

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 50 (1975)

**Artikel:** Semisimple Algebras and a Cancellation Law.  
**Autor:** Rao, M.L. Ranga  
**DOI:** <https://doi.org/10.5169/seals-38798>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 20.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Semisimple Algebras and a Cancellation Law

M. L. RANGA RAO

## Introduction

Let  $R$  be a commutative ring and  $A$  be an  $R$ -algebra which is finitely generated as  $R$ -module.<sup>1)</sup> For any ideal  $I$  of  $R$  the image of the canonical mapping  $I \otimes A \rightarrow A$  is an ideal of  $A$ , denoted by  $IA$  and called a scalar ideal. The algebra  $A$  is called an ideal algebra if  $I \rightarrow IA$  defines a bijection from the ideals of  $R$  to the ideals of  $A$  [8].  $A$  is called a semisimple  $R$ -algebra if any finitely generated  $A$ -module (left as well as right) is  $(A, R)$  projective in the sense of Hochschild's relative homology. A theorem of Endo-Watanabe [2] states that for central semisimple algebras the maximal ideals are of the form  $mA$ , where  $m$  is a maximal ideal of  $R$ . Since Azumaya algebras are semisimple and also ideal algebras, it is natural to ask whether or not central, projective, semisimple algebras are ideal algebras. In [8], it was shown that this is indeed the case, if  $R$  is a Dedekind domain.

In this note, we obtain a class of ideals which must be scalar ideals. We show that any ideal of a central, projective, semisimple algebra, lying over a radical ideal, is a scalar ideal. If in addition  $R$  is noetherian, we prove that any ideal lying over an irreducible ideal, must necessarily be a scalar ideal.

Finally, we prove two types of Skolem-Noether theorems over semilocal rings, and deduce a "Cancellation Law" for a class of semisimple algebras. In particular, we show that if  $A_1$  and  $A_2$  are projective, central, semisimple algebras, then  $M_n(A_1) \simeq M_n(A_2)$  implies  $A_1 \simeq A_2$ . Such "Cancellation Laws" for Azumaya algebras were studied by Knus [5], Ojanguren-Sridharan [6] and Roy-Sridharan [9].

For standard concepts and results regarding semisimple algebras and Azumaya algebras, we refer to Hattori [4], Endo-Watanabe [2, 3] and Bass [1].

## 1. Ideals of a Semisimple Algebra

We begin with a slight generalization of a result of Endo-Watanabe [2].

**LEMMA 1.1.** *Let  $(R, m)$  be a local ring such that every finitely generated ideal*

---

<sup>1)</sup> Throughout this paper we assume that all algebras are finitely generated as modules. We call an  $R$ -algebra projective if it is projective as  $R$ -module.  $\otimes$  will stand for  $\otimes_R$ . By an ideal, we always mean a two sided ideal.

has a non trivial annihilator. Let  $\Lambda$  be a central algebra over  $R$  with a unique maximal ideal  $m\Lambda$ . Then  $\Lambda$  is Azumaya if it is a generator as  $R$ -module.

*Proof.* Since  $\Lambda/m\Lambda$  is simple, it suffices to show that  $R/m$  is the center of  $\Lambda/m\Lambda$ . Let  $\bar{\lambda}$  be in the center of  $\Lambda/m\Lambda$  and  $\lambda$  be a representative of  $\bar{\lambda}$  in  $\Lambda$ . Then we have  $\lambda\alpha - \alpha\lambda \in m\Lambda$  for all  $\alpha \in \Lambda$ . As  $\Lambda$  is finitely generated as  $R$ -module, it follows that  $\lambda\alpha - \alpha\lambda \in I\Lambda$  for all  $\alpha \in \Lambda$ , where  $I$  is a finitely generated ideal of  $R$ . Let  $0 \neq s \in R$  such that  $sI = 0$ . Then we have  $s(\lambda\alpha - \alpha\lambda) \in sI\Lambda = 0$  for all  $\alpha$  and therefore  $s\lambda \in R$ .

As  $\Lambda$  is a generator as  $R$ -module,  $\Lambda = R \oplus M$  for some  $R$ -module  $M$ . If  $\lambda = r + v$ , it is clear that  $sv = 0$ . Therefore  $v$  generates a proper ideal in  $\Lambda$  and must necessarily be contained in  $m\Lambda$ . So we have  $\bar{\lambda} = \bar{r} \in R/m$ .

**PROPOSITION 1.2.** *Let  $R$  be either a ring of Krull dimension zero or a total quotient ring of a noetherian ring. If  $\Lambda$  is a projective, central, semisimple  $R$ -algebra, then it is Azumaya and in particular an ideal algebra.*

*Proof.* By [2], the maximal ideals of  $\Lambda$  are of the form  $m\Lambda$ , where  $m$  is a maximal ideal of  $R$ . Also every finitely generated ideal of  $R_m$  has a non trivial annihilator. By Lemma 1.1.,  $\Lambda$  is locally Azumaya and therefore is an Azumaya algebra.

The following is easy to verify

**LEMMA 1.3.** *Let  $R_S$  be the total quotient ring of  $R$ . If  $\Lambda$  is a projective  $R$ -algebra such that any non zero ideal of  $\Lambda_S$  contains a non zero element of  $R_S$ , then every non zero ideal of  $\Lambda$  also contains a non zero element of  $R$ .*

**PROPOSITION 1.4.** *Let  $\Lambda$  be a projective, central, semisimple  $R$ -algebra. Any ideal of  $\Lambda$ , lying over a radical ideal of  $R$ , is a scalar ideal.*

*Proof.* First suppose that  $\mathfrak{U}$  is any prime ideal of  $\Lambda$ . Let  $\mathfrak{U} \cap R = P$ . Since  $\Lambda_p$  is again an  $R_p$ -central semisimple algebra [4],  $\Lambda_p/P\Lambda_p$  is a simple ring. As  $\Lambda/P\Lambda$  is  $R/P$ -projective and  $R_p/PR_p$  is the quotient field of  $R/P$ , by Lemma 1.3. every non zero ideal of  $\Lambda/P\Lambda$  contains a non zero element of  $R/P$ . Thus the image of  $\mathfrak{U}$  in  $\Lambda/P\Lambda$  is zero, i.e.  $\mathfrak{U} = P\Lambda$ .

Now, let  $\mathfrak{U}$  be an ideal of  $\Lambda$  such that  $\mathfrak{U} \cap R = I$  is a radical ideal ( $\sqrt{I} = I$ ). Let  $P$  be any prime ideal of  $R$  containing  $I$ .  $\mathfrak{U}_p \cap R_p = I_p$  and hence,  $\mathfrak{U}_p$  is a proper ideal of  $\Lambda_p$ .  $P\Lambda_p$ , being the unique maximal ideal of  $\Lambda_p$ , must contain  $\mathfrak{U}_p$ . Contracting to  $\Lambda$ , we have  $\mathfrak{U} \subset P\Lambda$ . Thus  $\mathfrak{U} \subset \bigcap P\Lambda$  where the intersection is taken over all prime ideals containing  $I$ . Because  $\Lambda$  is  $R$ -projective,  $\mathfrak{U} \subset \bigcap P\Lambda = (\bigcap P)\Lambda = I\Lambda$ , the desired conclusion follows.

**LEMMA 1.5.** *Let  $(R, m)$  be a commutative quasi-Frobenius local ring.  $\Lambda$  be a projective algebra over  $R$  with a unique maximal ideal  $m\Lambda$ . Then  $\Lambda$  is also quasi-Frobenius. Furthermore every non zero ideal of  $\Lambda$  contains a non zero element of  $R$ .*

*Proof.* We have  $\text{Ext}_R^1(R/m, R) = 0$ .  $\Lambda$ , being finitely generated as  $R$ -module, is artinian. Also  $\Lambda$  is  $R$ -free. Therefore  $\Lambda \otimes \text{Ext}_R^1(R/m, R) \simeq \text{Ext}_\Lambda^1(\Lambda/m\Lambda, \Lambda) = 0$ .  $\Lambda/m\Lambda$  is an artinian simple ring and hence any two simple left  $\Lambda$ -modules are isomorphic. Moreover,  $\Lambda/m\Lambda \simeq S \oplus \cdots \oplus S$  where  $S$  is a simple left  $\Lambda$ -module. Thus  $\text{Ext}_\Lambda^1(S, \Lambda) = 0$ . Therefore  $\Lambda$  is left  $\Lambda$ -injective [7].

To prove the second assertion, consider the annihilator of  $m\Lambda$  in  $\Lambda$ . It is precisely the annihilator of  $m$  in  $\Lambda$ . Since  $\Lambda$  is  $R$ -projective, we have  $(0 :_\Lambda m) = (0 :_R m) \Lambda = I\Lambda$ , where  $I$  is the annihilator of  $m$  in  $R$ . As  $\Lambda$  is quasi-Frobenius, it satisfies the “annihilator condition”. Therefore  $I\Lambda$ , the annihilator of the unique maximal ideal, must necessarily be the unique minimal ideal of  $\Lambda$ . Thus every non zero ideal of  $\Lambda$  contains at least  $I$ .

Finally, here is the main result of this section:

**PROPOSITION 1.6.** *Suppose  $R$  is a noetherian ring and let  $\Lambda$  be a projective, central, semisimple  $R$ -algebra. Then any ideal of  $\Lambda$ , lying over an irreducible ideal of  $R$ , is a scalar ideal.*

*Proof.* Let  $\mathfrak{A}$  be an ideal of  $\Lambda$  such that  $\mathfrak{A} \cap R = I$  is a  $P$ -irreducible ideal. It is enough to show that every non zero ideal of  $\Lambda/I\Lambda$  contains a non zero element of  $R/I$ .

It is clear that  $IR_p$  is a  $PR_p$ -irreducible ideal of  $R_p$  and hence,  $R_p/IR_p$  is an artinian ring with a unique minimal ideal. Therefore it is quasi-Frobenius [Kaplanski’s “commutative rings”]. Then the  $R_p/IR_p$ -algebra  $\Lambda_p/I\Lambda_p$  satisfies all the conditions of Lemma 1.5. Thus every non zero ideal of  $\Lambda_p/I\Lambda_p$  contains a non zero element of  $R_p/IR_p$ . Since  $R_p/IR_p$  is the total quotient ring of  $R/I$ , by Lemma 1.3 we have the desired conclusion.

*Remark 1.* It is known that if  $\Lambda$  is an  $R$ -ideal algebra,  $\Lambda$  is necessarily a faithfully flat  $R$ -module [8]. Since there exists non projective, central, semisimple algebras over artinian rings [3], not every central semisimple algebra is an ideal algebra.

*Remark 2.* In Lemma 1.5., we have not assumed that  $\Lambda$  is semisimple. Thus we have actually shown that, if  $P\Lambda$  is a prime ideal of a projective  $R$ -algebra  $\Lambda$ , then any ideal of  $\Lambda$ , lying over a  $P$ -irreducible ideal of  $R$ , is necessarily a scalar ideal.

## 2. $\otimes$ with an Azumaya Algebra

**LEMMA 2.1.** *Let  $A$  be an Azumaya algebra over  $R$ . Then there exists an Azumaya  $R$ -algebra  $B$  such that  $A \otimes B \simeq M_n(R)$ .*

*Proof.* Since  $A$  is a faithfully projective  $R$ -module, there exists a faithfully projective  $R$ -module  $Q$ , such that  $A \otimes Q \simeq R^n$  for some  $n$  [1]. Setting  $B = A^0 \otimes \text{End}_R(Q, Q)$ , we have  $A \otimes B \simeq M_n(R)$ .

**LEMMA 2.2.** *Let  $\Lambda$  and  $A$  be  $R$ -algebras (not necessarily finitely generated) and also assume that  $A$  is faithfully flat as  $R$ -module. Then  $(\mathfrak{A} \otimes A) \cap \Lambda = \mathfrak{A}$  for any ideal  $\mathfrak{A}$  of  $\Lambda$ .*



*Proof.* Consider the following commutative diagram

$$\begin{array}{ccccccc}
 0 \rightarrow & \mathfrak{U} & \rightarrow & \Lambda & \rightarrow & \Lambda/\mathfrak{U} & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \mathfrak{U} \otimes A & \rightarrow & \Lambda \otimes A & \rightarrow & \Lambda/\mathfrak{U} \otimes A & \rightarrow 0
 \end{array}$$

The lower row is exact and the vertical maps ( $a \rightarrow a \otimes 1$ ) are monomorphisms because  $A$  is faithfully flat. The assertion follows by diagram chasing.

The following is a natural generalization of a classical result and may be of some independent interest

**PROPOSITION 2.3.** *Let  $\Lambda$  be an  $R$ -algebra (not necessarily finitely generated) and  $A$  be an Azumaya  $R$ -algebra. Then  $\mathfrak{U} \rightarrow \mathfrak{U} \otimes A$  defines a bijection from the ideals of  $\Lambda$  to the ideals of  $\Lambda \otimes A$ .*

*Proof.* As  $A$  is a faithfully projective  $R$ -module, the above map is a monomorphism [Lemma 2.2]. Let  $J$  be any ideal of  $\Lambda \otimes A$ . Suppose  $B$  is an Azumaya  $R$ -algebra such that  $A \otimes B \simeq M_n(R)$  [Lemma 2.1]. Then  $J \otimes B$  is an ideal of  $\Lambda \otimes M_n(R)$  and hence, must be of the form  $\mathfrak{U} \otimes M_n(R)$  where  $\mathfrak{U} = (J \otimes B) \cap \Lambda$ . Thus  $J \otimes B = \mathfrak{U} \otimes A \otimes B$ . Since  $B$  is also faithfully projective, we have  $J = \mathfrak{U} \otimes A$ .

**PROPOSITION 2.4.** *Suppose  $\Lambda$  is a faithful, semisimple algebra over  $R$  whose maximal ideals are scalar ideals. For any Azumaya  $R$ -algebra  $A$ ,  $\Lambda \otimes A$  is also a faithful, semisimple algebra whose maximal ideals are scalar ideals.*

*Proof.* It is known that  $\Lambda \otimes A$  is a semisimple algebra [4]. The rest of the assertion is an immediate consequence of the above proposition.

### 3. Skolem-Noether Theorems

Throughout this section,  $R$  denotes a semilocal ring. Also all semisimple algebras considered in this section are faithful semisimple algebras whose maximal ideals are scalar ideals. Of course any central semisimple algebra is of this type. This class also includes some non central semisimple algebras.

**LEMMA 3.1.** *Let  $m_1, m_2, \dots, m_n$  be the maximal ideals of  $R$  and  $J = \bigcap m_i$ . If  $\Lambda$  is a semisimple algebra over  $R$ , then  $\Lambda/J\Lambda \simeq \Lambda/m_1\Lambda \times \dots \times \Lambda/m_n\Lambda$ .*

*Proof.* Since  $m_i\Lambda$  are the maximal ideals of  $\Lambda$ , the lemma is a consequence of the Chinese Remainder Theorem, provided that  $J\Lambda = \bigcap m_i\Lambda$ .

$\Lambda/J\Lambda$  is a semisimple  $R/J$ -algebra [4]. Since  $R/J$  is a classical semisimple ring, it is easy to verify that  $\Lambda/J\Lambda$  must also be a classical semisimple ring. Hence, its Jacobson

radical, which is zero, must be the intersection of its maximal ideals. Therefore  $J\Lambda = \bigcap m_i \Lambda$ .

**PROPOSITION 3.2.** *Suppose  $\Lambda$  is a semisimple algebra over  $R$ . If  $P_1$  and  $P_2$  are two finitely generated  $R$ -projective left  $\Lambda$ -modules such that  $\text{rank}_R P_1 = \text{rank}_R P_2$ , then  $P_1 \simeq P_2$ .*

*Proof.* Since  $\Lambda$  is semisimple,  $P_1$  and  $P_2$ , being  $R$ -projective, are  $\Lambda$ -projective [4]. So it is sufficient to show that  $P_1/J\Lambda \simeq P_2/J\Lambda$  where  $J$  is the Jacobson radical of  $R$ . Thus we can assume that  $R = R_1 \times \cdots \times R_n$ , where each  $R_i$  is a field.

Lemma 3.1 implies that  $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$  where each  $\Lambda_i$  is a simple  $R_i$ -algebra. This induces a decomposition on  $P_1$  and  $P_2$ . Let  $P_1 = P_{11} \oplus \cdots \oplus P_{1n}$  and  $P_2 = P_{21} \oplus \cdots \oplus P_{2n}$ . Then  $P_{1i}$  and  $P_{2i}$  are  $\Lambda_i$ -modules which have the same dimension over  $R_i$  (because of rank condition) and hence, must be  $\Lambda_i$ -isomorphic. Thus  $P_1 \cong P_2$ .

**THEOREM 1.** *Let  $\Lambda$  be a semisimple algebra over  $R$  and  $A$  be an Azumaya  $R$ -algebra. Suppose  $f, g: \Lambda \rightarrow A$  are  $R$ -algebra mappings. Then there exists an inner automorphism  $\theta$  of  $A$  such that  $g = \theta \circ f$ .*

*Proof.* It is enough to show the existence of a unit  $u$  of  $A$  such that  $uf(\lambda) = g(\lambda)u$  for all  $\lambda \in \Lambda$ . Let  ${}_f A$  and  ${}_g A$  denote the left  $\Lambda$ -module structures on  $A$  through  $f$  and  $g$  respectively. Consider the left  $\Lambda \otimes A^0$ -modules  ${}_f A_A$  and  ${}_g A_A$ . Since  $\Lambda \otimes A^0$  is again semisimple, whose maximal ideals are scalar ideals [Prop. 2.4], and  $A$  is  $R$ -projective, by Prop. 3.2, there exists  $\eta: {}_f A_A \rightarrow {}_g A_A$  which is a  $\Lambda \otimes A^0$ -isomorphism.  $\eta$ , being a right  $A$ -linear isomorphism, must be of the form  $\eta(\bar{\lambda}) = u\bar{\lambda}$  where  $u$  is a unit. The left  $\Lambda$ -linearity implies  $uf(\lambda) = g(\lambda)u$  for all  $\lambda \in \Lambda$ .

**THEOREM 2.** *Let  $\Lambda$  be a projective semisimple algebra over  $R$  and  $A$  be an Azumaya  $R$ -algebra. Let  $f, g: \Lambda \rightarrow A$  be  $R$ -monomorphisms. Then there exists an inner automorphism  $\theta$  of  $A$  such that  $g = \theta \circ f$ .*

*Proof.* Since it is assumed that  $\Lambda$  is  $R$ -projective one has only to consider the left  $\Lambda \otimes A^0$ -modules  ${}_f \Lambda_A$  and  ${}_g \Lambda_A$  and imitate the proof of Theorem 1.

**CANCELLATION LAW:** *Let  $\Lambda_1$  and  $\Lambda_2$  be projective semisimple algebras over  $R$ . Suppose  $A$  is an Azumaya  $R$ -algebra such that  $A \otimes \Lambda_1 \simeq A \otimes \Lambda_2$ . Then  $\Lambda_1 \simeq \Lambda_2$ .*

*Proof.* Let  $h: A \otimes \Lambda_1 \rightarrow A \otimes \Lambda_2$  be an isomorphism. We have two monomorphisms of  $A$  into  $A \otimes \Lambda_2$  namely,  $f: A \rightarrow A \otimes 1 \subset A \otimes \Lambda_2$  and  $g: A \rightarrow A \otimes 1 \subset A \otimes \Lambda_1 \rightarrow A \otimes \Lambda_2$ . By the above theorem, there exists an automorphism  $\theta$  of  $A \otimes \Lambda_2$  such that  $g = \theta \circ f$ . Hence the commutants of  $f(A)$  and  $g(A)$  in  $A \otimes \Lambda_2$  are isomorphic. But these commutants are isomorphic to  $\Lambda_2$  and  $\Lambda_1$  respectively, which proves the proposition.

## REFERENCES

- [1] BASS, H., *Topics in algebraic K-theory*, Tata notes no. 41, Bombay, 1967.
- [2] ENDO, S. and WATANABE, Y., *The centers of a semisimple algebra over a commutative ring*, Nagoya Math. J. 30 (1967), 285–293.
- [3] ———, *The centers of a semisimple algebra over a commutative ring II*, Nagoya Math. J. 39 (1970), 1–6.
- [4] HATTORI, A., *Semisimple algebras over a commutative ring*, J. Math. Soc. Japan 15 (1963), 404–419.
- [5] KNUS, M. A., *Algèbres d’Azumaya et modules projectifs*, Comment. Math. Helv. 45 (1970), 372–383.
- [6] OJANGUREN, M. and SRIDHARAN, R., *Cancellation of Azumaya algebras*, J. Algebra 18 (1971), 501–505.
- [7] RAMRAS, M. B., *Maximal orders over regular local rings of dimension 2*, dissertation, Brandeis University, 1967.
- [8] RANGA. RAO, M. L., *Azumaya, semisimple and ideal algebras*, Bull. Amer. Math. Soc. 78 (1972).
- [9] ROY, A. and SRIDHARAN, R., *Derivations in Azumaya algebras*, J. Math. Kyoto Univ. 7 (1963), 161–167.

*Forschungsinstitut für Mathematik  
ETH  
8006 Zürich, Switzerland*

Current Address:

*Department of Mathematics  
Osmania University  
Hyderabad  
India*

Received August 14, 1973/November 12, 1974