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On the Covariant Differential of an Almost Hermitian Structure

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This paper deals with the covariant differential ∇J of an almost hermitian structure J on a Riemannian manifold M with metric g . The connection with respect to which ∇J is defined is the Riemannian connection on M . The extension of the notion of antisymmetrization and symmetrization from tensor fields of type (o, q) to those of type (p, q) for $p > 0$ allows us to decompose ∇J in an antisymmetric part A and a symmetric part S . In this way, we find new formulas which have the feature of stressing the relationship between ∇J and the torsion τ of J as well as the fundamental 2-form ω or better, its exterior derivative $d\omega$. The results are essentially based first on the Palais formula which gives the exterior derivative of a q -form via Lie product and covariant derivative and second on a theorem ([2], p. 149) which gives the exterior derivative $d\alpha$ of a q -form α as the antisymmetric part of the covariant differential $\nabla\alpha$ of α , provided the connection has vanishing torsion. As an application of our formulas we give a characterization of so called nearly Kähler manifolds ([1]) via the fundamental 2-form ω . We also give a very simple proof of the characterization of a Kähler manifold given by the vanishing of ∇J or of $d\omega$ and τ . We finally prove a lemma which gives a nice interpretation of the torsion of J when the fundamental 2-form is closed, that is, in the case of an almost Kähler manifold.

§1. The Covariant Differential of a Tensor Field

Let t be a given tensor field of type (p, q) on a C^∞ manifold M . We shall simply write $t \in T_M(p, q)$ or $t \in T(p, q)$.

Suppose there is also a linear connection ∇ given on M . Then, as in [2], we can define the covariant differential of t as the tensor field $\nabla t \in T(p, q+1)$ defined by

$$\nabla t(X_1, \dots, X_q, X) = (\nabla_X t)(X_1, \dots, X_q) \tag{1.1}$$

where $\nabla_X t$ denotes the covariant derivative of the tensor field t and X_1, \dots, X_q, X are in the Lie algebra $\mathfrak{X}(M)$ of vector fields on M .

Because ∇_X is a derivation commuting with every contraction, we have

THEOREM 1.1 ([2], p. 124). *If $t \in T(p, q)$ then, for $X_i, X \in \mathfrak{X}(M)$*

$$\nabla t(X_1, \dots, X_q, X) = \nabla_X(t(X_1, \dots, X_q)) - \sum_{k=1}^q t(X_1, \dots, \nabla_{X_i} X_k, \dots, X_q).$$

EXAMPLE 1. Take $t=g$ a Riemann metric on M . g is in $T(0, 2)$, so ∇g is in $T(0, 3)$ and for $X, Y, Z, \in \mathfrak{X}(M)$ we have

$$\nabla g(X, Y, Z) = (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

One of the features of a Riemannian metric g is to be parallel i.e. $\nabla g = 0$. For convenience we shall write $\langle X, Y \rangle$ instead of $g(X, Y)$.

EXAMPLE 2. Take $t=J$ an almost complex structure on M . This is a tensor field of type $(1, 1)$ whose square J^2 equals minus the identity. ∇J is in $T(1, 2)$ and

$$\nabla J(X, Y) = (\nabla_Y J)X = \nabla_Y(JX) - J\nabla_Y X.$$

One of the features of a Kähler structure J on M is to be parallel with respect to ∇ : $\nabla J = 0$.

§2. Extension of Antisymmetrization and Symmetrization

It is known [1] how to define for a covariant tensor field t in $T(0, q)$ the *alternation* At of t . It is a tensor field of the same type defined by

$$(At)(X_1, \dots, X_q) = \frac{1}{q!} \sum_{\pi \in P_q} \varepsilon(\pi) t(X_{\pi(1)}, \dots, X_{\pi(q)}) \quad (2.1)$$

where P_q is the group of permutations of $\{1, 2, \dots, q\}$ and $\varepsilon(\pi)$ is the signe of the permutation π . Notice that At is a skew-symmetric tensor field and t is skew-symmetric if and only if $At = t$.

On the other hand one also defines for $t \in T(0, q)$ the *symmetrization* St of t by

$$St(X_1, \dots, X_q) = \frac{1}{q!} \sum_{\pi \in P_q} t(X_{\pi(1)}, \dots, X_{\pi(q)}). \quad (2.2)$$

Here is St a symmetric tensor field and t is symmetric if and only if $St = t$.

We now make the straightforward extension of the above notions to tensor fields of type (p, q) for $p > 0$. In this case $t(X_1, \dots, X_q)$ is no longer a real function on M , but a p -contravariant tensor field. Nevertheless, all algebraic operations needed for definitions (2.1) and (2.2) still make sense in the module of p -contravariant tensor fields.

EXAMPLE 1. Take $t=\tau \in T(1, 2)$ the torsion of an almost complex structure J defined by

$$\tau(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Here we have $A\tau = \tau$ and $S\tau = 0$.

EXAMPLE 2. Take $t = K \in T(1, 3)$ where $K(X, Y, Z) = R(X, Y)Z$ and $R \in T(1, 3)$ is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$.

Because $R(X, Y) = -R(Y, X)$ we have

$$(AK)(X, Y, Z) = \frac{1}{3}\xi(K(X, Y, Z))$$

(ξ means cyclic sum) and $SK = 0$. Furthermore if the torsion of ∇ vanishes, we have $AK = 0$ in view of the first Bianchi identity.

§3. Almost Hermitian Manifolds

From now on, we assume that (M, J, g) is an almost hermitian manifold, that is, the almost complex structure J on M gives in each point p of M an isometry of $T_p(M)$, the tangent space at M in p , i.e.

$$\langle JX, JY \rangle = \langle X, Y \rangle \quad \forall X, Y \in T_p(M). \tag{3.1}$$

Denoting by ∇ the Riemannian connection on M , we can compute ∇J which is a tensor field of type $(1, 2)$ and we can write the following decomposition

$$\nabla J = A(VJ) + S(VJ). \tag{3.2}$$

Needless to say, this is only possible because we are in $T(p, q)$ with $q = 2$. Example 2 above shows what can happen with $q \neq 2$.

For convenience we shall write A for $A(\nabla J)$ and S for $S(\nabla J)$ and establish several formulas relating A and/or S with various tensor fields one can define on an almost hermitian manifold.

I. The Torsion τ and A

THEOREM 3.1. *If A denotes the antisymmetric part of the covariant differential ∇J of the almost hermitian structure J on M , then*

$$\frac{1}{2}J\tau(X, Y) = A(X, Y) - A(JY, JY). \tag{3.3}$$

Proof. By definition $A(X, Y) = \frac{1}{2}\{\nabla J(X, Y) - \nabla J(Y, X)\}$. So

$$2\{A(X, Y) - A(JX, JY)\} = \nabla J(X, Y) - \nabla J(Y, X) - \nabla J(JX, JY) + \nabla J(JY, JX).$$

In view of (1.1), the right hand side becomes

$$(\nabla_Y J)X - (\nabla_X J)Y - (\nabla_{JY} J)(JX) + (\nabla_{JX} J)(JY).$$

which is, according to Theorem 1.1, the same as

$$\nabla_Y JX - J\nabla_Y X - \nabla_X JY + J\nabla_X Y + \nabla_{JY} X + J\nabla_{JY} JX - \nabla_{JX} Y - J\nabla_{JX} JY. \quad (3.4)$$

The Riemannian connection having torsion zero, we have $\nabla_X Y - \nabla_Y X = [X, Y]$ and multiplication of (3.4) by J gives (3.3).

From Theorem 3.1 we trivially get the following

COROLLARY 3.2. *If the covariant differential ∇J of an almost hermitian structure J is symmetric then J is integrable.*

II. The Fundamental 2-form ω and A

The fundamental 2-form ω on an almost hermitian manifold (M, J, g) is defined by

$$\omega(X, Y) = \langle JX, Y \rangle \quad \text{for } X, Y \in \mathfrak{X}(M). \quad (3.5)$$

The torsion of the Riemann connection being zero, we know ([2], chap. III) that $d\omega = A(\nabla\omega)$. For vector fields X, Y , and Z on M , we have therefore:

$$6d\omega(X, Y, Z) = 6A(\nabla\omega)(X, Y, Z) = \mathfrak{s}(\nabla\omega(X, Y, Z)) - \mathfrak{s}(\nabla\omega(Y, X, Z)) \quad (3.6)$$

where \mathfrak{s} denotes cyclic sum. By definition of the covariant differential and (3.5) we have

$$\begin{aligned} \nabla\omega(X, Y, Z) &= (\nabla_Z\omega)(X, Y) = Z\omega(X, Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y) \\ &= \langle \nabla_Z JX, Y \rangle + \langle JX, \nabla_Z Y \rangle - \langle J\nabla_Z X, Y \rangle - \langle JX, \nabla_Z Y \rangle \\ &= \langle \nabla_Z JX, Y \rangle - \langle J\nabla_Z X, Y \rangle = \langle (\nabla_Z J)X, Y \rangle = \langle \nabla J(X, Z), Y \rangle. \end{aligned}$$

Hence

$$\nabla\omega(X, Y, Z) = \langle \nabla J(X, Z), Y \rangle \quad (3.7)$$

In view of (2.1) and (3.2) we have the following

THEOREM 3.3. *On an almost hermitian manifold (M, J, g) the exterior differential $d\omega$ of the fundamental 2-form ω and the antisymmetric part of the covariant differential ∇J are related by the formula*

$$3d\omega(X, Y, Z) = -\mathfrak{s}(\langle A(X, Y), Z \rangle). \quad (3.8)$$

Together with Corollary 3.2 this result implies

COROLLARY 3.4. *If the covariant differential ∇J of an almost hermitian structure*

J is symmetric, then the fundamental 2-form ω is closed and *J* is integrable, i.e. (M, J, g) is a Kähler manifold.

There is another interesting relation between $d\omega$, *A* and ∇J ; namely, we have

THEOREM 3.5. *On any almost hermitian manifold (M, J, q) we have for the fundamental 2-form ω :*

$$3d\omega(X, Y, Z) = -2\langle A(X, Y), Z \rangle + \langle \nabla J(X, Z), Y \rangle. \quad (3.9)$$

Proof. By the Palais formula, one has for any 2-form ω

$$3d\omega(X, Y, Z) = X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$

By the definition of ω and because g is parallel with respect to ∇ , together with the fact, once again, that the torsion of ∇ vanishes, we get the stated result.

III. A Relation Between *A* and *S* and an Identity for *S*

From Theorem 3.3 and Theorem 3.5 above we deduce the

COROLLARY 3.6. *If *A* (resp. *S*) is the antisymmetric (resp. symmetric) part of the covariant differential ∇J of an almost hermitian structure then, for vector fields *X*, *Y*, *Z* on *M*, we have*

$$\langle S(X, Y), Z \rangle = \langle A(X, Z), Y \rangle + \langle A(Y, Z), X \rangle. \quad (3.10)$$

Proof. From (3.2), (3.8) and (3.9) one has

$$\mathfrak{s}(\langle A(X, Y), Z \rangle) = 2\langle A(X, Y), Z \rangle + \langle A(X, Z), Y \rangle + \langle S(X, Z), Y \rangle.$$

But *A* is antisymmetric, so $A(X, Z) = -A(Z, X)$ and (3.10) follows by permuting *Y* and *Z*.

The antisymmetry of *A* has another consequence:

COROLLARY 3.7. *The symmetric part *S* of the covariant differential ∇J of an almost hermitian structure satisfies the following identity*

$$\mathfrak{s}(\langle S(X, Y), Z \rangle) = 0 \quad (3.11)$$

where \mathfrak{s} denotes cyclic sum.

Proof. Write the left hand side of (3.11) with (3.10) and use the antisymmetry of *A*.

§4. A few Remarks

1. Because of (3.11) we can rewrite Theorem 3.3 in the following form:

THEOREM 3.3'. *On an almost hermitian manifold (M, J, g) the exterior differential $d\omega$ of the fundamental 2-form ω and the covariant differential ∇J of J are related by the formula*

$$3d\omega(X, Y, Z) = -s(\langle \nabla J(X, Y), Z \rangle). \quad (4.1)$$

2. An almost hermitian manifold (M, J, g) for which $S=0$ is already known [1] as *nearly Kähler manifold*. An alternative condition is given by the following

THEOREM 4.1. *A nearly Kähler manifold (M, J, g) is characterized by the condition*

$$\nabla\omega = d\omega. \quad (4.2)$$

Proof. We have to show that this condition is equivalent to $S=0$. Suppose (4.2) is true. Then $\nabla\omega$ is antisymmetric. By (3.7) $\nabla\omega(X, Y, Z) = \langle \nabla J(X, Z), Y \rangle$ and $\nabla\omega(Z, Y, X) = \langle \nabla J(Z, X), Y \rangle = -\langle \nabla J(X, Z), Y \rangle$ which implies that ∇J itself is antisymmetric, i.e. $S=0$.

On the other hand, $S=0$ implies $\nabla J = A$ and by (3.10) $\langle \nabla J(X, Z), Y \rangle + \langle \nabla J(Y, Z), X \rangle = 0$ which in turn gives by antisymmetry of ∇J : $\langle \nabla J(Z, X), Y \rangle + \langle \nabla J(Z, X), Y \rangle = 0$ or with (3.7), $\nabla\omega(Z, Y, X) + \nabla\omega(Z, X, Y) = 0$. But (even if $S \neq 0$) one has $\nabla\omega(X, Y, Z) + \nabla\omega(Y, X, Z) = 0$, and (3.6) gives the result.

3. Of the antisymmetric part A and the symmetric part S of ∇J , the former plays the most important role. It allows us to give as an application a very simple proof of the following theorem. Compare with [3] (chap. IX).

THEOREM 4.2. *An almost hermitian manifold (M, J, g) is a Kähler manifold (i.e. $\tau=0$ and $d\omega=0$) if and only if the covariant differential ∇J vanishes.*

Proof. If $\nabla J=0$ then $A=0$ which implies $\tau=0$ by (3.3) and $d\omega=0$ by (3.8). On the other hand, from Lemma 4.3 below and $d\omega=0$ we get $A=0$ and by (3.10), $S=0$.

LEMMA 4.3. *For an almost Kähler manifold, the antisymmetric part A of the covariant differential ∇J is essentially the torsion of J : more precisely we have*

$$4A = J\tau.$$

Proof. $d\omega=0$ implies by (3.9)

$$\langle \nabla J(X, Z), Y \rangle = 2\langle A(X, Y), Z \rangle.$$

Substituting JX to X and JY to Y and adding, we get

$$\langle \nabla J(X, Z), Y \rangle + \langle \nabla J(JX, Z), JY \rangle = 2 \langle A(X, Y) + A(JX, JY), Z \rangle. \quad (4.3)$$

But the left hand side vanishes because $\nabla J(JX, Y) = -J\nabla J(X, Y)$ as it is easy to see and J is an isometry. To get the desired result one just has to add (4.3) to (3.3).

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