the cases $|G|=36$ or 60.

If $|G|=36$, G has a normal series all of whose components are elementary p-groups; the only possibilities which do not immediately show $G \in \mathscr{G}_2$ or $\mathscr{G}_3$ are $\langle \mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_2 \rangle$, $\langle \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_3 \rangle$, $\langle \mathbf{Z}_2, \mathbf{Z}_3^2, \mathbf{Z}_2 \rangle$ and $\langle \mathbf{Z}_3, \mathbf{Z}_2^2, \mathbf{Z}_3 \rangle$. Since the only extension of $\mathbf{Z}_2$ by $\mathbf{Z}_3$ is $\mathbf{Z}_6$, the first case reduces to $\langle \mathbf{Z}_3, \mathbf{Z}_6, \mathbf{Z}_2 \rangle$, or $G \in \mathscr{G}_3$, and the second case to $\langle \mathbf{Z}_6, \mathbf{Z}_6 \rangle$, which has the same form as the first case. Similarly, the third case reduces to $\langle \mathbf{Z}_2 \oplus \mathbf{Z}_3^2, \mathbf{Z}_2 \rangle$ or $G \in \mathscr{G}_3$. In the fourth case, either G is of type $\langle \mathbf{Z}_3, \mathbf{Z}_2^2 \oplus \mathbf{Z}_3 \rangle$ (and $G \in \mathscr{G}_3$) or G is of type $\langle \mathbf{Z}_3, A_4 \rangle$. Since A_4 has no subgroup of index 2, $\mathbf{Z}_3$ must be in the center of G. Thus, G is also of the form $\langle \mathbf{Z}_3 \oplus \mathbf{Z}_2^2, \mathbf{Z}_3 \rangle$, and $G \in \mathscr{G}_2$.

If $|G|=60$, there are eight possibilities for the components of a normal series which do not immediately show $G \in \mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_5$. Four of them, $\langle \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{Z}_2 \rangle$, $\langle \mathbf{Z}_2, \mathbf{Z}_5, \mathbf{Z}_3, \mathbf{Z}_2 \rangle$, $\langle \mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_5, \mathbf{Z}_2 \rangle$ and $\langle \mathbf{Z}_5, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_2 \rangle$ imply $G \in \mathscr{G}_2$, $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_5$, respectively, since $\mathbf{Z}_{15}$ is the only group of type $\langle \mathbf{Z}_3, \mathbf{Z}_5 \rangle$ or $\langle \mathbf{Z}_5, \mathbf{Z}_3 \rangle$, etc. If G is of type $\langle \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_5 \rangle$ or $\langle \mathbf{Z}_2, \mathbf{Z}_5, \mathbf{Z}_2, \mathbf{Z}_3 \rangle$, then it is of type $\langle \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_{10} \rangle$ or $\langle \mathbf{Z}_2, \mathbf{Z}_5, \mathbf{Z}_6 \rangle$, cases which were considered above ($G \in \mathscr{G}_2$). A group of type $\langle \mathbf{Z}_3, \mathbf{Z}_2^2, \mathbf{Z}_5 \rangle$ is in $\mathscr{G}_3$, since $\mathbf{Z}_2^2 \oplus \mathbf{Z}_5$ is the only group of type $\langle \mathbf{Z}_2^2, \mathbf{Z}_5 \rangle$. If G is of type $\langle \mathbf{Z}_5, \mathbf{Z}_2^2, \mathbf{Z}_3 \rangle$, then $G/\mathbf{Z}_5 \cong \mathbf{Z}_2^2 \oplus \mathbf{Z}_3$ (so $G \in \mathscr{G}_5$), or $G/\mathbf{Z}_5 \cong A_4$. In this last case G is a semi-direct product; the only homomorphism $A_4 \to \operatorname{Aut}(\mathbf{Z}_5) \cong \mathbf{Z}_4$ is the trivial one, so $G \cong \mathbf{Z}_5 \oplus A_4 \in \mathscr{G}_2$. □

In answering the question of which positive-dimensional compact Lie groups have smooth fixed-point free action on disks, the corollary to Theorem 6 shows that it suffices to construct finite contractible equivariant CW complexes with no fixed points. When the group is nonabelian and connected, the following proposition further simplifies the problem.

PROPOSITION 8. *Assume the compact Lie group G is non-abelian and connected. If there is a fixed-point free finite G-equivariant* CW *complex X which is $\mathbf{Z}_2$-acyclic, then G has a smooth fixed-point free action on a disk.*

Proof. By a theorem of Hsiang and Hsiang [6, p. 366], G has an irreducible representation on $\mathbf{R}^{2k+1}$ for some $k \geqslant 1$. This induces a smooth fixed-point free action of G on the space $\mathbf{R}P^{2k}$.

As mentioned above, Illman [7] shows that a smooth action on a compact manifold has the structure of an equivariant CW complex. Thus, $X * \mathbf{R}P^{3k}$ has the structure of a $(G \times G)$-equivariant CW complex, with no isotopy subgroup containing the diagonal. Since X is $\mathbf{Z}_2$-acyclic and $\mathbf{R}P^{2k}$ is $\mathbf{Z}_p$-acyclic for all odd primes p, $X * \mathbf{R}P^{2k}$ is contractible. The corollary to Theorem 6 applies: $G \times G$ has a smooth action on a disk with no isotopy subgroup containing the diagonal, which thus restricts to a smooth fixed-point free action of G. $\square$

Finally, the following is given as an example that such actions do occur:

COROLLARY. *$SO(3)$ has a smooth fixed-point free action on a disk.*

Proof. By Proposition 8, it will suffice to construct a finite $\mathbf{Z}_2$-acyclic $SO(3)$-equivariant CW complex X without fixed-points. Let $S_4 \subseteq SO(3)$ be the subgroup of matrices with a single non-zero entry in each row and column. Let $O(2) \subseteq SO(3)$ be the subgroup of matrices with a (± 1) in the upper left-hand corner, and set $D_4 = S_4 \cap O(2)$. Let X be the complex constructed by attaching $(SO(3)/D_4) \times I$ to the disjoint union $(SO(3)/S_4) \cup (SO(3)/O(2))$ via the projections induced by inclusions of subgroups.

The Meyer-Vietoris sequence in reduced homology takes the form:

$$\begin{aligned} &\to \tilde{H}_n(SO(3)/D_4) \to \tilde{H}_n(SO(3)/S_4) \oplus \tilde{H}_n(SO(3)/O(2)) \\ &\to \tilde{H}_n(X) \to \tilde{H}_{n-1}(SO(3)/D_4) \to . \end{aligned}$$

To prove that X is $\mathbf{Z}_2$-acyclic, it will suffice to show that the kernel and cokernel of the map

$$\tilde{H}_n(SO(3)/D_4) \to \tilde{H}_n(SO(3)/S_4) \oplus \tilde{H}_n(SO(3)/O(2)) \tag{1}$$

consist of odd torsion for all n. Except in dimensions 1 and 3, all of these groups are zero. Since $SO(3)/O(2) \cong P^3$ is the two-dimensional, and

$$SO(3)/D_4 \to SO(3)/S_4$$

is a three fold cover of orientable three-manifolds, (1) is a monomorphism $\mathbf{Z} \to \mathbf{Z}$ with cokernel $\mathbf{Z}_3$ in dimension 3.

When $n=1$, $\pi_1(SO(3)/D_4)$ is the generalized quaternionic group of order 16, with abelianization $\mathbf{Z}_2^2$. The space $SO(3)/S_4$ has fundamental group the binary octahedral subgroup of S^3, with abelianization $\mathbf{Z}_2$. Thus, (1) takes the form

$$\mathbf{Z}_2^2 \to \mathbf{Z}_2 + \mathbf{Z}_2$$

and a straightforward computation shows it to be an isomorphism. □

REFERENCES

[1] BREDON, *Introduction to compact transformation groups*, (Acad. Press, 1972).
[2] CURTIS and REINER, *Representation theory of finite groups and associative algebras*, (Interscience, 1962).
[3] FLOYD and RICHARDSON, *An action of a finite group on an n-cell without stationary points*, Bull. Amer. Math. Soc. *65* (1959), 73–76.
[4] GREEVER, *Stationary points for finite transformation groups*, Duke Math. Journal *27* (1960, 163–170.
[5] HALL, *A note on solvable groups*, J. London Math. Soc. *3* (1928), 98–105.
[6] HSIANG and HSIANG, *On the compact subgroups of the diffeomorphism groups of Kervaire spheres*, Ann. of Math. *85* (1967), 359–369.
[7] ILLMAN, *Equivariant algebraic topology*, Thesis, Princeton University (1972).
[8] JONES, *The converse to the fixed point theorem of P. A. Smith: I*, Ann. of Math. *94* (1971), 52–68.
[9] OLIVER, *Smooth fixed point free actions of compact Lie groups on disks*, Thesis, Princeton University (1974).
[10] RIM, *Modules over finite groups*, Ann. of Math. *69* (1959), 700–712.
[11] SERRE, *Représentations linéaires des groupes finis*, (Hermann, 1967).
[12] SMITH, *Fixed-points of periodic transformations*, Amer. Math. Soc. Coll. Pub. XXVII (1942), 350–373.
[13] WHITEHEAD, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. *45* (1938–39), 243–327.

Matematisk Institut Universitetsparken
Ny Munkegade
8000 Aarhus, C, Danmark

Received July 31, 1974