

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 50 (1975)

**Artikel:** Unstable K-Theories of the Algebraic Closure of a Finite Field.  
**Autor:** Friedlander, Eric M.  
**DOI:** <https://doi.org/10.5169/seals-38801>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 20.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Unstable $K$ -Theories of the Algebraic Closure of a Finite Field

ERIC M. FRIEDLANDER<sup>1)</sup>

We show that the Brauer lifting of the standard modular representation of  $GL_n(\mathbb{F}_q)$  on  $\mathbb{F}^{\oplus n}$  to a virtual complex representation has a curious non-stable version (where  $\mathbb{F}_q$  is a finite field of characteristic  $p > 0$  with algebraic closure  $\mathbb{F}$ ). Namely, the map  $BGL_n(\mathbb{F}_q) \rightarrow BGL_{\infty}^{\text{top}}(\mathbb{C})$  associated to this Brauer lifting factors through a map  $BGL_n(\mathbb{F}_q) \rightarrow BGL_n^{\text{top}}(\mathbb{C})$ . This unstable lifting is also achieved for  $BSO_n(\mathbb{F}_q)$  and  $BSp_{2k}(\mathbb{F}_q)$ . Using the induced maps  $BGL_n(\mathbb{F})^+ \rightarrow BGL_n^{\text{top}}(\mathbb{C})$ ,  $BSO_n(\mathbb{F})^+ \rightarrow BSO_n^{\text{top}}(\mathbb{C})$ ,  $BSp_{2k}(\mathbb{F})^+ \rightarrow BSp_{2k}^{\text{top}}(\mathbb{C})$  we determine the unstable algebraic  $K$ -groups  $\pi_i(BGL_n(\mathbb{F})^+)$ ,  $\pi_i(BSO_n(\mathbb{F})^+)$ , and  $\pi_i(BSp_{2k}(\mathbb{F})^+)$  explicitly in terms of the homotopy groups of the corresponding classical groups.

In section 1, we exhibit natural map from  $BG_n(\mathbb{F})$  to the prime-to- $p$  pro-finite completion of  $BG_n^{\text{top}}(\mathbb{C})$ , where  $G_n$  denotes either  $GL_n$ ,  $SL_n$ ,  $O_n$ ,  $SO_n$  or  $Sp_n$  (where  $p = \text{char}(\mathbb{F})$  is odd for  $O_n$  and  $n = 2k$  for  $Sp_n$ ). This map induces isomorphisms in  $\mathbb{Z}/l\mathbb{Z}$  cohomology for  $l$  prime to  $p$ . Because  $BG_n(\mathbb{F})$  is  $\mathbb{Z}/p\mathbb{Z}$ -acyclic, this map determines a map  $\eta_n: BG_n(\mathbb{F}) \rightarrow BG_n^{\text{top}}(\mathbb{C})$ . We verify that the composition  $BG_n(\mathbb{F}_q) \rightarrow BG_n(\mathbb{F}) \rightarrow BG_n^{\text{top}}(\mathbb{C}) \rightarrow BG_{\infty}^{\text{top}}(\mathbb{C})$  corresponds to Brauer lifting.

In section 2, we show that  $\pi_i(BG(\mathbb{F})^+)$  is directly computable from  $\pi_i(BG_n^{\text{top}}(\mathbb{C}))$  and  $\pi_{i+1}(BG_n^{\text{top}}(\mathbb{C}))$ . More precisely,  $\eta_n^+: BG_n(\mathbb{F})^+ \rightarrow BG_n^{\text{top}}(\mathbb{C})$  is shown to be the fibre of localization at  $p = \text{char}(\mathbb{F})$ ,  $BG_n^{\text{top}}(\mathbb{C}) \rightarrow BG_n^{\text{top}}(\mathbb{C})_{(p)}$ . As a consequence, the sequences

$$\cdots \rightarrow BG_n(\mathbb{F})^+ \rightarrow BG_{n+1}(\mathbb{F})^+ \rightarrow BG_{n+2}(\mathbb{F})^+ \rightarrow \cdots$$

are seen to be “intrinsic spherical fibrations” with fibres prime-to- $p$  torsion spheres.

We gratefully acknowledge several valuable conversations with G. Lusztig.

## 1. Unstable Brauer Lifting

We let  $G_n$  denote either  $GL_n$ ,  $SL_n$ ,  $O_n$ ,  $SO_n$ , or  $Sp_n$  (with  $n = 2k$  for  $Sp_n$ ), so that  $G_{n,R}$  is the corresponding group scheme defined over  $\text{Spec } R$  for any (commutative with identity) ring  $R$ . We denote by  $G_n^{\text{top}}(\mathbb{C})$  the corresponding classical topological

---

<sup>1)</sup> Partially supported by the N.S.F., by the U.S.-France Exchange of Scientists Program, and by I.H.E.S.

group. In the orthogonal case, we recall that  $O_{n,R}$  is the algebraic sub-group scheme of  $GL_{n,R}$  preserving the quadratic form  $X_1X_{k+1} + \cdots + X_kX_{2k}$  if  $n=2k$  and  $X_1X_{k+1} + \cdots + X_kX_{2k} + X_{2k+1}^2$  if  $n=2k+1$ ;  $SO_{n,R}$  is the sub-group scheme of  $GL_{n,R}$  defined over  $\text{Spec } \mathbb{Z}$  as the reduction of the connected component of the identity of  $O_{n,R}$ .

We fix a prime  $p$  and let  $F$  denote the algebraic closure of the finite field  $F_p$ . For any power of  $p$ ,  $q=p^d$ , we let  $F_q$  denote the subfield of  $F$  with  $q$  elements. If  $p=2$ , we exclude the case  $G_n=O_n$  so that  $G_{n,F}$  is an algebraic group. We denote by  $W\{G_{n,F}\}$  the simplicial algebraic variety with  $(W\{G_{n,F}\})_k = (G_{n,F})^{\times k}$  and with face and degeneracy maps obtained by deleting and inserting a factor; we let  $\bar{W}\{G_{n,F}\}$  denote  $W\{G_{n,F}\}/G_{n,F}$ .

In [1], the rigid etale homotopy type of a noetherian scheme or noetherian simplicial scheme was introduced (denoted by  $(\ )_{\text{ret}}$ ), and the following was proved:

- a) If  $H \subset G_{n,F}$  is a finite algebraic subgroup, then  $(W\{G_{n,F}\}/H)_{\text{ret}}$  is naturally homotopy equivalent to  $BH$ , the classifying space of  $H$  viewed as a discrete group.
- b) If  $(\ )^A$  denotes pro-finite completion prime-to-char( $F$ ) and if  $G_{n,F}$  is connected, then  $(\bar{W}\{G_{n,F}\}_{\text{ret}})^A$  is weakly homotopy equivalent to  $(BG_n^{\text{top}}(C))^A$  via maps dependent only on a choice of embedding of the Witt vectors of  $F$  into  $C$ .

We can actually verify that the maps determining the weak equivalence of b) depend only on a choice of embedding of  $F^*$  into  $C^*$  (using the fact that these maps are determined by their effect on cohomology and reducing to the case  $n=1$ ).

In the case of  $O_{n,F}$  and char( $F$ ) odd, property b) above remains valid: the natural maps relating  $\bar{W}\{G_{n,F}\}_{\text{ret}}$  and  $BG_n^{\text{top}}(C)$  exist even if  $G_n$  is not connected; moreover,  $W\{O_{n,F}\}/SO_{n,F} \rightarrow W\{O_{n,F}\}/O_{n,F} = \bar{W}\{O_{n,F}\}$  is a double covering and  $(W\{SO_{n,F}\}/\times/SO_{n,F})_{\text{ret}} \rightarrow (W\{O_{n,F}\}/SO_{n,F})_{\text{ret}}$  is a weak equivalence.

Properties a) and b) enable us to define

$$\chi_{n,q}: BG_n(F_q) \rightarrow \varprojlim (BG_n^{\text{top}}(C))^A \quad (1.1)$$

where  $G_n(F_q)$  is the discrete group of points of  $G_{n,F}$  rational over  $F_q$  and where  $\varprojlim (\ )$  is the inverse limit of an inverse system in the homotopy category of spaces with finite homotopy groups as in [4]. Namely, to obtain  $\chi_{n,q}$ , compose the maps

$$\begin{aligned} (W\{G_{n,F}\}/G_n(F_q))_{\text{ret}} &\rightarrow (W\{G_{n,F}\}/G_{n,F})_{\text{ret}} = \bar{W}\{G_{n,F}\}_{\text{ret}} \\ \bar{W}\{G_{n,F}\}_{\text{ret}} &\rightarrow \varprojlim (\bar{W}\{G_{n,F}\}_{\text{ret}})^A, \end{aligned}$$

and the homotopy equivalence

$$\varprojlim (\bar{W}\{G_{n,F}\}_{\text{ret}})^A \rightarrow \varprojlim (BG_n^{\text{top}}(C))^A$$

induced by the weak homotopy equivalence of b).

The following proposition will enable us to compute the  $Z/lZ$  cohomology of the discrete group  $G_n(\mathbf{F}) = \varinjlim G_n(\mathbf{F}_q)$ .

**PROPOSITION 1.2.** *Let  $G_{n,\mathbf{F}}$  denote one of the following connected algebraic groups:  $GL_{n,\mathbf{F}}$ ,  $SL_{n,\mathbf{F}}$ ,  $SO_{n,\mathbf{F}}$ , or  $Sp_{n,\mathbf{F}}$  (with  $n=2k$  for  $Sp_n$ ). Let  $\phi^q: G_{n,\mathbf{F}} \rightarrow G_{n,\mathbf{F}}$  denote the geometric frobenius defined over  $\mathbf{F}_q$ , with  $q$  a power of  $p = \text{char}(\mathbf{F})$ . Let  $l$  be an integer not divisible by  $p$ . There exists an integer  $d > 1$  such that the map*

$$1 \cdot \phi^q \cdots \phi^{q^{d-1}}: G_{n,\mathbf{F}} \rightarrow G_{n,\mathbf{F}}$$

*defined as  $\mu \circ (1 \times \phi^q \times \cdots \times \phi^{q^{d-1}}): G_{n,\mathbf{F}} \rightarrow (G_{n,\mathbf{F}})^{\times d} \rightarrow G_{n,\mathbf{F}}$  induces the zero map in  $Z/lZ$  cohomology.*

*Proof.* We first assume that  $G_{n,\mathbf{F}} = GL_{n,\mathbf{F}}$ . Recall that

$$(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*: H^{2k}(BGL_{\infty}^{\text{top}}(C), Z/lZ) \rightarrow H^{2k}(BGL_{\infty}^{\text{top}}(C), Z/lZ)$$

where  $+: BGL_{\infty}^{\text{top}}(C) \times BGL_{\infty}^{\text{top}}(C) \rightarrow BGL_{\infty}^{\text{top}}(C)$  is induced by tensor product of line bundles, is multiplication by  $(1 + q + \cdots + q^{d-1})^k$ . Let  $d$  be the order of  $q$  in  $(Z/l^{e+1}Z)^*$ , where  $e$  is the exponent of  $l$  in  $q-1$ . Then  $l$  divides  $1 + q + \cdots + q^{d-1}$  so that  $(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*$  is the zero map. Since the generators of the exterior algebra  $H^*(GL_{\infty}^{\text{top}}(C), Z/lZ)$  totally transgress to the generators of the polynomial algebra  $H^*(BGL_{\infty}^{\text{top}}(C), Z/lZ)$ , we conclude that

$$\Omega(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})^*: H^*(GL_{\infty}^{\text{top}}(C), Z/lZ) \rightarrow H^*(GL_{\infty}^{\text{top}}(C), Z/lZ)$$

is the zero map.

We claim that the following diagram determines a commutative square in  $Z/lZ$  cohomology:

$$\begin{array}{ccc} (GL_{n,\mathbf{F}})_{\text{ret}} & \xrightarrow{1 \cdot \phi^q \cdots \phi^{q^{d-1}}} & (GL_{n,\mathbf{F}})_{\text{ret}} \\ \theta \downarrow & & \downarrow \theta \\ GL_{\infty}^{\text{top}}(C)^A & \xrightarrow{\Omega(1 + \Psi^q + \cdots + \Psi^{q^{d-1}})} & GL_{\infty}^{\text{top}}(C)^A \end{array} \quad (1.2.1)$$

where  $\theta$  is determined by maps  $(GL_{n,\mathbf{F}})_{\text{ret}} \rightarrow (GL_{n,\text{Witt}(\mathbf{F})})_{\text{ret}}$ ,  $(GL_{n,\mathbf{C}})_{\text{ret}} \rightarrow \times (GL_{n,\text{Witt}(\mathbf{F})})_{\text{ret}}$ ,  $GL_n^{\text{top}}(C) \rightarrow (GL_{n,\mathbf{C}})_{\text{ret}}$ , and  $GL_n^{\text{top}}(C) \rightarrow GL_{\infty}^{\text{top}}(C)$ ; and where  $( )^A$  is pro-finite, prime-to- $p$  completion. We verify that (1.2.1) induces a commutative square in  $Z/lZ$  cohomology by observing that each of the maps above is induced by a homomorphism of groups in an appropriate category so that  $\theta^A$  likewise commutes with the group structures on  $((GL_{n,\mathbf{F}})_{\text{ret}})^A$  and  $GL_{\infty}^{\text{top}}(C)^A$ ; and by observing that  $(\theta \circ \phi^q)^* = (\Omega(\Psi^q)^A \circ \theta)^*$ , as can be seen by reducing to the case  $n=1$  since  $(BGL_1^{\text{top}}(C))^{\times n} \rightarrow BGL_n^{\text{top}}(C)$  induces an injection in cohomology. Therefore, the



proposition follows for  $G_{n,\mathbb{F}} = \mathrm{GL}_{n,\mathbb{F}}$ , by observing that  $\theta$  induces a surjection in  $Z/lZ$  cohomology, since  $\varprojlim (\mathrm{GL}_{n,\mathbb{F}})_{\mathrm{ret}}^A \rightarrow \varprojlim (\mathrm{GL}_n^{\mathrm{top}}(\mathbb{C}))^A$  is a homotopy equivalence.

We immediately conclude the proposition for  $G_{n,\mathbb{F}} = \mathrm{SL}_{n,\mathbb{F}}$  or  $\mathrm{Sp}_{n,\mathbb{F}}$  (with  $n=2k$  for  $\mathrm{Sp}$ ), or for  $G_{n,\mathbb{F}}$  and  $l$  odd since  $\mathrm{SL}_{n,\mathbb{F}} \rightarrow \mathrm{GL}_{n,\mathbb{F}}$ ,  $\mathrm{Sp}_{n,\mathbb{F}} \rightarrow \mathrm{GL}_{n,\mathbb{F}}$ , and  $\mathrm{SO}_{n,\mathbb{F}} \rightarrow \mathrm{GL}_{n,\mathbb{F}}$  are group homomorphisms commuting with frobenius and inducing surjections in  $Z/lZ$  cohomology (with  $l$  odd for  $\mathrm{SO}_{n,\mathbb{F}}$ ).

For  $p$  odd and  $l$  a power of 2, we observe that  $(\mathrm{BO}_1^{\mathrm{top}}(\mathbb{C}))^{\times n} \rightarrow \mathrm{BO}_n^{\mathrm{top}}(\mathbb{C})$  induces injections in  $Z/lZ$  cohomology and  $\Psi^q$  acts trivially on  $\mathrm{BO}_1^{\mathrm{top}}(\mathbb{C})$ . Since  $H^*(\mathrm{BO}_n^{\mathrm{top}}(\mathbb{C}), Z/lZ) \rightarrow H^*(\mathrm{BSO}_n^{\mathrm{top}}(\mathbb{C}), Z/lZ)$  is surjective, we conclude that  $\phi^q$  acts trivially on  $H^*(\mathrm{SO}_{n,\mathbb{F}}, Z/lZ)$ . Hence, if  $l$  divides  $d$ ,  $(1 + \phi^q + \cdots + \phi^{q^{d-1}})^*$  is the zero map.

**PROPOSITION 1.3.** *Let  $G_{n,\mathbb{F}}$  denote one of the following algebraic groups:  $\mathrm{GL}_{n,\mathbb{F}}$ ,  $\mathrm{SL}_{n,\mathbb{F}}$ ,  $\mathrm{O}_{n,\mathbb{F}}$  ( $\mathrm{char}(\mathbb{F})$  odd),  $\mathrm{SO}_{n,\mathbb{F}}$ , or  $\mathrm{Sp}_{n,\mathbb{F}}$  (with  $n=2k$  for  $\mathrm{Sp}_n$ ). Let  $G_n(k)$  denote the discrete group of points of  $G_{n,\mathbb{F}}$  rational over  $k/\mathbb{F}_{\mathbb{F}}$ , where  $p = \mathrm{char}(\mathbb{F})$ . Then the direct limit of the maps*

$$\chi_{n,q}: \mathrm{BG}_n(\mathbb{F}_q) \rightarrow \varprojlim \mathrm{BG}_n^{\mathrm{top}}(\mathbb{C})^A$$

of (1.1),

$$\chi_n: \mathrm{BG}_n(\mathbb{F}) \rightarrow \varprojlim (\mathrm{BG}_n^{\mathrm{top}}(\mathbb{C}))^A$$

induce isomorphisms in  $Z/lZ$  cohomology for all  $l$  relatively prime to  $p$ , where  $(\ )^A$  denotes pro-finite, prime-to- $p$  completion.

*Proof.* Since  $\mathrm{O}_1(\mathbb{F}) = \mathrm{O}_1^{\mathrm{top}}(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$  for  $p$  odd, the proposition for  $G_{n,\mathbb{F}} = \mathrm{O}_{n,\mathbb{F}}$  easily follows from the theorem for  $G_{n,\mathbb{F}} = \mathrm{SO}_{n,\mathbb{F}}$ . Hence, we may assume  $G_{n,\mathbb{F}}$  to be connected.

Consider the following commutative diagram

$$\begin{array}{ccc} G_{n,\mathbb{F}}/G_n(\mathbb{F}_q) & \rightarrow & G_{n,\mathbb{F}}/G_n(\mathbb{F}_{q'}) \\ \downarrow & & \downarrow \\ W\{G_{n,\mathbb{F}}\}/G_n(\mathbb{F}_q) & \rightarrow & W\{G_{n,\mathbb{F}}\}/G_n(\mathbb{F}_{q'}) \\ \downarrow & & \downarrow \\ \mathcal{W}\{G_{n,\mathbb{F}}\} & = & \mathcal{W}\{G_{n,\mathbb{F}}\} \end{array} \quad (1.3.1)$$

where the top arrow fits in a commutative square

$$\begin{array}{ccc} G_{n,\mathbb{F}}/G_n(\mathbb{F}_q) & \xrightarrow{\quad} & G_{n,\mathbb{F}}/G_n(\mathbb{F}_{q'}) \\ 1/\phi^q \downarrow & & \downarrow 1/\phi^{q'} \\ G_{n,\mathbb{F}} & \xrightarrow{1 \cdot \phi^q \cdots \phi^{q'/q}} & G_{n,\mathbb{F}} \end{array}$$

where  $q'$  is a power of  $q$ . By a basic result of [1] (Corollary 2.6), (1.3.1) determines a map of “Serre spectral sequences”

$$\begin{aligned} E_2^{s,t}(\mathbf{F}_{q'}) &= H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t((G_{n,\mathbf{F}}/G_n(\mathbf{F}_{q'}))_{\text{ret}}, Z/lZ)) \\ &\Rightarrow H^{s+t}(\text{BG}_n(\mathbf{F}_{q'}), Z/lZ) \\ E_2^{s,t}(\mathbf{F}_q) &= H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t((G_{n,\mathbf{F}}/G_n(\mathbf{F}_q))_{\text{ret}}, Z/lZ)) \\ &\Rightarrow H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ) \end{aligned} \quad (1.3.3)$$

where we have used the homotopy equivalence between  $\text{BG}_n(\mathbf{F}_q)$  and  $(W\{G_{n,\mathbf{F}}\}/G_n(\mathbf{F}_q))_{\text{ret}}$ . Because  $E_2^{s,t}(\mathbf{F}_q)$  is finite for all  $s, t$  and all powers  $q$  of  $p$ , (1.3.3) determines an inverse limit spectral sequence

$$E_2^{s,t} = \varprojlim H^s(\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}}, H^t(G_{n,\mathbf{F}}/G_n(\mathbf{F}_q), Z/lZ)) \Rightarrow \varprojlim H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ).$$

By Proposition 1.2 and the fact that the vertical arrows of (1.3.2) are isomorphisms (the “Lang isomorphisms” of [1]), there exists  $d > 1$  such that  $E_2^{s,t}(\mathbf{F}_{q'}) \rightarrow E_2^{s,t}(\mathbf{F}_q)$  has image 0 for  $t > 0$ , any  $s$ , and  $q' = q^d$ . Moreover,

$$H^{s+t}(\text{BG}_n(\mathbf{F}), Z/lZ) \simeq \varprojlim H^{s+t}(\text{BG}_n(\mathbf{F}_q), Z/lZ).$$

The proposition now follows from the degeneration of the spectral sequence  $E^{s,t}$  and the fact that  $(\bar{W}\{G_{n,\mathbf{F}}\})_{\text{ret}} \rightarrow \varprojlim (\bar{W}\{G_{n,\mathbf{F}}\}_{\text{ret}})$  induces isomorphisms in  $Z/lZ$  cohomology (since

$$H^*(\varprojlim (\text{BGL}_n^{\text{top}}(\mathbf{C}))^A, Z/lZ) \simeq H^*(\text{BGL}_n^{\text{top}}(\mathbf{C}), Z/lZ) \quad [4]).$$

We record for future use the following proposition.

**PROPOSITION 1.4.** *Let  $G_n(\mathbf{F})$  denote the discrete group of rational points of  $G_{n,\mathbf{F}}$ , where  $G_{n,\mathbf{F}}$  denotes either  $\text{GL}_{n,\mathbf{F}}$ ,  $\text{SL}_{n,\mathbf{F}}$ ,  $\text{O}_{n,\mathbf{F}}$ ,  $\text{SO}_{n,\mathbf{F}}$ , or  $\text{Sp}_{n,\mathbf{F}}$  (with  $\text{char}(\mathbf{F}) = p$  odd for  $\text{O}_n$  and  $n = 2k$  for  $\text{Sp}_n$ ). Then, for every  $i > 0$ ,*

$$H^i(\text{BG}_n(\mathbf{F}), Z/pZ) = 0 = H^i(\text{BG}_n(\mathbf{F}), \mathbf{Q}).$$

*Proof.* For  $\mathbf{Q}$  coefficients, the proposition is an easy consequence of the fact that  $\text{BG}_n(\mathbf{F})$  is the direct limit of classifying spaces of finite groups. For  $Z/pZ$  coefficients, the proposition is the stable version of the vanishing theorem given in [3]; a more elementary proof is carried out in detail in [1] for all cases except  $\text{SO}_n(\mathbf{F})$  with  $n$  odd or  $p = 2$ .

In the following theorem, we see that  $\chi_n$  of Proposition 1.3 provides a sharper form of Brauer lifting as generalized by Quillen to orthogonal and symplectic representations [2].

**THEOREM 1.5.** *Let  $G_n$  denote either  $GL_n$ ,  $SL_n$ ,  $O_n$ ,  $SO_n$ , or  $Sp_n$  (with  $p = \text{char}(\mathbf{F})$  odd for  $O_n$  and  $n=2k$  for  $Sp_n$ ). The maps  $\chi_n$  of Proposition 1.3 determine maps (uniquely up to homotopy)*

$$\eta_n: BG_n(\mathbf{F}) \rightarrow BG_n^{\text{top}}(\mathbf{C})$$

*which induce isomorphisms in  $Z/lZ$  cohomology for  $(l, p)=1$ . Furthermore, the composition of  $\eta_n$  with the natural inclusions*

$$BG_n(\mathbf{F}_q) \rightarrow BG_n(\mathbf{F}) \rightarrow BG_n^{\text{top}}(\mathbf{C}) \rightarrow BG_\infty^{\text{top}}(\mathbf{C})$$

*corresponds to the virtual complex bundle on  $BG_n(\mathbf{F}_q)$  obtained as the Brauer lifting of the standard modular representation of  $G_n(\mathbf{F}_q)$  on  $F^{\oplus n}$  for the embedding  $\mathbf{F}^* \rightarrow \mathbf{C}^*$  chosen for  $\chi_n$ .*

*Proof.* By Proposition 1.4,  $H_i(BG_n(\mathbf{F}), \mathbf{Z})$  is a prime-to- $p$  torsion abelian group for  $i > 0$ . If  $H$  is such a prime-to- $p$  torsion abelian group and  $A$  is a finitely generated abelian group, then

$$\text{Hom}(H, A) = \text{Hom}(H, \mathbf{p}_{-p}(A)) = \text{Hom}(H, \varprojlim (A)^A)$$

where  $\mathbf{p}_{-p}(A)$  is the prime-to- $p$  torsion subgroup of  $A$  and  $\varprojlim (A)^A$  is the discrete group given as the inverse limit of the prime-to- $p$  completion of  $A$ . Moreover,

$$\text{Ext}_Z^1(H, A) \simeq \text{Ext}_Z^1(H, \varprojlim (A)^A)$$

since  $\text{Ext}_Z^1(H, Z) = \text{Hom}(H, \mathbf{Q}/Z) = \text{Hom}(H, \bigoplus_{l \neq p} (\mathbf{Q}_l/Z_l)) = \text{Ext}_Z^1(H, \varprojlim (Z^A))$ .

Since

$$\varprojlim (\pi_i(BG_n^{\text{top}}(\mathbf{C}))^A) \simeq \pi_i(\varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A)$$

we conclude that for all  $i, j > 0$

$$H^i(BG_n(\mathbf{F}), \pi_j(BG_n^{\text{top}}(\mathbf{C}))) \simeq H^i(BG_n(\mathbf{F}), \pi_j(\varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A)). \quad (1.5.1)$$

The existence and uniqueness of  $\eta_n$  is then obtained by inductively working up the Postnikov towers of  $BG_n^{\text{top}}(\mathbf{C})$  and  $\varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A$ , since (1.5.1) implies that  $\chi_n$  determines maps into the Postnikov truncations of  $BG_n^{\text{top}}(\mathbf{C})$  which are homotopy compatible via homotopy compatible homotopies. Since  $BG_n^{\text{top}}(\mathbf{C}) \rightarrow \varprojlim (BG_n^{\text{top}}(\mathbf{C}))^A$  induces isomorphisms in  $Z/lZ$  cohomology for  $(l, p)=1$ , so does  $\eta_n$  by Proposition 1.3.

To prove the second assertion comparing  $\eta_n$  with Brauer lifting, it suffices to consider  $G_n = GL_n$ , since  $G_n \rightarrow GL_n$  induces homotopy commutative squares

$$\begin{array}{ccc} BG_n(F) & \xrightarrow{\eta_n} & BG_n^{\text{top}}(C) \\ \downarrow & & \downarrow \\ BGL_n(F) & \xrightarrow{\eta_n} & BGL_n^{\text{top}}(C) \end{array}$$

and since  $[BG_n(F_q), BG_\infty^{\text{top}}(C)] \rightarrow [BG_n(F_q), BGL_\infty^{\text{top}}(C)]$  is injective ([2], 5.1.6). Let  $\eta'_n$  denote the composition  $BGL_n(F) \rightarrow BGL_n^{\text{top}}(C) \rightarrow BGL_\infty^{\text{top}}(C)$  and let  $\beta_n$  denote the map  $BGL_n(F) \rightarrow BGL_\infty^{\text{top}}(C)$  determined by Brauer lifting. One readily verifies that  $\eta'_1$  and  $\beta_1$  are induced by the chosen embedding

$$F^* = GL_1(F) \rightarrow GL_1^{\text{top}}(C) = C^*.$$

Since  $B(GL_1(F)^{\times n}) \rightarrow BGL_n(F)$  induces an injection in  $Z/lZ$  cohomology by Proposition 1.3 for  $(l, p) = 1$ , we conclude

$$(\eta'_n)^* = \beta_n^*: H^*(BGL_\infty^{\text{top}}(C), Z/lZ) \rightarrow H^*(BGL_n(F), Z/lZ).$$

Since  $H^*(BGL_n(F), Z_l)$  has no torsion for any prime  $l \neq p$  by Proposition 1.3, the compositions

$$BGL_n(F) \rightrightarrows BGL_\infty^{\text{top}}(C) \rightarrow (\varprojlim (BGL_\infty^{\text{top}}(C))^{\hat{}})^{(i)}$$

determined by  $\eta'_n$  and  $\beta_n$  are therefore homotopic for all  $i$ , where  $(\ )^{\hat{}}$  denotes pro- $l$  completion and  $(\ )^{(i)}$  denotes the  $i$ -th Postnikov truncation. As argued above for the existence and uniqueness of  $\eta_n$ ,  $\eta'_n$  and  $\beta_n$  thus determine homotopic maps from  $BGL_n(F)$  to  $(BGL_\infty^{\text{top}}(C))^{(i)}$  for all  $i$ . This implies that the compositions

$$BGL_n(F_q) \rightarrow BGL_n(F) \rightrightarrows BGL_\infty^{\text{top}}(C)$$

determined by  $\eta'_n$  and  $\beta_n$  are homotopic as asserted, since the groups  $H^i(BGL_n(F_q), \pi_j(BGL_\infty^{\text{top}}(C)))$  are finite for all  $i, j > 0$  so that maps from  $BGL_n(F_q)$  into  $BGL_\infty^{\text{top}}(C)$  are determined by maps into the Postnikov tower of  $BGL_\infty^{\text{top}}(C)$ .

## 2. $\mathbf{P}-p$ Torsion Completion and Unstable $K$ -theories

As in section 1, we fix a prime  $p$  and let  $F$  denote the algebraic closure of  $F_p$ . We let  $\mathbf{P}-p$  denote the set of all primes except  $p$ .

**DEFINITION 2.1.** Let  $X$  be a connected nilpotent space with  $\pi_1(X)$  a  $\mathbf{P}-p$

torsion abelian group. Then the  $\mathbf{P}-p$  torsion completion of  $X$ ,

$$t: \tau_{\mathbf{P}-p}(X) \rightarrow X$$

is the homotopy fibre of localization at  $p$ ,  $X \rightarrow X_{(p)}$ .

Using obstruction theory, one can readily verify that any map from a CW complex  $Z$  with  $\mathbf{P}-p$  torsion homology groups to  $X$  factors uniquely (up to homotopy) through  $t: \tau_{\mathbf{P}-p}(X) \rightarrow X$  (the obstruction to lifting a map “up along the Postnikov tower”, the obstruction to lifting a homotopy between two such liftings, and the obstruction to lifting a homotopy between two homotopies all vanish), if the torsion subgroup  ${}_{\mathbf{P}-p}(\pi_i(X))$  is a direct summand of  $\pi_i(X)$  for all  $i > 0$ .

Our principal result is the following theorem.

**THEOREM 2.2.** *Let  $G_n$  denote either  $GL_n$ ,  $SL_n$ ,  $O_n$ ,  $SO_n$ , or  $Sp_n$  (with  $p = \text{char}(\mathbf{F})$  odd for  $O_n$  and  $n = 2k$  for  $Sp_n$ ). Let  $BG_n(\mathbf{F}) \rightarrow BG_n(\mathbf{F})^+$  denote Quillen's plus construction with respect to the commutator subgroup  $[G_n(\mathbf{F}), G_n(\mathbf{F})]$  of  $\pi_1(BG_n(\mathbf{F})) = G_n(\mathbf{F})$ . Then the map*

$$\eta_n^+: BG_n(\mathbf{F})^+ \rightarrow BG_n^{\text{top}}(\mathbf{C})$$

*induced by  $\eta_n$  of Theorem 1.5 is the  $\mathbf{P}-p$  torsion completion of  $BG_n^{\text{top}}(\mathbf{C})$ .*

*Proof.* Since  $H_i(BG_n(\mathbf{F})) \simeq H_i(BG_n(\mathbf{F})^+)$  is  $\mathbf{P}-p$  torsion for  $i > 0$  by Proposition 1.4,  $\eta_n^+$  factors uniquely as  $\eta_n^+ = t \circ \varrho_n^+$ , where  $t$  is  $\mathbf{P}-p$  torsion completion and

$$\varrho_n^+: BG_n(\mathbf{F})^+ \rightarrow \tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})).$$

By Theorem 1.5,  $\eta_n^+$  induces isomorphisms in  $Z/lZ$  cohomology for  $(l, p) = 1$ . Moreover,  $t$  induces isomorphisms in  $Z/lZ$  cohomology because the Serre spectral sequence for the fibration

$$\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})) \rightarrow BG_n^{\text{top}}(\mathbf{C}) \rightarrow BG_n^{\text{top}}(\mathbf{C})_{(p)} \quad (2.2.1)$$

and  $Z/lZ$  coefficients degenerates and  $\pi_1(BG_n^{\text{top}}(\mathbf{C})_{(p)}) = 0$ . Therefore  $\varrho_n^+$  induces isomorphisms in  $Z/lZ$  cohomology so that

$$(\varrho_n^+)_*: H_*(BG_n(\mathbf{F})^+) \simeq H_*(\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C}))). \quad (2.2.2)$$

(for  $i > 0$ ,  $H_i(\tau_{\mathbf{P}-p}(BG_n^{\text{top}}(\mathbf{C})))$  is  $\mathbf{P}-p$  torsion by the degeneration of the Serre spectral sequence for (2.2.1) with  $Z/pZ$  and  $\mathbf{Q}$  coefficients).

For  $G_n$  equal to  $SL_n$ ,  $SO_n$ , or  $Sp_n$ , (2.2.2) plus the Whitehead theorem imply that  $\varrho_n^+$  is a homotopy equivalence. Moreover, (2.2.2) implies that  $\varrho_n^+$  induces isomorphisms of fundamental groups for  $G_n$  equal to  $GL_n$  or  $O_n$ . Since the map on universal cover-

ings induced by  $\varrho_n^+$  for  $G_n = GL_n$  (respectively,  $G_n = O_n$ ) is  $\varrho_n^+$  for  $SL_n$  (resp.,  $SO_n$ ),  $\varrho_n^+$  is also a homotopy equivalence for  $G_n$  equal to  $GL_n$  or  $O_n$ .

The fact that localization of spaces localizes homotopy groups enables us to immediately derive the following evaluation of unstable algebraic  $K$ -groups.

**COROLLARY 2.3.** *Let  $G_n$  be as in Theorem 2.2. Then*

$$\pi_i(BG_n(\mathbf{F})^+) \simeq_{\mathbf{P}-p} (\pi_i(BG_n^{\text{top}}(\mathbf{C}))) \oplus (\pi_{i+1}(BG_n^{\text{top}}(\mathbf{C})) \otimes \bigoplus_{l \neq p} (\mathbf{Q}_l/Z_l)).$$

We conclude with the following proposition asserting that the sequences

$$\cdots \rightarrow BG_n(\mathbf{F})^+ \rightarrow BG_{n+1}(\mathbf{F})^+ \rightarrow BG_{n+2}(\mathbf{F})^+ \rightarrow \cdots$$

determine “intrinsic”  $\mathbf{P}-p$  torsion spherical fibrations in the sense of Sullivan ([4]). For notational simplicity, we state and prove the proposition for  $G_n = GL_n$ .

**PROPOSITION 2.4.** *The homotopy fibre of*

$$i_n: BGL_n(\mathbf{F})^+ \rightarrow BGL_{n+1}(\mathbf{F})^+$$

*is  $\tau_{\mathbf{P}-p}(S^{2n+1})$ . Moreover, the pullback of the  $\tau_{\mathbf{P}-p}(S^{2n+3})$  fibration  $i_{n+1}$  via the map  $i_{n+1} \circ i_n$  is fibre homotopy equivalent to the fibre-wise join of  $i_{n-1}$  and the trivial  $\tau_{\mathbf{P}-p}(S^3)$  fibration over  $BGL_n(\mathbf{F})^+$ .*

*Proof.* Because localization is an exact functor on abelian groups, localization preserves fibrations with simply connected base and total space. Thus  $\tau_{\mathbf{P}-p}(\ )$  also preserves such fibrations, implying that the fibre of  $i_n: BGL_n(\mathbf{F})^+ \rightarrow BGL_{n+1}(\mathbf{F})^+$  is  $\tau_{\mathbf{P}-p}(S^{2n+1})$ .

Let  $S^i, S^j$  be spheres of dimension  $i, j > 1$ . Any representatives of  $\mathbf{P}-p$  torsion completions

$$\tau_{\mathbf{P}-p}(S^i) \rightarrow S^i, \quad \tau_{\mathbf{P}-p}(S^j) \rightarrow S^j$$

determine a map of joins

$$\phi: \tau_{\mathbf{P}-p}(S^i) * \tau_{\mathbf{P}-p}(S^j) \rightarrow \tau_{\mathbf{P}-p}(S^i * S^j).$$

Using the equality of excisive pairs,  $(CS^i, S^i) \times (CS^j, S^j) = (CS^i \times CS^j, S^i * S^j)$ , we conclude that  $\phi$  induces isomorphisms in  $Z/lZ$  cohomology. Thus,  $\phi$  is a homotopy equivalence. Consequently,  $\tau_{\mathbf{P}-p}(\ )$  preserves the fibre-wise join of sphere fibrations over a simply connected base. Since  $\tau_{\mathbf{P}-p}(\ )$  preserves fibrations and thus homotopy theoretic fibre products, the second assertion now follows from the well known corresponding “intrinsic” property of the sphere fibrations  $BG_n^{\text{top}}(\mathbf{C}) \rightarrow BG_{n+1}^{\text{top}}(\mathbf{C})$ .

## REFERENCES

- [1] FRIEDLANDER, E., *Computations of K-theories of finite fields*, to appear
- [2] QUILLEN, D., *The Adams conjecture*, *Topology* 10 (1971), 67–80.
- [3] ———, *On the cohomology and K-theory of the general linear groups over a finite field*, *Ann. of Math.* 96 (1972), 552–586.
- [4] SULLIVAN, D., *Genetics of homotopy theory and the Adams conjecture*, *Ann. of Math.* 100 (1974), 1–79.

*Department of Mathematics*  
*Princeton University*

Received September 27, 1974.