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Unstable K -Theories of the Algebraic Closure of a Finite Field

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We show that the Brauer lifting of the standard modular representation of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}^{\oplus n}$ to a virtual complex representation has a curious non-stable version (where \mathbb{F}_q is a finite field of characteristic $p > 0$ with algebraic closure \mathbb{F}). Namely, the map $BGL_n(\mathbb{F}_q) \rightarrow BGL_n^{\text{top}}(\mathbb{C})$ associated to this Brauer lifting factors through a map $BGL_n(\mathbb{F}_q) \rightarrow BGL_n^{\text{top}}(\mathbb{C})$. This unstable lifting is also achieved for $BSO_n(\mathbb{F}_q)$ and $BSp_{2k}(\mathbb{F}_q)$. Using the induced maps $BGL_n(\mathbb{F})^+ \rightarrow BGL_n^{\text{top}}(\mathbb{C})$, $BSO_n(\mathbb{F})^+ \rightarrow BSO_n^{\text{top}}(\mathbb{C})$, $BSp_{2k}(\mathbb{F})^+ \rightarrow BSp_{2k}^{\text{top}}(\mathbb{C})$ we determine the unstable algebraic K -groups $\pi_i(BGL_n(\mathbb{F})^+)$, $\pi_i(BSO_n(\mathbb{F})^+)$, and $\pi_i(BSp_{2k}(\mathbb{F})^+)$ explicitly in terms of the homotopy groups of the corresponding classical groups.

In section 1, we exhibit natural map from $BG_n(\mathbb{F})$ to the prime-to- p pro-finite completion of $BG_n^{\text{top}}(\mathbb{C})$, where G_n denotes either GL_n , SL_n , O_n , SO_n or Sp_n (where $p = \text{char}(\mathbb{F})$ is odd for O_n and $n = 2k$ for Sp_n). This map induces isomorphisms in $\mathbb{Z}/l\mathbb{Z}$ cohomology for l prime to p . Because $BG_n(\mathbb{F})$ is $\mathbb{Z}/p\mathbb{Z}$ -acyclic, this map determines a map $\eta_n: BG_n(\mathbb{F}) \rightarrow BG_n^{\text{top}}(\mathbb{C})$. We verify that the composition $BG_n(\mathbb{F}_q) \rightarrow BG_n(\mathbb{F}) \rightarrow BG_n^{\text{top}}(\mathbb{C}) \rightarrow BG_n^{\text{top}}(\mathbb{C})$ corresponds to Brauer lifting.

In section 2, we show that $\pi_i(BG(\mathbb{F})^+)$ is directly computable from $\pi_i(BG_n^{\text{top}}(\mathbb{C}))$ and $\pi_{i+1}(BG_n^{\text{top}}(\mathbb{C}))$. More precisely, $\eta_n^+: BG_n(\mathbb{F})^+ \rightarrow BG_n^{\text{top}}(\mathbb{C})$ is shown to be the fibre of localization at $p = \text{char}(\mathbb{F})$, $BG_n^{\text{top}}(\mathbb{C}) \rightarrow BG_n^{\text{top}}(\mathbb{C})_{(p)}$. As a consequence, the sequences

$$\dots \rightarrow BG_n(\mathbb{F})^+ \rightarrow BG_{n+1}(\mathbb{F})^+ \rightarrow BG_{n+2}(\mathbb{F})^+ \rightarrow \dots$$

are seen to be “intrinsic spherical fibrations” with fibres prime-to- p torsion spheres.

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1. Unstable Brauer Lifting

We let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with $n = 2k$ for Sp_n), so that $G_{n,R}$ is the corresponding group scheme defined over $\text{Spec} R$ for any (commutative with identity) ring R . We denote by $G_n^{\text{top}}(\mathbb{C})$ the corresponding classical topological

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group. In the orthogonal case, we recall that $O_{n,R}$ is the algebraic sub-group scheme of $GL_{n,R}$ preserving the quadratic form $X_1X_{k+1} + \dots + X_kX_{2k}$ if $n=2k$ and $X_1X_{k+1} + \dots + X_kX_{2k} + X_{2k+1}^2$ if $n=2k+1$; $SO_{n,R}$ is the sub-group scheme of $GL_{n,R}$ defined over $\text{Spec } \mathbf{Z}$ as the reduction of the connected component of the identity of $O_{n,R}$.

We fix a prime p and let \mathbf{F} denote the algebraic closure of the finite field \mathbf{F}_p . For any power of p , $q=p^d$, we let \mathbf{F}_q denote the subfield of \mathbf{F} with q elements. If $p=2$, we exclude the case $G_n=O_n$ so that $G_{n,\mathbf{F}}$ is an algebraic group. We denote by $W\{G_{n,\mathbf{F}}\}$ the simplicial algebraic variety with $(W\{G_{n,\mathbf{F}}\})_k = (G_{n,\mathbf{F}})^{\times k}$ and with face and degeneracy maps obtained by deleting and inserting a factor; we let $\bar{W}\{G_{n,\mathbf{F}}\}$ denote $W\{G_{n,\mathbf{F}}\}/G_{n,\mathbf{F}}$.

In [1], the rigid etale homotopy type of a noetherian scheme or noetherian simplicial scheme was introduced (denoted by $(\)_{\text{ret}}$), and the following was proved:

- a) If $H \subset G_{n,\mathbf{F}}$ is a finite algebraic subgroup, then $(W\{G_{n,\mathbf{F}}\}/H)_{\text{ret}}$ is naturally homotopy equivalent to BH , the classifying space of H viewed as a discrete group.
- b) If $(\)^A$ denotes pro-finite completion prime-to-char(\mathbf{F}) and if $G_{n,\mathbf{F}}$ is connected, then $(\bar{W}\{G_{n,\mathbf{F}}\})_{\text{ret}}^A$ is weakly homotopy equivalent to $(BG_n^{\text{top}}(\mathbf{C}))^A$ via maps dependent only on a choice of embedding of the Witt vectors of \mathbf{F} into \mathbf{C} .

We can actually verify that the maps determining the weak equivalence of b) depend only on a choice of embedding of \mathbf{F}^* into \mathbf{C}^* (using the fact that these maps are determined by their effect on cohomology and reducing to the case $n=1$).

In the case of $O_{n,\mathbf{F}}$ and char(\mathbf{F}) odd, property b) above remains valid: the natural maps relating $\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}$ and $BG_n^{\text{top}}(\mathbf{C})$ exist even if G_n is not connected; moreover, $W\{O_{n,\mathbf{F}}\}/SO_{n,\mathbf{F}} \rightarrow W\{O_{n,\mathbf{F}}\}/O_{n,\mathbf{F}} = \bar{W}\{O_{n,\mathbf{F}}\}$ is a double covering and $(W\{SO_{n,\mathbf{F}}\}/\times/SO_{n,\mathbf{F}})_{\text{ret}} \rightarrow (W\{O_{n,\mathbf{F}}\}/SO_{n,\mathbf{F}})_{\text{ret}}$ is a weak equivalence.

Properties a) and b) enable us to define

$$\chi_{n,q}: BG_n(\mathbf{F}_q) \rightarrow \varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A \tag{1.1}$$

where $G_n(\mathbf{F}_q)$ is the discrete group of points of $G_{n,\mathbf{F}}$ rational over \mathbf{F}_q and where $\varprojlim (\)$ is the inverse limit of an inverse system in the homotopy category of spaces with finite homotopy groups as in [4]. Namely, to obtain $\chi_{n,q}$, compose the maps

$$\begin{aligned} (W\{G_{n,\mathbf{F}}\}/G_n(\mathbf{F}_q))_{\text{ret}} &\rightarrow (W\{G_{n,\mathbf{F}}\}/G_{n,\mathbf{F}})_{\text{ret}} = \bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}} \\ \bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}} &\rightarrow \varprojlim (\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}})^A, \end{aligned}$$

and the homotopy equivalence

$$\varprojlim (\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}})^A \rightarrow \varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A$$

induced by the weak homotopy equivalence of b).

The following proposition will enable us to compute the Z/lZ cohomology of the discrete group $G_n(\mathbf{F}) = \varinjlim G_n(\mathbf{F}_q)$.

PROPOSITION 1.2. *Let $G_{n,\mathbf{F}}$ denote one of the following connected algebraic groups: $GL_{n,\mathbf{F}}$, $SL_{n,\mathbf{F}}$, $SO_{n,\mathbf{F}}$, or $Sp_{n,\mathbf{F}}$ (with $n=2k$ for Sp_n). Let $\phi^q: G_{n,\mathbf{F}} \rightarrow G_{n,\mathbf{F}}$ denote the geometric frobenius defined over \mathbf{F}_q , with q a power of $p = \text{char}(\mathbf{F})$. Let l be an integer not divisible by p . There exists an integer $d > 1$ such that the map*

$$1 \cdot \phi^q \cdots \phi^{q^{d-1}}: G_{n,\mathbf{F}} \rightarrow G_{n,\mathbf{F}}$$

defined as $\mu \circ (1 \times \phi^q \times \cdots \times \phi^{q^{d-1}}): G_{n,\mathbf{F}} \rightarrow (G_{n,\mathbf{F}})^{\times d} \rightarrow G_{n,\mathbf{F}}$ induces the zero map in Z/lZ cohomology.

Proof. We first assume that $G_{n,\mathbf{F}} = GL_{n,\mathbf{F}}$. Recall that

$$(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*: H^{2k}(BGL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ) \rightarrow H^{2k}(BGL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ)$$

where $+$: $BGL_{\infty}^{\text{top}}(\mathbf{C}) \times BGL_{\infty}^{\text{top}}(\mathbf{C}) \rightarrow BGL_{\infty}^{\text{top}}(\mathbf{C})$ is induced by tensor product of line bundles, is multiplication by $(1 + q + \cdots + q^{d-1})^k$. Let d be the order of q in $(Z/l^{e+1}Z)^*$, where e is the exponent of l in $q-1$. Then l divides $1 + q + \cdots + q^{d-1}$ so that $(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*$ is the zero map. Since the generators of the exterior algebra $H^*(GL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ)$ totally transgress to the generators of the polynomial algebra $H^*(BGL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ)$, we conclude that

$$\Omega(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*: H^*(GL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ) \rightarrow H^*(GL_{\infty}^{\text{top}}(\mathbf{C}), Z/lZ)$$

is the zero map.

We claim that the following diagram determines a commutative square in Z/lZ cohomology:

$$\begin{array}{ccc} (GL_{n,\mathbf{F}})_{\text{ret}} & \xrightarrow{1 \cdot \phi^q \cdots \phi^{q^{d-1}}} & (GL_{n,\mathbf{F}})_{\text{ret}} \\ \theta \downarrow & & \downarrow \theta \\ GL_{\infty}^{\text{top}}(\mathbf{C})^A & \xrightarrow{\Omega(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})} & GL_{\infty}^{\text{top}}(\mathbf{C})^A \end{array} \tag{1.2.1}$$

where θ is determined by maps $(GL_{n,\mathbf{F}})_{\text{ret}} \rightarrow (GL_{n,\text{Witt}(\mathbf{F})})_{\text{ret}}$, $(GL_{n,\mathbf{C}})_{\text{ret}} \rightarrow \times (GL_{n,\text{Witt}(\mathbf{F})})_{\text{ret}}$, $GL_n^{\text{top}}(\mathbf{C}) \rightarrow (GL_{n,\mathbf{C}})_{\text{ret}}$, and $GL_n^{\text{top}}(\mathbf{C}) \rightarrow GL_{\infty}^{\text{top}}(\mathbf{C})$; and where $()^A$ is pro-finite, prime-to- p completion. We verify that (1.2.1) induces a commutative square in Z/lZ cohomology by observing that each of the maps above is induced by a homomorphism of groups in an appropriate category so that θ^A likewise commutes with the group structures on $((GL_{n,\mathbf{F}})_{\text{ret}})^A$ and $GL_{\infty}^{\text{top}}(\mathbf{C})^A$; and by observing that $(\theta \circ \phi^q)^* = (\Omega(\Psi^q)^A \circ \theta)^*$, as can be seen by reducing to the case $n=1$ since $(BGL_1^{\text{top}}(\mathbf{C}))^{\times n} \rightarrow BGL_n^{\text{top}}(\mathbf{C})$ induces an injection in cohomology. Therefore, the

proposition follows for $G_{n,\mathbb{F}} = \text{GL}_{n,\mathbb{F}}$, by observing that θ induces a surjection in Z/lZ cohomology, since $\lim_{\leftarrow} (\text{GL}_{n,\mathbb{F}})_{\text{ret}}^A \rightarrow \lim_{\leftarrow} (\text{GL}_n^{\text{top}}(\mathbb{C}))^A$ is a homotopy equivalence.

We immediately conclude the proposition for $G_{n,\mathbb{F}} = \text{SL}_{n,\mathbb{F}}$ or $\text{Sp}_{n,\mathbb{F}}$ (with $n = 2k$ for Sp), or for $G_{n,\mathbb{F}}$ and l odd since $\text{SL}_{n,\mathbb{F}} \rightarrow \text{GL}_{n,\mathbb{F}}$, $\text{Sp}_{n,\mathbb{F}} \rightarrow \text{GL}_{n,\mathbb{F}}$, and $\text{SO}_{n,\mathbb{F}} \rightarrow \text{GL}_{n,\mathbb{F}}$ are group homomorphisms commuting with Frobenius and inducing surjections in Z/lZ cohomology (with l odd for $\text{SO}_{n,\mathbb{F}}$).

For p odd and l a power of 2, we observe that $(\text{BO}_1^{\text{top}}(\mathbb{C}))^{\times n} \rightarrow \text{BO}_n^{\text{top}}(\mathbb{C})$ induces injections in Z/lZ cohomology and Ψ^q acts trivially on $\text{BO}_1^{\text{top}}(\mathbb{C})$. Since $H^*(\text{BO}_n^{\text{top}}(\mathbb{C}), Z/lZ) \rightarrow H^*(\text{BSO}_n^{\text{top}}(\mathbb{C}), Z/lZ)$ is surjective, we conclude that ϕ^q acts trivially on $H^*(\text{SO}_{n,\mathbb{F}}, Z/lZ)$. Hence, if l divides d , $(1 + \phi^q + \dots + \phi^{q^{d-1}})^*$ is the zero map.

PROPOSITION 1.3. *Let $G_{n,\mathbb{F}}$ denote one of the following algebraic groups: $\text{GL}_{n,\mathbb{F}}$, $\text{SL}_{n,\mathbb{F}}$, $\text{O}_{n,\mathbb{F}}$ ($\text{char}(\mathbb{F})$ odd), $\text{SO}_{n,\mathbb{F}}$, or $\text{Sp}_{n,\mathbb{F}}$ (with $n = 2k$ for Sp_n). Let $G_n(k)$ denote the discrete group of points of $G_{n,\mathbb{F}}$ rational over k/\mathbb{F}_F , where $p = \text{char}(\mathbb{F})$. Then the direct limit of the maps*

$$\chi_{n,q}: \text{BG}_n(\mathbb{F}_q) \rightarrow \lim_{\leftarrow} \text{BG}_n^{\text{top}}(\mathbb{C})^A$$

of (1.1),

$$\chi_n: \text{BG}_n(\mathbb{F}) \rightarrow \lim_{\leftarrow} (\text{BG}_n^{\text{top}}(\mathbb{C}))^A$$

induce isomorphisms in Z/lZ cohomology for all l relatively prime to p , where $()^A$ denotes pro-finite, prime-to- p completion.

Proof. Since $\text{O}_1(\mathbb{F}) = \text{O}_1^{\text{top}}(\mathbb{C}) = Z/2Z$ for p odd, the proposition for $G_{n,\mathbb{F}} = \text{O}_{n,\mathbb{F}}$ easily follows from the theorem for $G_{n,\mathbb{F}} = \text{SO}_{n,\mathbb{F}}$. Hence, we may assume $G_{n,\mathbb{F}}$ to be connected.

Consider the following commutative diagram

$$\begin{array}{ccc} G_{n,\mathbb{F}}/G_n(\mathbb{F}_q) & \rightarrow & G_{n,\mathbb{F}}/G_n(\mathbb{F}_{q'}) \\ \downarrow & & \downarrow \\ W\{G_{n,\mathbb{F}}\}/G_n(\mathbb{F}_q) & \rightarrow & W\{G_{n,\mathbb{F}}\}/G_n(\mathbb{F}_{q'}) \\ \downarrow & & \downarrow \\ \bar{W}\{G_{n,\mathbb{F}}\} & = & \bar{W}\{G_{n,\mathbb{F}}\} \end{array} \tag{1.3.1}$$

where the top arrow fits in a commutative square

$$\begin{array}{ccc} G_{n,\mathbb{F}}/G_n(\mathbb{F}_q) & \longrightarrow & G_{n,\mathbb{F}}/G_n(\mathbb{F}_{q'}) \\ \downarrow 1/\phi^q & & \downarrow 1/\phi^{q'} \\ G_{n,\mathbb{F}} & \xrightarrow{1 \cdot \phi^q \dots \phi^{q'/q}} & G_{n,\mathbb{F}} \end{array}$$

where q' is a power of q . By a basic result of [1] (Corollary 2.6), (1.3.1) determines a map of ‘‘Serre spectral sequences’’

$$\begin{aligned} E_2^{s,t}(\mathbf{F}_{q'}) &= H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t((G_{n,\mathbf{F}}/G_n(\mathbf{F}_{q'}))_{\text{ret}}, Z/lZ)) \\ &\Rightarrow H^{s+t}(\text{BG}_n(\mathbf{F}_{q'}), Z/lZ) \\ E_2^{s,t}(\mathbf{F}_q) &= H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t((G_{n,\mathbf{F}}/G_n(\mathbf{F}_q))_{\text{ret}}, Z/lZ)) \\ &\Rightarrow H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ) \end{aligned} \quad (1.3.3)$$

where we have used the homotopy equivalence between $\text{BG}_n(\mathbf{F}_q)$ and $(W\{G_{n,\mathbf{F}}\}/G_n(\mathbf{F}_q))_{\text{ret}}$. Because $E_2^{s,t}(\mathbf{F}_q)$ is finite for all s, t and all powers q of p , (1.3.3) determines an inverse limit spectral sequence

$$E_2^{s,t} = \lim_{\leftarrow} H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t(G_{n,\mathbf{F}}/G_n(\mathbf{F}_q), Z/lZ)) \Rightarrow \lim_{\leftarrow} H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ).$$

By Proposition 1.2 and the fact that the vertical arrows of (1.3.2) are isomorphisms (the ‘‘Lang isomorphisms’’ of [1]), there exists $d > 1$ such that $E_2^{s,t}(\mathbf{F}_{q'}) \rightarrow E_2^{s,t}(\mathbf{F}_q)$ has image 0 for $t > 0$, any s , and $q' = q^d$. Moreover,

$$H^{s+t}(\text{BG}_n(\mathbf{F}), Z/lZ) \simeq \lim_{\leftarrow} H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ).$$

The proposition now follows from the degeneration of the spectral sequence $E^{s,t}$ and the fact that $(\bar{W}\{G_{n,\mathbf{F}}\})_{\text{ret}} \rightarrow \lim_{\leftarrow} (\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}})$ induces isomorphisms in Z/lZ cohomology (since

$$H^*(\lim_{\leftarrow} (\text{BGL}_n^{\text{top}}(\mathbf{C}))^A, Z/lZ) \simeq H^*(\text{BGL}_n^{\text{top}}(\mathbf{C}), Z/lZ) \quad [4]).$$

We record for future use the following proposition.

PROPOSITION 1.4. *Let $G_n(\mathbf{F})$ denote the discrete group of rational points of $G_{n,\mathbf{F}}$, where $G_{n,\mathbf{F}}$ denotes either $\text{GL}_{n,\mathbf{F}}$, $\text{SL}_{n,\mathbf{F}}$, $\text{O}_{n,\mathbf{F}}$, $\text{SO}_{n,\mathbf{F}}$, or $\text{Sp}_{n,\mathbf{F}}$ (with $\text{char}(\mathbf{F}) = p$ odd for O_n and $n = 2k$ for Sp_n). Then, for every $i > 0$,*

$$H^i(\text{BG}_n(\mathbf{F}), Z/pZ) = 0 = H^i(\text{BG}_n(\mathbf{F}), \mathbf{Q}).$$

Proof. For \mathbf{Q} coefficients, the proposition is an easy consequence of the fact that $\text{BG}_n(\mathbf{F})$ is the direct limit of classifying spaces of finite groups. For Z/pZ coefficients, the proposition is the stable version of the vanishing theorem given in [3]; a more elementary proof is carried out in detail in [1] for all cases except $\text{SO}_n(\mathbf{F})$ with n odd or $p = 2$.

In the following theorem, we see that χ_n of Proposition 1.3 provides a sharper form of Brauer lifting as generalized by Quillen to orthogonal and symplectic representations [2].

THEOREM 1.5. *Let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with $p = \text{char}(\mathbf{F})$ odd for O_n and $n=2k$ for Sp_n). The maps χ_n of Proposition 1.3 determine maps (uniquely up to homotopy)*

$$\eta_n: \mathbf{B}G_n(\mathbf{F}) \rightarrow \mathbf{B}G_n^{\text{top}}(\mathbf{C})$$

which induce isomorphisms in Z/lZ cohomology for $(l, p) = 1$. Furthermore, the composition of η_n with the natural inclusions

$$\mathbf{B}G_n(\mathbf{F}_q) \rightarrow \mathbf{B}G_n(\mathbf{F}) \rightarrow \mathbf{B}G_n^{\text{top}}(\mathbf{C}) \rightarrow \mathbf{B}G_\infty^{\text{top}}(\mathbf{C})$$

corresponds to the virtual complex bundle on $\mathbf{B}G_n(\mathbf{F}_q)$ obtained as the Brauer lifting of the standard modular representation of $G_n(\mathbf{F}_q)$ on $F^{\oplus n}$ for the embedding $\mathbf{F}^ \rightarrow \mathbf{C}^*$ chosen for χ_n .*

Proof. By Proposition 1.4, $H_i(\mathbf{B}G_n(\mathbf{F}), Z)$ is a prime-to- p torsion abelian group for $i > 0$. If H is such a prime-to- p torsion abelian group and A is a finitely generated abelian group, then

$$\text{Hom}(H, A) = \text{Hom}(H, \mathbf{p}_{-p}(A)) = \text{Hom}(H, \varprojlim (A)^A)$$

where $\mathbf{p}_{-p}(A)$ is the prime-to- p torsion subgroup of A and $\varprojlim (A)^A$ is the discrete group given as the inverse limit of the prime-to- p completion of A . Moreover,

$$\text{Ext}_Z^1(H, A) \simeq \text{Ext}_Z^1(H, \varprojlim (A)^A)$$

since $\text{Ext}_Z^1(H, Z) = \text{Hom}(H, \mathbf{Q}/Z) = \text{Hom}(H, \bigoplus_{l \neq p} (\mathbf{Q}_l/Z_l)) = \text{Ext}_Z^1(H, \varprojlim (Z^A))$.

Since

$$\varprojlim (\pi_i(\mathbf{B}G_n^{\text{top}}(\mathbf{C}))^A) \simeq \pi_i(\varprojlim (\mathbf{B}G_n^{\text{top}}(\mathbf{C}))^A)$$

we conclude that for all $i, j > 0$

$$H^i(\mathbf{B}G_n(\mathbf{F}), \pi_j(\mathbf{B}G_n^{\text{top}}(\mathbf{C}))) \simeq H^i(\mathbf{B}G_n(\mathbf{F}), \pi_j(\varprojlim (\mathbf{B}G_n^{\text{top}}(\mathbf{C}))^A)). \quad (1.5.1)$$

The existence and uniqueness of η_n is then obtained by inductively working up the Postnikov towers of $\mathbf{B}G_n^{\text{top}}(\mathbf{C})$ and $\varprojlim (\mathbf{B}G_n^{\text{top}}(\mathbf{C}))^A$, since (1.5.1) implies that χ_n determines maps into the Postnikov truncations of $\mathbf{B}G_n^{\text{top}}(\mathbf{C})$ which are homotopy compatible via homotopy compatible homotopies. Since $\mathbf{B}G_n^{\text{top}}(\mathbf{C}) \rightarrow \varprojlim (\mathbf{B}G_n^{\text{top}}(\mathbf{C}))^A$ induces isomorphisms in Z/lZ cohomology for $(l, p) = 1$, so does η_n by Proposition 1.3.

To prove the second assertion comparing η_n with Brauer lifting, it suffices to consider $G_n = GL_n$, since $G_n \rightarrow GL_n$ induces homotopy commutative squares

$$\begin{array}{ccc} BG_n(\mathbb{F}) & \xrightarrow{\eta_n} & BG_n^{\text{top}}(\mathbb{C}) \\ \downarrow & & \downarrow \\ BGL_n(\mathbb{F}) & \xrightarrow{\eta_n} & BGL_n^{\text{top}}(\mathbb{C}) \end{array}$$

and since $[BG_n(\mathbb{F}_q), BG_\infty^{\text{top}}(\mathbb{C})] \rightarrow [BG_n(\mathbb{F}_q), BGL_\infty^{\text{top}}(\mathbb{C})]$ is injective ([2], 5.1.6). Let η'_n denote the composition $BGL_n(\mathbb{F}) \rightarrow BGL_n^{\text{top}}(\mathbb{C}) \rightarrow BGL_\infty^{\text{top}}(\mathbb{C})$ and let β_n denote the map $BGL_n(\mathbb{F}) \rightarrow BGL_\infty^{\text{top}}(\mathbb{C})$ determined by Brauer lifting. One readily verifies that η'_1 and β_1 are induced by the chosen embedding

$$\mathbb{F}^* = GL_1(\mathbb{F}) \rightarrow GL_1^{\text{top}}(\mathbb{C}) = \mathbb{C}^* .$$

Since $B(GL_1(\mathbb{F})^{\times n}) \rightarrow BGL_n(\mathbb{F})$ induces an injection in Z/lZ cohomology by Proposition 1.3 for $(l, p) = 1$, we conclude

$$(\eta'_n)^* = \beta_n^* : H^*(BGL_\infty^{\text{top}}(\mathbb{C}), Z/lZ) \rightarrow H^*(BGL_n(\mathbb{F}), Z/lZ) .$$

Since $H^*(BGL_n(\mathbb{F}), Z_l)$ has no torsion for any prime $l \neq p$ by Proposition 1.3, the compositions

$$BGL_n(\mathbb{F}) \rightrightarrows BGL_\infty^{\text{top}}(\mathbb{C}) \rightarrow (\varprojlim (BGL_\infty^{\text{top}}(\mathbb{C}))^{\hat{}})^{(i)}$$

determined by η'_n and β_n are therefore homotopic for all i , where $(\)^{\hat{}}$ denotes pro- l completion and $(\)^{(i)}$ denotes the i -th Postnikov truncation. As argued above for the existence and uniqueness of η_n , η'_n and β_n thus determine homotopic maps from $BGL_n(\mathbb{F})$ to $(BGL_\infty^{\text{top}}(\mathbb{C}))^{(i)}$ for all i . This implies that the compositions

$$BGL_n(\mathbb{F}_q) \rightarrow BGL_n(\mathbb{F}) \rightrightarrows BGL_\infty^{\text{top}}(\mathbb{C})$$

determined by η'_n and B_n are homotopic as asserted, since the groups $H^i(BGL_n(\mathbb{F}_q), \pi_j(BGL_\infty^{\text{top}}(\mathbb{C})))$ are finite for all $i, j > 0$ so that maps from $BGL_n(\mathbb{F}_q)$ into $BGL_\infty^{\text{top}}(\mathbb{C})$ are determined by maps into the Postnikov tower of $BGL_\infty^{\text{top}}(\mathbb{C})$.

2. $\mathbf{P}-p$ Torsion Completion and Unstable K -theories

As in section 1, we fix a prime p and let \mathbb{F} denote the algebraic closure of \mathbb{F}_p . We let $\mathbf{P}-p$ denote the set of all primes except p .

DEFINITION 2.1. Let X be a connected nilpotent space with $\pi_1(X)$ a $\mathbf{P}-p$

torsion abelian group. Then the $\mathbf{P}-p$ torsion completion of X ,

$$t: \tau_{\mathbf{P}-p}(X) \rightarrow X$$

is the homotopy fibre of localization at p , $X \rightarrow X_{(p)}$.

Using obstruction theory, one can readily verify that any map from a CW complex Z with $\mathbf{P}-p$ torsion homology groups to X factors uniquely (up to homotopy) through $t: \tau_{\mathbf{P}-p}(X) \rightarrow X$ (the obstruction to lifting a map “up along the Postnikov tower”, the obstruction to lifting a homotopy between two such liftings, and the obstruction to lifting a homotopy between two homotopies all vanish), if the torsion subgroup ${}_{\mathbf{P}-p}(\pi_i(X))$ is a direct summand of $\pi_i(X)$ for all $i > 0$.

Our principal result is the following theorem.

THEOREM 2.2. *Let G_n denote either GL_n , SL_n , O_n , SO_n , or Sp_n (with $p = \text{char}(\mathbf{F})$ odd for O_n and $n = 2k$ for Sp_n). Let $BG_n(\mathbf{F}) \rightarrow BG_n(\mathbf{F})^+$ denote Quillen's plus construction with respect to the commutator subgroup $[G_n(\mathbf{F}), G_n(\mathbf{F})]$ of $\pi_1(BG_n(\mathbf{F})) = G_n(\mathbf{F})$. Then the map*

$$\eta_n^+ : BG_n(\mathbf{F})^+ \rightarrow BG_n^{\text{top}}(\mathbf{C})$$

induced by η_n of Theorem 1.5 is the $\mathbf{P}-p$ torsion completion of $BG_n^{\text{top}}(\mathbf{C})$.

Proof. Since $H_i(BG_n(\mathbf{F})) \simeq H_i(BG_n(\mathbf{F})^+)$ is $\mathbf{P}-p$ torsion for $i > 0$ by Proposition 1.4, η_n^+ factors uniquely as $\eta_n^+ = t \circ \varrho_n^+$, where t is $\mathbf{P}-p$ torsion completion and

$$\varrho_n^+ : BG_n(\mathbf{F})^+ \rightarrow \tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})).$$

By Theorem 1.5, η_n^+ induces isomorphisms in Z/lZ cohomology for $(l, p) = 1$. Moreover, t induces isomorphisms in Z/lZ cohomology because the Serre spectral sequence for the fibration

$$\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})) \rightarrow BG_n^{\text{top}}(\mathbf{C}) \rightarrow BG_n^{\text{top}}(\mathbf{C})_{(p)} \quad (2.2.1)$$

and Z/lZ coefficients degenerates and $\pi_1(BG_n^{\text{top}}(\mathbf{C})_{(p)}) = 0$. Therefore ϱ_n^+ induces isomorphisms in Z/lZ cohomology so that

$$(\varrho_n^+)_* : H_*(BG_n(\mathbf{F})^+) \simeq H_*(\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C}))). \quad (2.2.2)$$

(for $i > 0$, $H_i(\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})))$ is $\mathbf{P}-p$ torsion by the degeneration of the Serre spectral sequence for (2.2.1) with Z/pZ and \mathbf{Q} coefficients).

For G_n equal to SL_n , SO_n , or Sp_n , (2.2.2) plus the Whitehead theorem imply that ϱ_n^+ is a homotopy equivalence. Moreover, (2.2.2) implies that ϱ_n^+ induces isomorphisms of fundamental groups for G_n equal to GL_n or O_n . Since the map on universal cover-

ings induced by ϱ_n^+ for $G_n = GL_n$ (respectively, $G_n = O_n$) is ϱ_n^+ for SL_n (resp., SO_n), ϱ_n^+ is also a homotopy equivalence for G_n equal to GL_n or O_n .

The fact that localization of spaces localizes homotopy groups enables us to immediately derive the following evaluation of unstable algebraic K -groups.

COROLLARY 2.3. *Let G_n be as in Theorem 2.2. Then*

$$\pi_i(\mathbf{BG}_n(\mathbf{F})^+) \simeq_{\mathbf{P}-p} (\pi_i(\mathbf{BG}_n^{\text{top}}(\mathbf{C}))) \oplus (\pi_{i+1}(\mathbf{BG}_n^{\text{top}}(\mathbf{C}))) \otimes \bigoplus_{l \neq p} (\mathbf{Q}_l/\mathbf{Z}_l).$$

We conclude with the following proposition asserting that the sequences

$$\cdots \rightarrow \mathbf{BG}_n(\mathbf{F})^+ \rightarrow \mathbf{BG}_{n+1}(\mathbf{F})^+ \rightarrow \mathbf{BG}_{n+2}(\mathbf{F})^+ \rightarrow \cdots$$

determine ‘‘intrinsic’’ $\mathbf{P}-p$ torsion spherical fibrations in the sense of Sullivan ([4]). For notational simplicity, we state and prove the proposition for $G_n = GL_n$.

PROPOSITION 2.4. *The homotopy fibre of*

$$i_n: \mathbf{BGL}_n(\mathbf{F})^+ \rightarrow \mathbf{BGL}_{n+1}(\mathbf{F})^+$$

is $\tau_{\mathbf{P}-p}(S^{2n+1})$. Moreover, the pullback of the $\tau_{\mathbf{P}-p}(S^{2n+3})$ fibration i_{n+1} via the map $i_{n+1} \circ i_n$ is fibre homotopy equivalent to the fibre-wise join of i_{n-1} and the trivial $\tau_{\mathbf{P}-p}(S^3)$ fibration over $\mathbf{BGL}_n(\mathbf{F})^+$.

Proof. Because localization is an exact functor on abelian groups, localization preserves fibrations with simply connected base and total space. Thus $\tau_{\mathbf{P}-p}(\)$ also preserves such fibrations, implying that the fibre of $i_n: \mathbf{BGL}_n(\mathbf{F})^+ \rightarrow \mathbf{BGL}_{n+1}(\mathbf{F})^+$ is $\tau_{\mathbf{P}-p}(S^{2n+1})$.

Let S^i, S^j be spheres of dimension $i, j > 1$. Any representatives of $\mathbf{P}-p$ torsion completions

$$\tau_{\mathbf{P}-p}(S^i) \rightarrow S^i, \quad \tau_{\mathbf{P}-p}(S^j) \rightarrow S^j$$

determine a map of joins

$$\phi: \tau_{\mathbf{P}-p}(S^i) * \tau_{\mathbf{P}-p}(S^j) \rightarrow \tau_{\mathbf{P}-p}(S^i * S^j).$$

Using the equality of excisive pairs, $(CS^i, S^i) \times (CS^j, S^j) = (CS^i \times CS^j, S^i * S^j)$, we conclude that ϕ induces isomorphisms in $Z//Z$ cohomology. Thus, ϕ is a homotopy equivalence. Consequently, $\tau_{\mathbf{P}-p}(\)$ preserves the fibre-wise join of sphere fibrations over a simply connected base. Since $\tau_{\mathbf{P}-p}(\)$ preserves fibrations and thus homotopy theoretic fibre products, the second assertion now follows from the well known corresponding ‘‘intrinsic’’ property of the sphere fibrations $\mathbf{BG}_n^{\text{top}}(\mathbf{C}) \rightarrow \mathbf{BG}_{n+1}^{\text{top}}(\mathbf{C})$.

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