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# On the Covariant Differential of an Almost Hermitian Structure

### J. M. TERRIER

This paper deals with the covariant differential  $\nabla J$  of an almost hermitian structure J on a Riemannian manifold M with metric g. The connection with respect to which  $\nabla J$  is defined is the Riemannian connection on M. The extension of the notion of antisymmetrization and symmetrization from tensor fields of type (o, q) to those of type (p, q) for p>0 allows us to decompose  $\nabla J$  is an antisymmetric part A and a symmetric part S. In this way, we find new formulas which have the feature of stressing the relationship between  $\nabla J$  and the torsion  $\tau$  of J as well as the fundamental 2-form  $\omega$  or better, its exterior derivative  $d\omega$ . The results are essentially based first on the Palais formula which gives the exterior derivative of a q-form via Lie product and covariant derivative and second on a theorem ([2], p. 149) which gives the exterior derivative  $d\alpha$  of a q-form  $\alpha$  as the antisymmetric part of the covariant differential  $\nabla \alpha$ of  $\alpha$ , provided the connection has vanishing torsion. As an application of our formulas we give a characterization of so called nearly Kähler manifolds ([1]) via the fundamental 2-form  $\omega$ . We also give a very simple proof of the characterization of a Kähler manifold given by the vanishing of  $\nabla J$  or of  $d\omega$  and  $\tau$ . We finally prove a lemma which gives a nice interpretation of the torsion of J when the fundamental 2-form is closed, that is, in the case of an almost Kähler manifold.

### §1. The Covariant Differential of a Tensor Field

Let t be a given tensor field of type (p, q) on a  $C^{\infty}$  manifold M. We shall simply write  $t \in T_M(p, q)$  or  $t \in T(p, q)$ .

Suppose there is also a linear connection  $\nabla$  given on M. Then, as in [2], we can define the covariant differential of t as the tensor field  $\nabla t \in T(p, q+1)$  defined by

$$\nabla t(X_1, ..., X_q, X) = (\nabla_X t)(X_1, ..., X_q)$$
(1.1)

where  $\nabla_X t$  denotes the covariant derivative of the tensor field t and  $X_1, ..., X_q, X$  are in the Lie algebra  $\mathfrak{X}(M)$  of vector fields on M.

Because  $\nabla_X$  is a derivation commuting with every contraction, we have

THEOREM 1.1 ([2], p. 124). If 
$$t \in T(p, q)$$
 then, for  $X_i, X \in \mathfrak{X}(M)$ 

$$\nabla t(X_1,...,X_q,X) = \nabla_X(t(X_1,...,X_q)) - \sum_{k=1}^q t(X_1,...,\nabla_{X_i}X_k,...,X_q).$$

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EXAMPLE 1. Take t=g a Riemann metric on M. g is in T(0,2), so  $\nabla g$  is in T(0,3) and for  $X, Y, Z, \in \mathfrak{X}(M)$  we have

$$\nabla g(X, Y, Z) = (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

One of the features of a Riemannian metric g is to be parallel i.e.  $\nabla g = 0$ . For convenience we shall write  $\langle X, Y \rangle$  instead of g(X, Y).

**EXAMPLE 2.** Take t=J an almost complex structure on M. This is a tensor field of type (1, 1) whose square  $J^2$  equals minus the identity.  $\nabla J$  is in T(1, 2) and

$$\nabla J(X, Y) = (\nabla_Y J) X = \nabla_Y (JX) - J\nabla_Y X.$$

One of the features of a Kähler structure J on M is to be parallel with respect to  $\nabla : \nabla J = 0$ .

## §2. Extension of Antisymmetrization and Symmetrization

It is known [1] how to define for a covariant tensor field t in T(0, q) the alternation At of t. It is a tensor field of the same type defined by

$$(At) (X_1, ..., X_q) = \frac{1}{q!} \sum_{\pi \in P_q} \varepsilon(\pi) t(X_{\pi(1)}, ..., X_{\pi(q)})$$
 (2.1)

where  $P_q$  is the group of permutations of  $\{1, 2, ..., q\}$  and  $\varepsilon(\pi)$  is the signe of the permutation  $\pi$ . Notice that At is a skew-symmetric tensor field and t is skew-symmetric if and only if At = t.

On the other hand one also defines for  $t \in T(0, q)$  the symmetrization St of t by

$$St(X_1, ..., X_q) = \frac{1}{q!} \sum_{\pi \in P_q} t(X_{\pi(1)}, ..., X_{\pi(q)}).$$
 (2.2)

Here is St a symmetric tensor field and t is symmetric if and only if St = t.

We now make the straightforward extension of the above notions to tensor fields of type (p, q) for p > 0. In this case  $t(X_1, ..., X_q)$  is no longer a real function on M, but a p-contravariant tensor field. Nevertheless, all algebraic operations needed for definitions (2.1) and (2.2) still make sense in the module of p-contravariant tensor fields.

EXAMPLE 1. Take  $t=\tau \in T(1,2)$  the torsion of an almost complex structure J defined by

$$\tau(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Here we have  $A\tau = \tau$  and  $S\tau = 0$ .

EXAMPLE 2. Take  $t = K \in T(1, 3)$  where K(X, Y, Z) = R(X, Y) Z and  $R \in T(1, 3)$  is defined by  $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ .

Because R(X, Y) = -R(X, Y) we have

$$(AK)(X, Y, Z) = \frac{1}{3}\mathfrak{S}(K(X, Y, Z))$$

(s means cyclic sum) and SK=0. Furthermore if the torsion of  $\nabla$  vanishes, we have AK=0 in view of the first Bianchi identity.

# §3. Almost Hermitian Manifolds

From now on, we assume that (M, J, g) is an almost hermitian manifold, that is, the almost complex structure J on M gives in each point p of M an isometry of  $T_p(M)$ , the tangent space at M in p, i.e.

$$\langle JX, JY \rangle = \langle X, Y \rangle \quad \forall X, Y \in T_p(M).$$
 (3.1)

Denoting by  $\nabla$  the Riemannian connection on M, we can compute  $\nabla J$  which is a tensor field of type (1, 2) and we can write the following decomposition

$$\nabla J = A(VJ) + S(VJ). \tag{3.2}$$

Needless to say, this is only possible because we are in T(p, q) with q = 2. Example 2 above shows what can happen with  $q \neq 2$ .

For convenience we shall write A for  $A(\nabla J)$  and S for  $S(\nabla J)$  and establish several formulas relating A and/or S with various tensor fields one can define on an almost hermitian manifold.

### I. The Torsion $\tau$ and A

THEOREM 3.1. If A denotes the antisymmetric part of the covariant differential  $\nabla J$  of the almost hermitian structure J on M, then

$$\frac{1}{2}J\tau(X,Y) = A(X,Y) - A(JY,JY). \tag{3.3}$$

*Proof.* By definition  $A(X, Y) = \frac{1}{2} \{ \nabla J(X, Y) - \nabla J(Y, X) \}$ . So

$$2\{A(X, Y) - A(JX, JY)\} = \nabla J(X, Y) - \nabla J(Y, X) - \nabla J(JX, JY) + \nabla J(JY, JX).$$

In view of (1.1), the right hand side becomes

$$(\nabla_{Y}J) X - (\nabla_{X}J) Y - (\nabla_{JY}J) (JX) + (\nabla_{JX}J) (JY).$$

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which is, according to Theorem 1.1, the same as

$$\nabla_{Y}JX - J\nabla_{Y}X - \nabla_{X}JY + J\nabla_{X}Y + \nabla_{JY}X + J\nabla_{JY}JX - \nabla_{JX}Y - J\nabla_{JX}JY. \tag{3.4}$$

The Riemannian connection having torsion zero, we have  $\nabla_X Y - \nabla_Y X = [X, Y]$  and multiplication of (3.4) by J gives (3.3).

From Theorem 3.1 we trivially get the following

COROLLARY 3.2. If the covariant differential  $\nabla J$  of an almost hermitian structure J is symmetric then J is integrable.

### II. The Fundamental 2-form ω and A

The fundamental 2-form  $\omega$  on an almost hermitian manifold (M, J, g) is defined by

$$\omega(X, Y) = \langle JX, Y \rangle \quad \text{for} \quad X, Y \in \mathfrak{X}(M).$$
 (3.5)

The torsion of the Riemann connection being zero, we know ([2], chap. III) that  $d\omega = A(\nabla \omega)$ . For vector fields X, Y, and Z on M, we have therefore:

$$6d\omega(X, Y, Z) = 6A(\nabla\omega)(X, Y, Z) = \mathfrak{s}(\nabla\omega(X, Y, Z)) - \mathfrak{s}(\nabla\omega(Y, X, Z))$$
(3.6)

where s denotes cyclic sum. By definition of the covariant differential and (3.5) we have

$$\nabla \omega (X, Y, Z) = (\nabla_{Z}\omega) (X, Y) = Z\omega (X, Y) - \omega (\nabla_{Z}X, Y) - \omega (X, \nabla_{Z}Y)$$

$$= \langle \nabla_{Z}JX, Y \rangle + \langle JX, \nabla_{Z}Y \rangle - \langle J\nabla_{Z}X, Y \rangle - \langle JX, \nabla_{Z}Y \rangle$$

$$= \langle \nabla_{Z}JX, Y \rangle - \langle J\nabla_{Z}X, Y \rangle = \langle (\nabla_{Z}J) X, Y \rangle = \langle \nabla J(X, Z), Y \rangle.$$

Hence

$$\nabla \omega(X, Y, Z) = \langle \nabla J(X, Z), Y \rangle \tag{3.7}$$

In view of (2.1) and (3.2) we have the following

THEOREM 3.3. On an almost hermitian manifold (M, J, g) the exterior differential  $d\omega$  of the fundamental 2-form  $\omega$  and the antisymmetric part of the covariant differential  $\nabla J$  are related by the formula

$$3d\omega(X, Y, Z) = -\mathfrak{s}(\langle A(X, Y), Z \rangle). \tag{3.8}$$

Together with Corollary 3.2 this result implies

COROLLARY 3.4. If the covariant differential  $\nabla J$  of an almost hermitian structure

J is symmetric, then the fundamental 2-form  $\omega$  is closed and J is integrable, i.e. (M, J, g) is a Kähler manifold.

There is another interesting relation between  $d\omega$ , A and  $\nabla J$ ; namely, we have

THEOREM 3.5. On any almost hermitian manifold (M, J, q) we have for the fundamental 2-form  $\omega$ :

$$3d\omega(X, Y, Z) = -2\langle A(X, Y), Z \rangle + \langle \nabla J(X, Z), Y \rangle. \tag{3.9}$$

*Proof.* By the Palais formula, one has for any 2-form  $\omega$ 

$$3d\omega(X, Y, Z) = X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y)$$
$$-\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$

By the definition of  $\omega$  and because g is parallel with respect to  $\nabla$ , together with the fact, once again, that the torsion of  $\nabla$  vanishes, we get the stated result.

## III. A Relation Between A and S and an Identity for S

From Theorem 3.3 and Theorem 3.5 above we deduce the

COROLLARY 3.6. If A (resp. S) is the antisymmetric (resp. symmetric) part of the covariant differential  $\nabla J$  of an almost hermitian structure then, for vector fields X, Y, Z on M, we have

$$\langle S(X, Y), Z \rangle = \langle A(X, Z), Y \rangle + \langle A(Y, Z), X \rangle.$$
 (3.10)

Proof. From (3.2), (3.8) and (3.9) one has

$$\mathfrak{S}(\langle A(X, Y), Z \rangle) = 2\langle A(X, Y), Z \rangle + \langle A(X, Z), Y \rangle + \langle S(X, Z), Y \rangle.$$

But A is antisymmetric, so A(X, Z) = -A(Z, X) and (3.10) follows by permuting Y and Z.

The antisymmetry of A has another consequence:

COROLLARY 3.7. The symmetric part S of the covariant differential  $\nabla J$  of an almost hermitian structure satisfies the following identity

$$\mathfrak{s}(\langle S(X,Y),Z\rangle)=0 \tag{3.11}$$

where 5 denotes cyclic sum.

*Proof.* Write the left hand side of (3.11) with (3.10) and use the antisymmetry of A.

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# §4. A few Remarks

1. Because of (3.11) we can rewrite Theorem 3.3 in the following form:

THEOREM 3.3'. On an almost hermitian manifold (M, J, g) the exterior differential  $d\omega$  of the fundamental 2-form  $\omega$  and the covariant differential  $\nabla J$  of J are related by the formula

$$3d\omega(X, Y, Z) = -\mathfrak{s}(\langle \nabla J(X, Y), Z \rangle). \tag{4.1}$$

2. An almost hermitian manifold (M, J, g) for which S=0 is already known [1] as nearly Kähler manifold. An alternative condition is given by the following

THEOREM 4.1. A nearly Kähler manifold (M, J, g) is characterized by the condition  $\nabla \omega = d\omega$ . (4.2)

**Proof.** We have to show that this condition is equivalent to S=0. Suppose (4.2) is true. Then  $\nabla \omega$  is antisymmetric. By (3.7)  $\nabla \omega(X, Y, Z) = \langle \nabla J(X, Z), Y \rangle$  and  $\nabla \omega(Z, Y, X) = \langle \nabla J(Z, X), Y \rangle = -\langle \nabla J(X, Z), Y \rangle$  which implies that  $\nabla J$  itself is antisymmetric, i.e. S=0.

On the other hand, S=0 implies  $\nabla J=A$  and by (3.10)  $\langle \nabla J(X,Z), Y \rangle + \langle \nabla J(Y,Z), X \rangle = 0$  which in turn gives by antisymmetry of  $\nabla J$ :  $\langle \nabla J(Z,X), Y \rangle + \langle \nabla J(Z,X), Y \rangle = 0$  or with (3.7),  $\nabla \omega(Z,Y,X) + \nabla \omega(Z,X,Y) = 0$ . But (even if  $S \neq 0$ ) one has  $\nabla \omega(X,Y,Z) + \nabla \omega(Y,X,Z) = 0$ , and (3.6) gives the result.

3. Of the antisymmetric part A and the symmetric part S of  $\nabla J$ , the former plays the most important role. It allows us to give as an application a very simple proof of the following theorem. Compare with [3] (chap. IX).

THEOREM 4.2. An almost hermitian manifold (M, J, g) is a Kähler manifold (i.e.  $\tau = 0$  and  $d\omega = 0$ ) if and only if the covariant differential  $\nabla J$  vanishes.

*Proof.* If  $\nabla J = 0$  then A = 0 which implies  $\tau = 0$  by (3.3) and  $d\omega = 0$  by (3.8). On the other hand, from Lemma 4.3 below and  $d\omega = 0$  we get A = 0 and by (3.10), S = 0.

LEMMA 4.3. For an almost Kähler manifold, the antisymmetric part A of the covariant differential  $\nabla J$  is essentially the torsion of J: more precisely we have

 $4A = J\tau$ .

*Proof.*  $d\omega = 0$  implies by (3.9)

$$(\nabla J(X,Z), Y) = 2\langle A(X,Y), Z \rangle.$$

Substituing JX to X and JY to Y and adding, we get

$$\langle \nabla J(X,Z), Y \rangle + \langle \nabla J(JX,Z), JY \rangle = 2 \langle A(X,Y) + A(JX,JY), Z \rangle.$$
 (4.3)

But the left hand side vanishes because  $\nabla J(JX, Y) = -J\nabla J(X, Y)$  as it is easy to see and J is an isometry. To get the desired result one just has to add (4.3) to (3.3).

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