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Unstable K-Theories of the Algebraic Closure of a Finite Field

ERIC M. FRIEDLANDER 1)

We show that the Brauer lifting of the standard modular representation of $GL_n(\mathbf{F}_q)$ on $\mathbf{F}^{\oplus n}$ to a virtual complex representation has a curious non-stable version (where \mathbf{F}_q is a finite field of characteristic p>0 with algebraic closure \mathbf{F}). Namely, the map $BGL_n(\mathbf{F}_q) \to BGL_\infty^{top}(\mathbf{C})$ associated to this Brauer lifting factors through a map $BGL_n(\mathbf{F}_q) \to BGL_n^{top}(\mathbf{C})$. This unstable lifting is also achieved for $BSO_n(\mathbf{F}_q)$ and $BSp_{2k}(\mathbf{F}_q)$. Using the induced maps $BGL_n(\mathbf{F})^+ \to BGL_n^{top}(\mathbf{C})$, $BSO_n(\mathbf{F})^+ \to BSp_{2k}^{top}(\mathbf{C})$ we determine the unstable algebraic K-groups $\pi_i(BGL_n(\mathbf{F})^+)$, $\pi_i(BSO_n(\mathbf{F})^+)$, and $\pi_i(BSp_{2k}(\mathbf{F})^+)$ explicitly in terms of the homotopy groups of the corresponding classical groups.

In section 1, we exhibit natural map from $BG_n(F)$ to the prime-to-p pro-finite completion of $BG_n^{top}(C)$, where G_n denotes either GL_n , SL_n , O_n , SO_n or Sp_n (where p = char(F) is odd for O_n and n = 2k for Sp_n). This map induces isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology for l prime to p. Because $BG_n(F)$ is $\mathbb{Z}/p\mathbb{Z}$ -acyclic, this map determines a map $\eta_n: BG_n(F) \to BG_n^{top}(C)$. We verify that the composition $BG_n(F_q) \to BG_n^{top}(C) \to BG_n^{top}(C) \to BG_n^{top}(C)$ corresponds to Brauer lifting.

In section 2, we show that $\pi_i(BG(F)^+)$ is directly computable from $\pi_i(BG_n^{top}(C))$ and $\pi_{i+1}(BG_n^{top}(C))$. More precisely, $\eta_n^+:BG_n(F)^+\to BG_n^{top}(C)$ is shown to be the fibre of localization at p=char(F), $BG_n^{top}(C)\to BG_n^{top}(C)_{(p)}$. As a consequence, the sequences

$$\cdots \rightarrow BG_n(\mathbf{F})^+ \rightarrow BG_{n+1}(\mathbf{F})^+ \rightarrow BG_{n+2}(\mathbf{F})^+ \rightarrow \cdots$$

are seen to be "intrinsic spherical fibrations" with fibres prime-to-p torsion spheres. We gratefully acknowledge several valuable conversations with G. Lusztig.

1. Unstable Brauer Lifting

We let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with n=2k for Sp_n), so that $G_{n,R}$ is the corresponding group scheme defined over Spec R for any (commutative with identity) ring R. We denote by $G_n^{top}(\mathbb{C})$ the corresponding classical topological

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group. In the orthogonal case, we recall that $O_{n,R}$ is the algebraic sub-group scheme of $GL_{n,R}$ preserving the quadratic form $X_1X_{k+1} + \cdots + X_kX_{2k}$ if n=2k and $X_1X_{k+1} + \cdots + X_kX_{2k} + X_{2k+1}^2$ if n=2k+1; $SO_{n,R}$ is the sub-group scheme of $GL_{n,R}$ defined over $Spec \mathbb{Z}$ as the reduction of the connected component of the identity of $O_{n,R}$.

We fix a prime p and let \mathbf{F} denote the algebraic closure of the finite field \mathbf{F}_p . For any power of p, $q=p^d$, we let \mathbf{F}_q denote the subfield of \mathbf{F} with q elements. If p=2, we exclude the case $\mathbf{G}_n=\mathbf{O}_n$ so that $\mathbf{G}_{n,\mathbf{F}}$ is an algebraic group. We denote by $W\{\mathbf{G}_{n,\mathbf{F}}\}$ the simplicial algebraic variety with $(W\{\mathbf{G}_{n,\mathbf{F}}\})_k=(\mathbf{G}_{n,\mathbf{F}})^{\times k}$ and with face and degeneracy maps obtained by deleting and inserting a factor; we let $\overline{W}\{\mathbf{G}_{n,\mathbf{F}}\}$ denote $W\{\mathbf{G}_{n,\mathbf{F}}\}/\mathbf{G}_{n,\mathbf{F}}$.

In [1], the rigid etale homotopy type of a noetherian scheme or noetherian simplicial scheme was introduced (denoted by $()_{ret}$), and the following was proved:

- a) If $H \subset G_{n,F}$ is a finite algebraic subgroup, then $(W\{G_{n,F}\}/H)_{ret}$ is naturally homotopy equivalent to BH, the classifying space of H viewed as a discrete group.
- b) If ()^A denotes pro-finite completion prime-to-char (F) and if $G_{n,F}$ is connected, then $(\overline{W}\{G_{n,F}\}_{ret})^A$ is weakly homotopy equivalent to $(BG_n^{top}(C))^A$ via maps dependent only on a choice of embedding of the Witt vectors of F into C.

We can actually verify that the maps determining the weak equivalence of b) depend only on a choice of embedding of F^* into C^* (using the fact that these maps are determined by their effect on cohomology and reducing to the case n=1).

In the case of $O_{n,F}$ and char (F) odd, property b) above remains valid: the natural maps relating $\overline{W}\{G_{n,F}\}_{ret}$ and $BG_n^{top}(C)$ exist even if G_n is not connected; moreover, $W\{O_{n,F}\}/SO_{n,F} \to W\{O_{n,F}\}/O_{n,F} = \overline{W}\{O_{n,F}\}$ is a double covering and $(W\{SO_{n,F}\}/\times/SO_{n,F})_{ret} \to (W\{O_{n,F}\}/SO_{n,F})_{ret}$ is a weak equivalence.

Properties a) and b) enable us to define

$$\chi_{n,q} : \mathrm{BG}_n(\mathbf{F}_q) \to \lim_{\longleftarrow} (\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}$$
(1.1)

where $G_n(F_q)$ is the discrete group of points of $G_{n,F}$ rational over F_q and where $\lim_{n \to \infty} ($

is the inverse limit of an inverse system in the homotopy category of spaces with finite homotopy groups as in [4]. Namely, to obtain $\chi_{n,q}$, compose the maps

$$(W\{G_{n,F}\}/G_n(F_q))_{ret} \to (W\{G_{n,F}\}/G_{n,F})_{ret} = \overline{W}\{G_{n,F}\}_{ret}$$

$$\overline{W}\{G_{n,F}\}_{ret} \to \lim_{\longleftarrow} (\overline{W}\{G_{n,F}\}_{ret})^A,$$

and the homotopy equivalence

$$\lim_{\longleftarrow} (\overline{W} \{G_{n, F}\}_{ret})^{\Lambda} \to \lim_{\longleftarrow} (BG_n^{top} (C))^{\Lambda}$$

induced by the weak homotopy equivalence of b).

The following proposition will enable us to compute the Z/lZ cohomology of the discrete group $G_n(F) = \lim_{n \to \infty} G_n(F_q)$.

PROPOSITION 1.2. Let $G_{n,F}$ denote one of the following connected algebraic groups: $GL_{n,F}$, $SL_{n,F}$, $SO_{n,F}$, or $Sp_{n,F}$ (with n=2k for Sp_n). Let $\phi^q: G_{n,F} \to G_{n,F}$ denote the geometric frobenius defined over F_q , with q a power of $p=\operatorname{char}(F)$. Let l be an integer not divisible by p. There exists an integer d>1 such that the map

$$1 \cdot \phi^q \cdots \phi^{q^{d-1}} : G_{n,F} \to G_{n,F}$$

defined as $\mu \circ (1 \times \phi^q \times \cdots \times \phi^{q^{d-1}}) : G_{n,F} \to (G_{n,F})^{\times d} \to G_{n,F}$ induces the zero map in $\mathbb{Z}/l\mathbb{Z}$ cohomology.

Proof. We first assume that $G_{n,F} = GL_{n,F}$. Recall that

$$(1 + \Psi^q + \dots + \Psi^{q^{d-1}})^*: H^{2k}(BGL^{top}_{\infty}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z}) \to H^{2k}(BGL^{top}_{\infty}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z})$$

where $+: \mathrm{BGL}_{\infty}^{\mathrm{top}}(\mathbf{C}) \times \mathrm{BGL}_{\infty}^{\mathrm{top}}(\mathbf{C}) \to \mathrm{BGL}_{\infty}^{\mathrm{top}}(\mathbf{C})$ is induced by tensor product of line bundles, is multiplication by $(1+q+\cdots+q^{d-1})^k$. Let d be the order of q in $(Z/l^{e+1}Z)^*$, where e is the exponent of l in q-1. Then l divides $1+q+\cdots+q^{d-1}$ so that $(1+\Psi^q+\cdots+\Psi^{q^{d-1}})^*$ is the zero map. Since the generators of the exterior algebra $H^*(\mathrm{GL}_{\infty}^{\mathrm{top}}(\mathbf{C}), Z/lZ)$ totally transgress to the generators of the polynomial algebra $H^*(\mathrm{BGL}_{\infty}^{\mathrm{top}}(\mathbf{C}), Z/lZ)$, we conclude that

$$\Omega\left(1+\Psi^{q}+\cdots+\Psi^{q^{d-1}}\right)^{*}:H^{*}\left(\mathrm{GL}_{\infty}^{\mathrm{top}}\left(\mathbf{C}\right),Z/lZ\right)\to H^{*}\left(\mathrm{GL}_{\infty}^{\mathrm{top}}\left(\mathbf{C}\right),Z/lZ\right)$$

is the zero map.

We claim that the following diagram determines a commutative square in Z/lZ cohomology:

$$(GL_{n, \mathbf{F}})_{ret} \xrightarrow{1 \cdot \phi^{q} \dots \phi^{q^{d-1}}} (GL_{n, \mathbf{F}})_{ret}$$

$$\downarrow \theta \qquad \qquad \downarrow \theta$$

$$GL_{\infty}^{top}(\mathbf{C})^{\Lambda} \xrightarrow{\Omega(1 + \Psi^{q} + \dots + \Psi^{q^{d-1}})} GL_{\infty}^{top}(\mathbf{C})^{\Lambda}$$

$$(1.2.1)$$

where θ is determined by maps $(GL_{n,F})_{ret} \to (GL_{n,Witt(F)})_{ret}$, $(GL_{n,C})_{ret} \to \times$ $(GL_{n,Witt(F)})_{ret}$, $GL_n^{top}(C) \to (GL_n,C)_{ret}$, and $GL_n^{top}(C) \to GL_\infty^{top}(C)$; and where ()^A is pro-finite, prime-to-p completion. We verify that (1.2.1) induces a commutative square in Z/lZ cohomology by observing that each of the maps above is induced by a homomorphism of groups in an appropriate category so that θ^A likewise commutes with the group structures on $((GL_{n,F})_{ret})^A$ and $GL_\infty^{top}(C)^A$; and by observing that $(\theta \circ \phi^q)^* = (\Omega(\Psi^q)^A \circ \theta)^*$, as can be seen by reducing to the case n=1 since $(BGL_n^{top}(C))^{\times n} \to BGL_n^{top}(C)$ induces an injection in cohomology. Therefore, the

proposition follows for $G_{n,F} = GL_{n,F}$, by observing that θ induces a surjection in Z/lZ cohomology, since $\lim_{\leftarrow} (GL_{n,F})_{ret}^{\Lambda} \to \lim_{\leftarrow} (GL_{n}^{top}(C))^{\Lambda}$ is a homotopy equivalence.

We immediately conclude the proposition for $G_{n,F} = SL_{n,F}$ or $Sp_{n,F}$ (with n = 2k for Sp), or for $G_{n,F}$ and l odd since $SL_{n,F} \to GL_{n,F}$, $Sp_{n,F} \to GL_{n,F}$, and $SO_{n,F} \to GL_{n,F}$ are group homomorphisms commuting with frobenius and inducing surjections in Z/lZ cohomology (with l odd for $SO_{n,F}$).

For p odd and l a power of 2, we observe that $(BO_1^{top}(\mathbb{C}))^{\times n} \to BO_n^{top}(\mathbb{C})$ induces injections in $\mathbb{Z}/l\mathbb{Z}$ cohomology and Ψ^q acts trivially on $BO_1^{top}(\mathbb{C})$. Since $H^*(BO_n^{top}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z}) \to H^*(BSO_n^{top}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z})$ is surjective, we conclude that ϕ^q acts trivially on $H^*(SO_{n,\mathbb{F}}, \mathbb{Z}/l\mathbb{Z})$. Hence, if l divides d, $(1 + \phi^q + \cdots + \phi^{q^{d-1}})^*$ is the zero map.

PROPOSITION 1.3. Let $G_{n,F}$ denote one of the following algebraic groups: $GL_{n,F}$, $SL_{n,F}$, $O_{n,F}$ (char(F) odd), $SO_{n,F}$, or $Sp_{n,F}$ (with n=2k for Sp_n). Let $G_n(k)$ denote the discrete group of points of $G_{n,F}$ rational over k/F_F , where p=char(F). Then the direct limit of the maps

$$\chi_{n, q} \colon \mathbf{BG}_{n}(\mathbf{F}_{q}) \to \lim_{\leftarrow} \mathbf{BG}_{n}^{\mathsf{top}}(\mathbf{C}))^{\Lambda}$$
of (1.1),
$$\chi_{n} \colon \mathbf{BG}_{n}(\mathbf{F}) \to \lim_{\leftarrow} (\mathbf{BG}_{n}^{\mathsf{top}}(\mathbf{C}))^{\Lambda}$$

induce isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology for all l relatively prime to p, where $\binom{n}{l}$ denotes pro-finite, prime-to-p completion.

Proof. Since $O_1(F) = O_1^{top}(C) = \mathbb{Z}/2\mathbb{Z}$ for p odd, the proposition for $G_{n,F} = O_{n,F}$ easily follows from the theorem for $G_{n,F} = SO_{n,F}$. Hence, we may assume $G_{n,F}$ to be connected.

Consider the following commutative diagram

$$G_{n, \mathbf{F}}/G_{n}(\mathbf{F}_{q}) \rightarrow G_{n, \mathbf{F}}/G_{n}(\mathbf{F}_{q'})$$

$$\downarrow \qquad \qquad \downarrow$$

$$W\{G_{n, \mathbf{F}}\}/G_{n}(\mathbf{F}_{q}) \rightarrow W\{G_{n, \mathbf{F}}\}/G_{n}(\mathbf{F}_{q'})$$

$$\downarrow \qquad \qquad \downarrow$$

$$W\{G_{n, \mathbf{F}}\} = W\{G_{n, \mathbf{F}}\}$$

$$(1.3.1)$$

where the top arrow fits in a commutative square

$$\begin{array}{ccc}
G_{n, F}/G_{n}(F_{q}) & \longrightarrow & G_{n, F}/G_{n}(F_{q'}) \\
\downarrow^{1/\phi^{q}} & & \downarrow^{1/\phi^{q'}} \\
G_{n, F} & \xrightarrow{1 \cdot \phi^{q} \dots \phi^{q'/q}} & G_{n, F}
\end{array}$$

where q' is a power of q. By a basic result of [1] (Corollary 2.6), (1.3.1) determines a map of "Serre spectral sequences"

$$E_{2}^{s,t}(\mathbf{F}_{q'}) = H^{s}(\overline{W}\{\mathbf{G}_{n,\mathbf{F}}\}_{ret}, H^{t}((\mathbf{G}_{n,\mathbf{F}}/\mathbf{G}_{n}(\mathbf{F}_{q'}))_{ret}, Z/lZ))$$

$$\Rightarrow H^{s+t}(\mathbf{B}\mathbf{G}_{n}(\mathbf{F}_{q'}), Z/lZ)$$

$$E_{2}^{s,t}(\mathbf{F}_{q}) = H^{s}(\overline{W}\{\mathbf{G}_{n,\mathbf{F}}\}_{ret}, H^{t}((\mathbf{G}_{n,\mathbf{F}}/\mathbf{G}_{n}(\mathbf{F}_{q}))_{ret}, Z/lZ))$$

$$\Rightarrow H^{s+t}(\mathbf{B}\mathbf{G}_{n}(\mathbf{F}_{q}), Z/lZ) \qquad (1.3.3)$$

where we have used the homotopy equivalence between $BG_n(\mathbf{F}_q)$ and $(W\{G_{n,\mathbf{F}}\}/G_n(\mathbf{F}_q))_{ret}$. Because $E_2^{s,t}(\mathbf{F}_q)$ is finite for all s,t and all powers q of p, (1.3.3) determines an inverse limit spectral sequence

$$E_2^{s,t} = \lim_{\leftarrow} H^s(\overline{W}\{G_{n,F}\}_{ret}, H^t(G_{n,F}/G_n(F_q), Z/lZ)) \Rightarrow \lim_{\leftarrow} H^{s+t}(BG_n(F_q), Z/lZ).$$

By Proposition 1.2 and the fact that the vertical arrows of (1.3.2) are isomorphisms (the "Lang isomorphisms" of [1]), there exists d>1 such that $E_2^{s,t}(\mathbf{F}_{q'}) \to E_2^{s,t}(\mathbf{F}_q)$ has image 0 for t>0, any s, and $q'=q^d$. Moreover,

$$H^{s+t}(\mathrm{BG}_n(\mathrm{F}), \mathbb{Z}/l\mathbb{Z}) \cong \lim_{\leftarrow} H^{s+t}(\mathrm{BG}_n(\mathrm{F}_q), \mathbb{Z}/l\mathbb{Z}).$$

The proposition now follows from the degeneration of the spectral sequence $E^{s,t}$ and the fact that $(\overline{W}\{G_{n,F}\})_{ret} \to \lim_{\leftarrow} (\overline{W}\{G_{n,F}\}_{ret})$ induces isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology (since

$$H^*\left(\lim_{L\to\infty}\left(\mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C})\right)^A, Z/lZ\right) \cong H^*\left(\mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C}), Z/lZ\right) \quad [4]\right).$$

We record for future use the following proposition.

PROPOSITION 1.4. Let $G_n(F)$ denote the discrete group of rational points of $G_{n,F}$, where $G_{n,F}$ denotes either $GL_{n,F}$, $SL_{n,F}$, $O_{n,F}$, $SO_{n,F}$, or $Sp_{n,F}$ (with char (F)=p odd for O_n and n=2k for Sp_n). Then, for every i>0,

$$H^{i}(BG_{n}(\mathbf{F}), \mathbb{Z}/p\mathbb{Z}) = 0 = H^{i}(BG_{n}(\mathbf{F}), \mathbf{Q}).$$

Proof. For \mathbb{Q} coefficients, the proposition is an easy consequence of the fact that $\mathrm{BG}_n(\mathbb{F})$ is the direct limit of classifying spaces of finite groups. For $\mathbb{Z}/p\mathbb{Z}$ coefficients, the proposition is the stable version of the vanishing theorem given in [3]; a more elementary proof is carried out in detail in [1] for all cases except $\mathrm{SO}_n(\mathbb{F})$ with n odd or p=2.

In the following theorem, we see that χ_n of Proposition 1.3 provides a sharper form of Brauer lifting as generalized by Quillen to orthogonal and symplectic representations [2].

THEOREM 1.5. Let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with p = char(F) odd for O_n and n = 2k for Sp_n). The maps χ_n of Proposition 1.3 determine maps (uniquely up to homotopy)

$$\eta_n: \mathrm{BG}_n(\mathbf{F}) \to \mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})$$

which induce isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology for (l, p) = 1. Furthermore, the composition of η_n with the natural inclusions

$$BG_n(F_a) \rightarrow BG_n(F) \rightarrow BG_n^{top}(C) \rightarrow BG_{\infty}^{top}(C)$$

corresponds to the virtual complex bundle on $BG_n(\mathbf{F}_q)$ obtained as the Brauer lifting of the standard modular representation of $G_n(\mathbf{F}_q)$ on $F^{\oplus n}$ for the embedding $\mathbf{F}^* \to \mathbf{C}^*$ chosen for χ_n .

Proof. By Proposition 1.4, $H_i(BG_n(F), Z)$ is a prime-to-p torsion abelian group for i>0. If H is such a prime-to-p torsion abelian group and A is a finitely generated abelian group, then

$$\operatorname{Hom}(H, A) = \operatorname{Hom}(H, \operatorname{P-p}(A)) = \operatorname{Hom}(H, \lim_{\longleftarrow} (A)^{A})$$

where $p_{-p}(A)$ is the prime-to-p torsion subgroup of A and $\lim_{\leftarrow} (A)^A$ is the discrete group given as the inverse limit of the prime-to-p completion of A. Moreover,

$$\operatorname{Ext}_{Z}^{1}(H, A) \cong \operatorname{Ext}_{Z}^{1}(H, \lim_{\longleftarrow} (A)^{A})$$

since
$$\operatorname{Ext}_{\mathbf{Z}}^{1}(H, Z) = \operatorname{Hom}(H, \mathbf{Q}/Z) = \operatorname{Hom}(H, \bigoplus_{l \neq p} (\mathbf{Q}_{l}/Z_{l})) = \operatorname{Ext}_{\mathbf{Z}}^{1}(H, \lim_{\leftarrow} (Z^{\Lambda})).$$

Since

$$\lim_{\longleftarrow} (\pi_i(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}) \cong \pi_i(\lim_{\longleftarrow} (\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda})$$

we conclude that for all i, j > 0

$$H^{i}(\mathrm{BG}_{n}(\mathbf{F}), \pi_{j}(\mathrm{BG}_{n}^{\mathrm{top}}(\mathbf{C}))) \simeq H^{i}(\mathrm{BG}_{n}(\mathbf{F}), \pi_{j}(\underline{\lim}(\mathrm{BG}_{n}^{\mathrm{top}}(\mathbf{C}))^{A})).$$
 (1.5.1)

The existence and uniqueness of η_n is then obtained by inductively working up the Postnikov towers of $BG_n^{top}(C)$ and $\lim_{\leftarrow} (BG_n^{top}(C))^A$, since (1.5.1) implies that χ_n determines maps into the Postnikov truncations of $BG_n^{top}(C)$ which are homotopy compatible via homotopy compatible homotopies. Since $BG_n^{top}(C) \to \lim_{\leftarrow} (BG_n^{top}(C))^A$ induces isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology for (l, p) = 1, so does η_n by Proposition 1.3.

To prove the second assertion comparing η_n with Brauer lifting, it suffices to consider $G_n = GL_n$, since $G_n \to GL_n$ induces homotopy commutative squares

$$\begin{array}{ccc} BG_n(F) & \xrightarrow{\eta_n} & BG_n^{top}(C) \\ \downarrow & & \downarrow \\ BGL_n(F) & \xrightarrow{\eta_n} & BGL_n^{top}(C) \end{array}$$

and since $[BG_n(F_q), BG_{\infty}^{top}(C)] \rightarrow [BG_n(F_q), BGL_{\infty}^{top}(C)]$ is injective ([2], 5.1.6). Let η'_n denote the composition $BGL_n(F) \rightarrow BGL_n^{top}(C) \rightarrow BGL_{\infty}^{top}(C)$ and let β_n denote the map $BGL_n(F) \rightarrow BGL_{\infty}^{top}(C)$ determined by Brauer lifting. One readily verifies that η'_1 and β_1 are induced by the chosen embedding

$$F^* = GL_1(F) \to GL_1^{top}(C) = C^*$$
.

Since $B(GL_1(F)^{\times n}) \to BGL_n(F)$ induces an injection in $\mathbb{Z}/l\mathbb{Z}$ cohomology by Proposition 1.3 for (l, p) = 1, we conclude

$$(\eta'_n)^* = \beta_n^* : H^* (BGL_\infty^{top}(C), \mathbb{Z}/l\mathbb{Z}) \to H^* (BGL_n(F)\mathbb{Z}/l\mathbb{Z}).$$

Since $H^*(BGL_n(F), Z_l)$ has no torsion for any prime $l \neq p$ by Proposition 1.3, the compositions

$$BGL_{n}\left(\mathbf{F}\right) \overset{\rightarrow}{\rightarrow} BGL_{\infty}^{top}\left(\mathbf{C}\right) \rightarrow \left(\lim_{\leftarrow} \left(BGL_{\infty}^{top}\left(\mathbf{C}\right)\right)^{\widehat{l}}\right)^{(i)}$$

determined by η'_n and β_n are therefore homotopic for all i, where $\binom{1}{i}$ denotes pro-l completion and $\binom{1}{i}$ denotes the i-th Postnikov truncation. As argued above for the existence and uniqueness of η_n , η'_n and β_n thus determine homotopic maps from $\mathrm{BGL}_n(\mathbf{F})$ to $(\mathrm{BGL}_\infty^{\mathrm{top}}(\mathbf{C}))^{(i)}$ for all i. This implies that the compositions

$$BGL_n(\mathbf{F}_q) \to BGL_n(\mathbf{F}) \xrightarrow{\longrightarrow} BGL_{\infty}^{top}(\mathbf{C})$$

determined by η'_n and B_n are homotopic as asserted, since the groups $H^i(BGL_n(\mathbb{F}_q), \pi_j(BGL_{\infty}^{top}(\mathbb{C}))$ are finite for all i, j > 0 so that maps from $BGL_n(\mathbb{F}_q)$ into $BGL_{\infty}^{top}(\mathbb{C})$ are determined by maps into the Postnikov tower of $BGL_{\infty}^{top}(\mathbb{C})$.

2. P-p Torsion Completion and Unstable K-theories

As in section 1, we fix a prime p and let \mathbf{F} denote the algebraic closure of \mathbf{F}_p . We let $\mathbf{P} - p$ denote the set of all primes except p.

DEFINITION 2.1. Let X be a connected nilpotent space with $\pi_1(X)$ a P-p

torsion abelian group. Then the P-p torsion completion of X,

$$t: \tau_{\mathbf{P}-p}(X) \to X$$

is the homotopy fibre of localization at $p, X \rightarrow X_{(p)}$.

Using obstruction theory, one can readily verify that any map from a CW complex Z with P-p torsion homology groups to X factors uniquely (up to homotopy) through $t:\tau_{P-p}(X)\to X$ (the obstruction to lifting a map "up along the Postnikov tower", the obstruction to lifting a homotopy between two such liftings, and the obstruction to lifting a homotopy between two homotopies all vanish), if the torsion subgroup $_{P-p}(\pi_i(X))$ is a direct summand of $\pi_i(X)$ for all i>0.

Our principal result is the following theorem.

THEOREM 2.2. Let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with p = char(F) odd for O_n and n = 2k for Sp_n). Let $BG_n(F) \to BG_n(F)^+$ denote Quillen's plus construction with respect to the commutator subgroup $[G_n(F), G_n(F)]$ of $\pi_1(BG_n(F)) = G_n(F)$. Then the map

$$\eta_n^+: \mathrm{BG}_n(\mathbf{F})^+ \to \mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})$$

induced by η_n of Theorem 1.5 is the **P**-p torsion completion of $BG_n^{top}(\mathbb{C})$.

Proof. Since $H_i(BG_n(F)) \cong H_i(BG_n(F)^+)$ is P-p torsion for i>0 by Proposition 1.4, η_n^+ factors uniquely as $\eta_n^+ = t \circ \varrho_n^+$, where t is P-p torsion completion and

$$\varrho_n^+$$
: $BG_n(\mathbf{F})^+ \to \tau_{\mathbf{P}-p}(BG_n^{top}(\mathbf{C}))$.

By Theorem 1.5, η_n^+ induces isomorphisms in Z/lZ cohomology for (l, p) = 1. Moreover, t induces isomorphisms in Z/lZ cohomology because the Serre spectral sequence for the fibration

$$\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})) \to \mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}) \to \mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})_{(p)} \tag{2.2.1}$$

and Z/lZ coefficients degenerates and $\pi_1(BG_n^{top}(C)_{(p)})=0$. Therefore ϱ_n^+ induces isomorphisms in Z/lZ cohomology so that

$$(\varrho_n^+)_*: H_*(\mathrm{BG}_n(\mathbf{F})^+) \cong H_*(\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))). \tag{2.2.2}$$

(for i>0, $H_i(\tau_{\mathbf{P}-p}(\mathbf{BG}_n^{\mathsf{top}}(\mathbf{C})))$ is $\mathbf{P}-p$ torsion by the degeneration of the Serre spectral sequence for (2.2.1) with $\mathbb{Z}/p\mathbb{Z}$ and \mathbb{Q} coefficients).

For G_n equal to SL_n , SO_n , or Sp_n , (2.2.2) plus the Whitehead theorem imply that ϱ_n^+ is a homotopy equivalence. Moreover, (2.2.2) implies that ϱ_n^+ induces isomorphisms of fundamental groups for G_n equal to GL_n or O_n . Since the map on universal cover-

ings induced by ϱ_n^+ for $G_n = GL_n$ (respectively, $G_n = O_n$) is ϱ_n^+ for SL_n (resp., SO_n), ϱ_n^+ is also a homotopy equivalence for G_n equal to GL_n or O_n .

The fact that localization of spaces localizes homotopy groups enables us to immediately derive the following evaluation of unstable algebraic K-groups.

COROLLARY 2.3. Let G_n be as in Theorem 2.2. Then

$$\pi_{i}(\mathrm{BG}_{n}(\mathbf{F})^{+}) \simeq_{\mathbf{P}-p}(\pi_{i}(\mathrm{BG}_{n}^{\mathrm{top}}(\mathbf{C}))) \oplus (\pi_{i+1}(\mathrm{BG}_{n}^{\mathrm{top}}(\mathbf{C})) \otimes \bigoplus_{l \neq p} (\mathbf{Q}_{l}/Z_{l})).$$

We conclude with the following proposition asserting that the sequences

$$\cdots \rightarrow BG_n(F)^+ \rightarrow BG_{n+1}(F)^+ \rightarrow BG_{n+2}(F)^+ \rightarrow \cdots$$

determine "intrinsic" P-p torsion spherical fibrations in the sense of Sullivan ([4]). For notational simplicity, we state and prove the proposition for $G_n = GL_n$.

PROPOSITION 2.4. The homotopy fibre of

$$i_n: BGL_n(\mathbf{F})^+ \to BGL_{n+1}(\mathbf{F})^+$$

is $\tau_{\mathbf{P}-p}(S^{2n+1})$. Moreover, the pullback of the $\tau_{\mathbf{P}-p}(S^{2n+3})$ fibration i_{n+1} via the map $i_{n+1} \circ i_n$ is fibre homotopy equivalent to the fibre-wise join of i_{n-1} and the trivial $\tau_{\mathbf{P}-p}(S^3)$ fibration over $\mathrm{BGL}_n(\mathbf{F})^+$.

Proof. Because localization is an exact functor on abelian groups, localization preserves fibrations with simply connected base and total space. Thus $\tau_{\mathbf{P}-p}$ () also preserves such fibrations, implying that the fibre of $i_n: \mathrm{BGL}_n(\mathbf{F})^+ \to \mathrm{BGL}_{n+1}(\mathbf{F})^+$ is $\tau_{\mathbf{P}-p}(S^{2n+1})$.

Let S^i , S^j be spheres of dimension i, j>1. Any representatives of $\mathbf{P}-p$ torsion completions

$$\tau_{\mathbf{P}^{-p}}(S^i) \to S^i, \qquad \tau_{\mathbf{P}^{-p}}(S^j) \to S^j$$

determine a map of joins

$$\phi: \tau_{\mathbf{P}-p}(S^i) * \tau_{\mathbf{P}-p}(S^j) \to \tau_{\mathbf{P}-p}(S^i * S^j).$$

Using the equality of excisive pairs, $(CS^i, S^i) \times (CS^j, S^j) = (CS^i \times CS^j, S^i * S^j)$, we conclude that ϕ induces isomorphisms in Z/lZ cohomology. Thus, ϕ is a homotopy equivalence. Consequently, $\tau_{\mathbf{P}-p}(\)$ preserves the fibre-wise join of sphere fibrations over a simply connected base. Since $\tau_{\mathbf{P}-p}(\)$ preserves fibrations and thus homotopy theoretic fibre products, the second assertion now follows from the well known corresponding "intrinsic" property of the sphere fibrations $BG_n^{top}(C) \to BG_{n+1}^{top}(C)$.

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