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On the Covariant Differential of an Almost Hermitian Structure

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This paper deals with the covariant differential ∇J of an almost hermitian structure J on a Riemannian manifold M with metric g. The connection with respect to which ∇J is defined is the Riemannian connection on M. The extension of the notion of antisymmetrization and symmetrization from tensor fields of type (o, q) to those of type (p, q) for $p>0$ allows us to decompose ∇J is an antisymmetric part A and a symmetric part S. In this way, we find new formulas which have the feature of stressing the relationship between ∇J and the torsion τ of J as well as the fundamental 2-form ω or better, its exterior derivative $d\omega$. The results are essentially based first on the Palais formula which gives the exterior derivative of a q-form via Lie product and covariant derivative and second on a theorem ([2], p. 149) which gives the exterior derivative $d\alpha$ of a q-form α as the antisymmetric part of the covariant differential $\nabla\alpha$ of α, provided the connection has vanishing torsion. As an application of our formulas we give a characterization of so called nearly Kähler manifolds ([1]) via the fundamental 2-form ω. We also give a very simple proof of the characterization of a Kähler manifold given by the vanishing of ∇J or of $d\omega$ and τ. We finally prove a lemma which gives a nice interpretation of the torsion of J when the fundamental 2-form is closed, that is, in the case of an almost Kähler manifold.

§1. The Covariant Differential of a Tensor Field

Let t be a given tensor field of type (p, q) on a C^∞ manifold M. We shall simply write $t\in T_M(p, q)$ or $t\in T(p, q)$.

Suppose there is also a linear connection ∇ given on M. Then, as in [2], we can define the covariant differential of t as the tensor field $\nabla t\in T(p, q+1)$ defined by

$$\nabla t(X_1,\ldots, X_q, X)=(\nabla_X t)(X_1,\ldots, X_q) \tag{1.1}$$

where $\nabla_X t$ denotes the covariant derivative of the tensor field t and $X_1,\ldots, X_q, X$ are in the Lie algebra $\mathfrak{X}(M)$ of vector fields on M.

Because ∇_X is a derivation commuting with every contraction, we have

THEOREM 1.1 ([2], p. 124). *If $t\in T(p, q)$ then, for $X_i, X\in\mathfrak{X}(M)$*

$$\nabla t(X_1,\ldots, X_q, X)=\nabla_X(t(X_1,\ldots, X_q))-\sum_{k=1}^{q} t(X_1,\ldots, \nabla_{X_t}X_k,\ldots, X_q).$$

EXAMPLE 1. Take $t=g$ a Riemann metric on M. g is in $T(0,2)$, so ∇g is in $T(0,3)$ and for $X, Y, Z, \in \mathfrak{X}(M)$ we have

$$\nabla g(X, Y, Z)=(\nabla_Z g)(X, Y)=Zg(X, Y)-g(\nabla_Z X, Y)-g(X, \nabla_Z Y).$$

One of the features of a Riemannian metric g is to be parallel i.e. $\nabla g=0$. For convenience we shall write $\langle X, Y\rangle$ instead of $g(X, Y)$.

EXAMPLE 2. Take $t=J$ an almost complex structure on M. This is a tensor field of type $(1,1)$ whose square J^2 equals minus the identity. ∇J is in $T(1,2)$ and

$$\nabla J(X, Y)=(\nabla_Y J)X=\nabla_Y(JX)-J\nabla_Y X.$$

One of the features of a Kähler structure J on M is to be parallel with respect to ∇: $\nabla J=0$.

§2. Extension of Antisymmetrization and Symmetrization

It is known [1] how to define for a covariant tensor field t in $T(0,q)$ the *alternation At* of t. It is a tensor field of the same type defined by

$$(At)(X_1, \dots, X_q)=\frac{1}{q!}\sum_{\pi\in P_q}\varepsilon(\pi)\,t(X_{\pi(1)}, \dots, X_{\pi(q)}) \tag{2.1}$$

where P_q is the group of permutations of $\{1, 2, \dots, q\}$ and $\varepsilon(\pi)$ is the signe of the permutation π. Notice that At is a skew-symmetric tensor field and t is skew-symmetric if and only if $At=t$.

On the other hand one also defines for $t\in T(0,q)$ the *symmetrization St* of t by

$$St(X_1, \dots, X_q)=\frac{1}{q!}\sum_{\pi\in P_q} t(X_{\pi(1)}, \dots, X_{\pi(q)}). \tag{2.2}$$

Here is St a symmetric tensor field and t is symmetric if and only if $St=t$.

We now make the straightforward extension of the above notions to tensor fields of type (p,q) for $p>0$. In this case $t(X_1, \dots, X_q)$ is no longer a real function on M, but a p-contravariant tensor field. Nevertheless, all algebraic operations needed for definitions (2.1) and (2.2) still make sense in the module of p-contravariant tensor fields.

EXAMPLE 1. Take $t=\tau\in T(1,2)$ the torsion of an almost complex structure J defined by

$$\tau(X, Y)=[X, Y]+J[JX, Y]+J[X, JY]-[JX, JY].$$

Here we have $A\tau=\tau$ and $S\tau=0$.

EXAMPLE 2. Take $t=K\in T(1,3)$ where $K(X, Y, Z)=R(X, Y)Z$ and $R\in T(1,3)$ is defined by $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$.

Because $R(X, Y)=-R(X, Y)$ we have

$$(AK)(X, Y, Z)=\tfrac{1}{3}\mathfrak{s}(K(X, Y, Z))$$

($\mathfrak{s}$ means cyclic sum) and $SK=0$. Furthermore if the torsion of ∇ vanishes, we have $AK=0$ in view of the first Bianchi identity.

§3. Almost Hermitian Manifolds

From now on, we assume that (M, J, g) is an almost hermitian manifold, that is, the almost complex structure J on M gives in each point p of M an isometry of $T_p(M)$, the tangent space at M in p, i.e.

$$\langle JX, JY\rangle=\langle X, Y\rangle \quad \forall X, Y\in T_p(M). \tag{3.1}$$

Denoting by ∇ the Riemannian connection on M, we can compute ∇J which is a tensor field of type (1, 2) and we can write the following decomposition

$$\nabla J=A(VJ)+S(VJ). \tag{3.2}$$

Needless to say, this is only possible because we are in $T(p, q)$ with $q=2$. Example 2 above shows what can happen with $q\neq 2$.

For convenience we shall write A for $A(\nabla J)$ and S for $S(\nabla J)$ and establish several formulas relating A and/or S with various tensor fields one can define on an almost hermitian manifold.

I. The Torsion τ and A

THEOREM 3.1. *If A denotes the antisymmetric part of the covariant differential ∇J of the almost hermitian structure J on M, then*

$$\tfrac{1}{2}J\tau(X, Y)=A(X, Y)-A(JY, JY). \tag{3.3}$$

Proof. By definition $A(X, Y)=\frac{1}{2}\{\nabla J(X, Y)-\nabla J(Y, X)\}$. So

$$2\{A(X, Y)-A(JX, JY)\}=\nabla J(X, Y)-\nabla J(Y, X)-\nabla J(JX, JY)+\nabla J(JY, JX).$$

In view of (1.1), the right hand side becomes

$$(\nabla_Y J)X-(\nabla_X J)Y-(\nabla_{JY}J)(JX)+(\nabla_{JX}J)(JY).$$

which is, according to Theorem 1.1, the same as

$$\nabla_Y JX - J\nabla_Y X - \nabla_X JY + J\nabla_X Y + \nabla_{JY} X + J\nabla_{JY} JX - \nabla_{JX} Y - J\nabla_{JX} JY. \tag{3.4}$$

The Riemannian connection having torsion zero, we have $\nabla_X Y - \nabla_Y X = [X, Y]$ and multiplication of (3.4) by J gives (3.3).

From Theorem 3.1 we trivially get the following

COROLLARY 3.2. *If the covariant differential ∇J of an almost hermitian structure J is symmetric then J is integrable.*

II. The Fundamental 2-form ω and A

The fundamental 2-form ω on an almost hermitian manifold (M, J, g) is defined by

$$\omega(X, Y) = \langle JX, Y\rangle \quad \text{for} \quad X, Y \in \mathfrak{X}(M). \tag{3.5}$$

The torsion of the Riemann connection being zero, we know ([2], chap. III) that $d\omega = A(\nabla\omega)$. For vector fields X, Y, and Z on M, we have therefore:

$$6d\omega(X, Y, Z) = 6A(\nabla\omega)(X, Y, Z) = \mathfrak{s}(\nabla\omega(X, Y, Z)) - \mathfrak{s}(\nabla\omega(Y, X, Z)) \tag{3.6}$$

where $\mathfrak{s}$ denotes cyclic sum. By definition of the covariant differential and (3.5) we have

$$\begin{aligned}\nabla\omega(X, Y, Z) &= (\nabla_Z\omega)(X, Y) = Z\omega(X, Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y)\\ &= \langle\nabla_Z JX, Y\rangle + \langle JX, \nabla_Z Y\rangle - \langle J\nabla_Z X, Y\rangle - \langle JX, \nabla_Z Y\rangle\\ &= \langle\nabla_Z JX, Y\rangle - \langle J\nabla_Z X, Y\rangle = \langle(\nabla_Z J)X, Y\rangle = \langle\nabla J(X, Z), Y\rangle.\end{aligned}$$

Hence

$$\nabla\omega(X, Y, Z) = \langle\nabla J(X, Z), Y\rangle \tag{3.7}$$

In view of (2.1) and (3.2) we have the following

THEOREM 3.3. *On an almost hermitian manifold (M, J, g) the exterior differential $d\omega$ of the fundamental 2-form ω and the antisymmetric part of the covariant differential ∇J are related by the formula*

$$3d\omega(X, Y, Z) = -\mathfrak{s}(\langle A(X, Y), Z\rangle). \tag{3.8}$$

Together with Corollary 3.2 this result implies

COROLLARY 3.4. *If the covariant differential ∇J of an almost hermitian structure*

J is symmetric, then the fundamental 2-form ω is closed and J is integrable, i.e. (M, J, g) is a Kähler manifold.

There is another interesting relation between $d\omega$, A and ∇J; namely, we have

THEOREM 3.5. *On any almost hermitian manifold (M, J, q) we have for the fundamental 2-form ω:*

$$3d\omega(X, Y, Z) = -2\langle A(X, Y), Z\rangle + \langle \nabla J(X, Z), Y\rangle. \tag{3.9}$$

Proof. By the Palais formula, one has for any 2-form ω

$$\begin{aligned} 3d\omega(X, Y, Z) = {} & X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ & -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned}$$

By the definition of ω and because g is parallel with respect to ∇, together with the fact, once again, that the torsion of ∇ vanishes, we get the stated result.

III. A Relation Between A and S and an Identity for S

From Theorem 3.3 and Theorem 3.5 above we deduce the

COROLLARY 3.6. *If A (resp. S) is the antisymmetric (resp. symmetric) part of the covariant differential ∇J of an almost hermitian structure then, for vector fields X, Y, Z on M, we have*

$$\langle S(X, Y), Z\rangle = \langle A(X, Z), Y\rangle + \langle A(Y, Z), X\rangle. \tag{3.10}$$

Proof. From (3.2), (3.8) and (3.9) one has

$$\mathfrak{s}(\langle A(X, Y), Z\rangle) = 2\langle A(X, Y), Z\rangle + \langle A(X, Z), Y\rangle + \langle S(X, Z), Y\rangle.$$

But A is antisymmetric, so $A(X, Z) = -A(Z, X)$ and (3.10) follows by permuting Y and Z.

The antisymmetry of A has another consequence:

COROLLARY 3.7. *The symmetric part S of the covariant differential ∇J of an almost hermitian structure satisfies the following identity*

$$\mathfrak{s}(\langle S(X, Y), Z\rangle) = 0 \tag{3.11}$$

where $\mathfrak{s}$ denotes cyclic sum.

Proof. Write the left hand side of (3.11) with (3.10) and use the antisymmetry of A.

§4. A few Remarks

1. Because of (3.11) we can rewrite Theorem 3.3 in the following form:

THEOREM 3.3′. *On an almost hermitian manifold* (M, J, g) *the exterior differential* $d\omega$ *of the fundamental* 2-*form* ω *and the covariant differential* ∇J *of* J *are related by the formula*

$$3d\omega(X, Y, Z) = -\mathfrak{s}(\langle \nabla J(X, Y), Z\rangle). \tag{4.1}$$

2. An almost hermitian manifold (M, J, g) for which $S=0$ is already known [1] as *nearly Kähler manifold*. An alternative condition is given by the following

THEOREM 4.1. *A nearly Kähler manifold* (M, J, g) *is characterized by the condition*

$$\nabla\omega = d\omega. \tag{4.2}$$

Proof. We have to show that this condition is equivalent to $S=0$. Suppose (4.2) is true. Then $\nabla\omega$ is antisymmetric. By (3.7) $\nabla\omega(X, Y, Z) = \langle \nabla J(X, Z), Y\rangle$ and $\nabla\omega(Z, Y, X) = \langle \nabla J(Z, X), Y\rangle = -\langle \nabla J(X, Z), Y\rangle$ which implies that ∇J itself is antisymmetric, i.e. $S=0$.

On the other hand, $S=0$ implies $\nabla J = A$ and by (3.10) $\langle \nabla J(X, Z), Y\rangle + \langle \nabla J(Y, Z), X\rangle = 0$ which in turn gives by antisymmetry of ∇J: $\langle \nabla J(Z, X), Y\rangle + \langle \nabla J(Z, X), Y\rangle = 0$ or with (3.7), $\nabla\omega(Z, Y, X) + \nabla\omega(Z, X, Y) = 0$. But (even if $S \neq 0$) one has $\nabla\omega(X, Y, Z) + \nabla\omega(Y, X, Z) = 0$, and (3.6) gives the result.

3. Of the antisymmetric part A and the symmetric part S of ∇J, the former plays the most important role. It allows us to give as an application a very simple proof of the following theorem. Compare with [3] (chap. IX).

THEOREM 4.2. *An almost hermitian manifold* (M, J, g) *is a Kähler manifold* (*i.e.* $\tau=0$ *and* $d\omega=0$) *if and only if the covariant differential* ∇J *vanishes.*

Proof. If $\nabla J=0$ then $A=0$ which implies $\tau=0$ by (3.3) and $d\omega=0$ by (3.8). On the other hand, from Lemma 4.3 below and $d\omega=0$ we get $A=0$ and by (3.10), $S=0$.

LEMMA 4.3. *For an almost Kähler manifold, the antisymmetric part* A *of the covariant differential* ∇J *is essentially the torsion of* J*: more precisely we have*

$$4A = J\tau.$$

Proof. $d\omega=0$ implies by (3.9)

$$\langle \nabla J(X, Z), Y\rangle = 2\langle A(X, Y), Z\rangle.$$

Substituing JX to X and JY to Y and adding, we get

$$\langle \nabla J(X, Z), Y\rangle + \langle \nabla J(JX, Z), JY\rangle = 2\langle A(X, Y) + A(JX, JY), Z\rangle. \tag{4.3}$$

But the left hand side vanishes because $\nabla J(JX, Y) = -J\nabla J(X, Y)$ as it is easy to see and J is an isometry. To get the desired result one just has to add (4.3) to (3.3).

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