Stable Vector Bundles over the Projective Orthogonal Groups

René P. Held and U. Suter

Introduction

Let G be a compact connected Lie group of rank r. If the fundamental group $\pi_1(G) = \pi$ is trivial, then Hodgkin [9] showed that the complex K-theory of G is an exterior algebra (over the integers) generated by r elements arising from the basic irreducible representations of G.

Now suppose that π is a non-trivial, *finite* group. Modulo torsion $K^*(G)$ is again an exterior algebra and therefore

$$K^*(G) \cong \{E_{\mathbf{Z}}(\alpha_1, \ldots, \alpha_r) \otimes T^*(G)\}/S(G),$$

where $\alpha_1, \ldots, \alpha_r \in K^1(G)$ are elements representing generators of the exterior algebra $K^*(G)/\mathrm{Tors}\, K^*(G)$, $T^*(G) = T^0(G) \oplus T^1(G)$ is a certain $\mathbf{Z}_2$-graded subalgebra of $K^*(G)$, generated by 1 and some elements of finite order, and $S(G)$ is the ideal generated by the "relations".

In the case when $\pi \cong \mathbf{Z}_p$, where p is a prime, the authors [8] proved that

$$T^*(G) \cong T^0(G) \cong R(\pi)/(j^*(I_{G_0})),$$

where $R(\pi)$ is the complex representation ring of the covering transformation group π of the universal covering $u: G_0 \to G$, $j^*: R(G_0) \to R(\pi)$ the homomorphism induced by the inclusion $j: \pi \hookrightarrow G_0$ and $(j^*(I_{G_0}))$ the ideal generated by j^*-image of the augmentation ideal I_{G_0} of $R(G_0)$. Furthermore $T^0(G)$ coincides with the image of the homomorphism $c^*: K^0(B_\pi) \to K^0(G)$ induced by the map $c: G \to B_\pi$ classifying the universal covering of G. The ideal $S(G)$ in this case is given by

$$S(G) = (\alpha_r \otimes \tilde{T}^0(G)),$$

where $T^0(G) \cong \mathbf{Z} \oplus \tilde{T}^0(G)$.

In this paper we propose to give a complete description of the ringstructure of the unitary K-theory for the family of the *projective orthogonal groups* $PSO(m)$. Note that if m is odd then we have $PSO(m) = SO(m)$; the ring $K^*(SO(m))$ is already known see [7], [8] or [6]. If m is even, say $m = 2n$, we shall distinguish between the "*cyclic*" case,

i.e. n odd and hence $\pi_1(\mathrm{PSO}(2n)) \cong Z_4$, and the "*non-cyclic*" case, i.e. n even and hence $\pi_1(\mathrm{PSO}(2n)) \cong Z_2 \times Z_2$. In the "cyclic" case it again turns out that $T^1(G)$ is zero and that $T^0(G)$ can be identified with the image $(c^*) \cong R(\pi)/(j^*(I_{G_0}))$, thus in this respect extending the results of [8]. However in the "non-cyclic" case it is no longer true that the ring $K^*(G)$ is generated by the image of the homomorphism c^* and the free generators $\alpha_1, \ldots, \alpha_r \in K^1(G)$. The enquiry after the generators of $K^*(\mathrm{PSO}(4t))$ then leads to the definition of a crucial stable vector bundle τ over the suspension of $\mathrm{PSO}(4t)$. The element $\tau \in K^1(\mathrm{PSO}(4t))$ will be given in terms of the *transfer maps* associated to the two *semi-spin coverings of $PSO(4t)$* (see (4.2)). The main result of this paper may then be paraphrased as follows (see (6.2), (7.2)).

Let $G = \mathrm{PSO}(2n)$, n even. Then $T^*(G) = T^0(G) \oplus T^1(G)$ is generated by 1 and elements ξ_1, $\xi_2 \in \mathrm{im}\, c^* \subset K^0(G)$ and $\tau \in K^1(G)$ such that the following relations hold

(i) *The elements ξ_1, $\xi_1\xi_2$ and $\xi_2\tau$ are of order 2^{k-1} where $k = v_2(n) + 2$. The element τ is of order 2^k whereas ξ_2 is of order 2^{n-1}.*

(ii) *$\xi_1^2 + 2\xi_1 = 0$, $\xi_2^2 + 2\xi_2 = 0$, $\tau^2 = 0$, $\tau\xi_1 + 2\tau = 0$.*

The ideal $S(G) \subset E_Z(\alpha_1, \ldots, \alpha_r) \otimes T^*(G)$ is generated by the following elements:

$$\alpha_{n-1} \otimes \xi_1,\ \alpha_n \otimes \xi_2,\ \alpha_{n-1} \otimes \tau,\ \alpha_n \otimes \tau,\ 1 \otimes 2^{k-1}\tau - \alpha_{n-1} \otimes 2^{n-2}\xi_2$$

and

$$1 \otimes \tau\xi_2 + 1 \otimes 2\tau - \alpha_n \otimes \xi_1.$$

(i.e. in $K^*(G)$ one has the relations $\alpha_{n-1}\xi_1 = 0$, $\alpha_n\xi_2 = 0$, $\alpha_{n-1}\tau = 0$, $\alpha_n\tau = 0$, $2^{k-1}\tau = 2^{n-2}\xi_2\alpha_{n-1}$, $\tau\xi_2 + 2\tau = \alpha_n\xi_1$.)

The proof of this result rests on the relationship between complex K-theory and the *complex representation ring* of a Lie group, the *Atiyah-transfer* homomorphism and a very detailed analysis of various *spectral sequences*.

The different geometric and "algebraic topological" features of $\mathrm{PSO}(4t+2)$ and $\mathrm{PSO}(4t)$ suggest that the two cases be looked at separately. In the layout of this paper the emphasis is put on the "non-cyclic" case (see section 1 to 6), whereas the main steps leading to the result in the "cyclic" case are just summarized; see section 7.

I. THE NON-CYCLIC CASE; $\pi_1(\mathrm{PSO}(2n)) \cong Z_2 \times Z_2$

1. Restricting Representation of Spin $(2n)$ to its Central Subgroups.

(1.1). Throughout Chapter I let $n \geqslant 6$ be an *even* integer and $k = v_2(n) + 2$, where $v_2(n)$ is the exponent of the highest power of 2 dividing n. The centre of $G_0 = \mathrm{Spin}(2n)$ is denoted by π. Hence $\pi \cong Z_2 \times Z_2$, and in accordance with Tits [11; p. 36] we choose generators z and z' of π. We shall consider the Lie groups of the form G_0/ω where

$\omega \cong \mathbf{Z}_2$ is one of the three possible subgroups of π. If $\omega = \omega_1$ is the subgroup generated by z we get the *semi-spin group* $G_1 = G_0/\omega_1$; if $\omega = \omega_3$ is generated by z' then it is well known that $G_0/\omega_3 = G_3$ is isomorphic to G_1. If $\omega = \omega_2$ is generated by $z \cdot z'$ – (diagonal subgroup of π) – we get the *special orthogonal group* $G_2 = G_0/\omega_2 = SO(2n)$. The *projective orthogonal group* $PSO(2n)$ is defined to be $G_0/\pi = G$.

(1.2). The complex representation ring $R(\pi)$ is generated, as a free abelian group, by 1, ϱ_1, ϱ_2 and ϱ_3 where the representations

$$\varrho_i : \pi \to S^1 \quad (i = 1, 2, 3)$$

are defined as follows:

$$\begin{aligned}
\varrho_1(z) &= -1 = \varrho_1(z') \\
\varrho_2(z) &= 1, \quad \varrho_2(z') = -1 \\
\varrho_3(z) &= -1, \quad \varrho_3(z') = 1
\end{aligned} \tag{1.3}$$

The representations ϱ_i, $(i = 1, 2, 3)$, satisfy

$$\varrho_i^2 = 1, \quad \varrho_1 \cdot \varrho_2 = \varrho_3 . \tag{1.4}$$

The augmentation ideal I_π of $R(\pi)$ is generated, as a free abelian group, by σ_1, σ_2 and σ_3 where $\sigma_i = \varrho_i - 1$ $(i = 1, 2, 3)$ with relations

$$\sigma_i^2 + 2\sigma_i = 0, \quad \sigma_1\sigma_2 + \sigma_1 + \sigma_2 = \sigma_3 . \tag{1.5}$$

The representation ring of $\omega_i \cong Z_2$, $(i = 1, 2)$, is given by

$$R(\omega_i) \cong Z[\theta_i]/(\theta_i^2 - 1)$$

where $\theta_i : \omega_i \to S^1$ is the canonical representation. The augmentation ideal I_{ω_i} is generated by $\kappa_i = \theta_i - 1$, with relation $\kappa_i^2 + 2\kappa_i = 0$.

The representation ring of G_0 is a polynomial ring

$$R(G_0) \cong Z[\lambda_1, \lambda_2, \dots, \lambda_n] \tag{1.6}$$

where the generator λ_s, $(s = 1, 2, \dots, n-2)$, is the s-th exterior power of the canonical representation $G_0 \xrightarrow{a_2} G_2 \hookrightarrow U(2n)$ (a_2 being the two-fold covering map of $G_2 = SO(2n)$), whereas λ_{n-1}, λ_n stand for the spin-representations Δ^+ and Δ^-. Hence the augmentation ideal I_{G_0} is, as a ring, generated by the elements

$$\tilde{\lambda}_s = \lambda_s - \dim \lambda_s \quad (s = 1, 2, \dots, n) . \tag{1.7}$$

Let $e_i : \omega_i \hookrightarrow \pi$, $(i = 1, 2)$, be the inclusion map. Denoting by $j : \pi \hookrightarrow G_0$ the inclusion of the centre, we define the map $j_i : \omega_i \hookrightarrow G_0$ to be $j_i = j \circ e_i$.

Thus the homomorphisms $e_i^* : R(\pi) \to R(\omega_i)$ are given by

$$
\begin{aligned}
e_1^*(\varrho_1) &= \theta_1 = e_1^*(\varrho_3), & e_1^*(\varrho_2) &= 1 \\
e_2^*(\varrho_2) &= \theta_2 = e_2^*(\varrho_3), & e_2^*(\varrho_1) &= 1.
\end{aligned}
\tag{1.8}
$$

According to [11; p. 36] the homomorphism $j^* : R(G_0) \to R(\pi)$ is determined by

$$
j^*(\lambda_s) =
\begin{cases}
\dbinom{2n}{s} \varrho_1, & \text{for } s \text{ } odd \text{ and } 1 \leqslant s < n-2 \\[2ex]
\dbinom{2n}{s}, & \text{for } s \text{ } even \text{ and } 1 < s \leqslant n-2
\end{cases}
\tag{1.9}
$$

$$
j^*(\lambda_{n-1}) = 2^{n-1} \varrho_2, \qquad j^*(\lambda_n) = 2^{n-1} \varrho_3.
$$

The maps $j_i^* : R(G_0) \to R(\mathbf{Z}_2)$, $(i = 1, 2)$, are given by (1.8), (1.9) and $j_1^* = e_1^* \circ j^*$, $j_2^* = e_2^* \circ j^*$.

A straight forward calculation using (1.8) and (1.9) establishes the following result.

(1.10) PROPOSITION. (i) *If* $J = (j^*(I_{G_0}))$ *is the ideal generated by* $j^*(I_{G_0})$, *then* $R(\pi)/J \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}} \oplus \mathbf{Z}_{2^{n-1}} \oplus \mathbf{Z}_{2^{k-1}}$, *where* $k = v_2(n) + 2$. *Generators for the three finite cyclic sumands may be represented by* σ_1, σ_2 *and* $\sigma_1 \sigma_2$ *respectively.*

(ii) *If* $J_1 = (j_1^*(I_{G_0}))$, *then* $R(\omega_1)/J_1 \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}}$, *with* κ_1 *representing a generator of* $\mathbf{Z}_{2^{k-1}}$.

(iii) *If* $J_2 = (j_2^*(I_{G_0}))$, *then* $R(\omega_2)/J_2 \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{n-1}}$, *with* κ_2 *representing a generator of* $\mathbf{Z}_{2^{n-1}}$.

(1.11) *Remark.* The canonical ring homomorphisms $h_i : R(\pi)/J \to R(\omega_i)/J_i$, $(i = 1, 2)$, are given by $h_1(\sigma_1) = \kappa_1$, $h_1(\sigma_2) = 0$ and $h_2(\sigma_1) = 0$, $h_2(\sigma_2) = \kappa_2$.

2. The Homomorphism in K-theory Induced by the Universal Covering of $G = \mathrm{PSO}(2n)$.

Let us begin with a few observations concerning the universal covering $u : M_0 \to M_0/\omega = M$ of a compact Lie group M of rank r, having finite fundamental group ω. Since $K^*(M_0)$ is torsion free (see [9]) the map $u^* : K^*(M) \to K^*(M_0)$ factors through $K^*(M)/\mathrm{Tors}\, K^*(M)$, thus giving rise to the homomorphism $\bar{u} : K^*(M)/\mathrm{Tors}\, K^*(M) \to K^*(M_0)$. As $\mathbf{Z}_2$-graded Hopf algebras, both $K^*(M)/\mathrm{Tors}\, K^*(M)$ and $K^*(M_0)$ are *exterior algebras* on the group of primitive elements denoted by P and P_0 respectively. The image of u^* is therefore a primitively generated exterior subalgebra of $K^*(M_0)$ and is determined by

$$
\bar{u}(P) = (\mathrm{im}\, u^*) \cap P_0.
$$

We now aim at giving a description of this latter group. There are elements v_1, v_2, ..., $v_r \in K^1(M)$ representing a basis of P and elements μ_1, μ_2, ..., $\mu_r \in P_0 \subset K^1(M_0)$ forming a basis of P_0 such that

$$u^*(v_s) = m_s \mu_s, \quad 0 < m_s \in Z, \quad (s = 1, 2, ..., r). \tag{2.1}$$

(2.2) LEMMA. *The product of the integers $m_1, m_2, ..., m_r$ is equal to the order of ω, i.e. $m_1 m_2 ... m_r = |\omega|$.*

Proof. In $K^*(M_0)$ we have $u^*(v_1 v_2 ... v_r) = m_1 m_2 ... m_r \cdot \lambda_1 \lambda_2 ... \lambda_r$. We shall prove that $u^*(v_1 v_2 ... v_r) = |\omega| \lambda_1 \lambda_2 ... \lambda_r$. This is seen as follows. For ordinary cohomology with integer coefficients the homomorphism u^* restricted to the top dimensional cohomology class of $H^*(M; Z)$ is multiplication by $|\omega|$. This together with the fact that both M_0 and M are parallelizable compact manifolds and hence stably reducible (see [1]) implies (2.2). (For a different proof of (2.2) see [8; section 2].)

(2.3). From (2.2) we conclude that the subgroup $(\operatorname{im} u^*) \cap P_0$ of P_0 has index $|\omega|$. The universal covering $u: M_0 \to M$ is classified by a map $c: M \to B_\omega$. We view

$$A = (M_0 \xrightarrow{u} M \xrightarrow{c} B_\omega)$$

– up to homotopy equivalence – as a principal fibre bundle over B_ω, u representing the homotopy class of the fibre inclusion; (see [5]). (The classifying map $B_\omega \to B_{M_0}$ of the M_0-bundle A is induced by the inclusion $j: \omega \to M_0$.)

According to [9] the α and β-constructions together with the K-theory exact sequence of the pair (M, M_0) give rise to the following commutative diagram.

$$\tag{2.4}$$

(For the definition of α see [2]).

(2.5) LEMMA. *The homomorphism $\bar{c}^* \circ \alpha: I_\omega \to K^*(M, M_0)$ factors through $I_\omega / I_\omega \cdot \operatorname{im} j^*$.*

Proof. In $K^0(M, M_0)$ products of the form $\xi \cdot \delta(\eta)$ vanish; [3; p. 87]. The lemma then follows from the commutativity of (2.4), i.e. from $\bar{c}^* \circ \alpha \circ j^* = -\delta \circ \beta$.

Let $F \subset I_{M_0}$ be the free abelian group generated by $\bar{\lambda}_s = \lambda_s - \dim \lambda_s$, $(s = 1, ..., r)$,

where $\lambda_1, ..., \lambda_r$ are the basic irreducible representations of M_0. By [9] the homomorphism β maps F isomorphically onto the group of primitive elements $P_0 \subset K^1(M_0)$. In the following we shall identify P_0 and F, in particular we shall write $\lambda \in P_0$ for any element $\beta(\lambda)$ with $\lambda \in F$.

With (2.4) and (2.5) we then get the commutative diagram

$$P_0 = F \xrightarrow{\ \delta \mid P_0\ } K^0(M, M_0)$$

$$\varphi \searrow \qquad \nearrow$$

$$I_\omega/I_\omega \cdot \mathrm{im}\, j^*$$

(2.6)

where φ is induced by j^*.
Hence

$$\ker \varphi \subseteq (\ker \delta) \cap P_0 = (\mathrm{im}\, u^*) \cap P_0 .$$

(2.7)

Recalling the notations introduced in section 1, we now revert to the three coverings $u: G_0 = \mathrm{Spin}(2n) \to \mathrm{PSO}(2n) = G$, $a_1: G_0 \to G_0/\omega_1 = G_1$ and $a_2: G_0 \to G_0/\omega_2 = \mathrm{SO}(2n)$. These coverings yield the following commutative diagram

$$F \searrow$$

$$\varphi \downarrow \qquad \searrow \varphi_i$$

$$I_\pi/I_\pi \cdot \mathrm{im}\, j^* \ \to\ I_{\omega_i}/I_{\omega_i} \cdot \mathrm{im}\, j_i^*$$

(2.8)

where φ, φ_i are induced by j^*, j_i^* respectively; $(i = 1, 2)$.

(2.9) PROPOSITION. *There is a basis* $\beta_1, ..., \beta_{n-2}, \gamma_{n-1}, \gamma_n$ *of* $F \subset I_{G_0}$ *such that*

(i) $\beta_1, ..., \beta_{n-2}, 2\gamma_{n-1}, 2\gamma_n$ *are a basis of* $\ker \varphi$

(ii) $\beta_1, ..., \beta_{n-2}, 2\gamma_{n-1}, \gamma_n$ *are a basis of* $\ker \varphi_1$

(iii) $\beta_1, ..., \beta_{n-2}, \gamma_{n-1}, 2\gamma_n$ *are a basis of* $\ker \varphi_2$.

Moreover, for $\beta_1, ..., \beta_{n-3}$ *and* γ_{n-1} *we can choose a linear combination of* $\lambda_1, ..., \lambda_{n-2}$ *whereas* $\beta_{n-2} = \Delta^+ - \Delta^-$ *and* $\gamma_n = \lambda_n = \Delta^- - \dim \Delta^-$; *(see* (1.7)).

We omit the proof of (2.9) which amounts to a plain computation based on (1.8), (1.9) and the relations (1.5).

It follows from (2.9) that the subgroup $\ker \varphi$ of $F = P_0$ has index 4 and we conclude with (2.3) and (2.7) that

$$\ker \varphi = (\mathrm{im}\, u^*) \cap P_0, \quad \text{and similarly} \quad \ker \varphi_i = (\mathrm{im}\, a_i^*) \cap P_0 .$$

(2.10)

The following proposition is then a consequence of (2.9), (2.10) and the commu-

tativity of the diagram

$$
\begin{array}{ccc}
 & G_1 & \\
a_1 \nearrow & & \searrow b_1 \\
G_0 \longrightarrow & & G \\
a_2 \searrow & & \nearrow b_2 \\
 & G_2 &
\end{array}
\qquad (2.11)
$$

where all the maps are canonical covering projections.

(2.12) PROPOSITION. *There are generators* $\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n$ *of the exterior algebra* $K^*(G_0)$ *and elements* $v_1, v_2, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n \in K^1(G)$, $v_1^{(i)}, \ldots, v_{n-2}^{(i)}, \varepsilon_{n-1}^{(i)}$, $\varepsilon_n^{(i)} \in K^1(G_i)$, $(i = 1, 2)$, *such that*

(i) *the elements* $v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n$ *generate an exterior algebra in* $K^*(G)$ *which, under projection, is isomorphic to* $K^*(G)/\mathrm{Tors}\, K^*(G)$. *Furthermore*

$$
u^*(v_s) = \beta_s, \quad (s = 1, \ldots, n-2); \qquad u^*(\varepsilon_{n-1}) = 2\gamma_{n-1}, \qquad u^*(\varepsilon_n) = 2\gamma_n.
$$

(ii) *the elements* $v_1^{(i)}, \ldots, v_{n-2}^{(i)}, \varepsilon_{n-1}^{(i)}, \varepsilon_n^{(i)}$ *generate an exterior algebra in* $K^*(G_i)$ *which, under projection, is isomorphic to* $K^*(G_i)/\mathrm{Tors}\, K^*(G_i)$, $(i = 1, 2)$. *Furthermore*

$$
a_i^*(v_s^{(i)}) = \beta_s, \quad (s = 1, \ldots, n-2), (i = 1, 2),
$$

and

$$
a_1^*(\varepsilon_{n-1}^{(1)}) = 2\gamma_{n-1}, \qquad a_1^*(\varepsilon_n^{(1)}) = \gamma_n, \qquad a_2^*(\varepsilon_{n-1}^{(2)}) = \gamma_{n-1}, \qquad a_2^*(\varepsilon_n^{(2)}) = 2\gamma_n
$$

whereas

$$
b_i^*(v_s) = v_s^{(i)}, \quad (s = 1, \ldots, n-2), (i = 1, 2)
$$

and

$$
b_1^*(\varepsilon_{n-1}) = \varepsilon_{n-1}^{(1)}, \qquad b_2^*(\varepsilon_n) = \varepsilon_n^{(2)}.
$$

(iii) *The above elements can be chosen such that with respect to the various transfer maps (see* [10]*) arising from* (2.11) *one has*

$$
(a_1)_*(\gamma_{n-1}) \equiv \varepsilon_{n-1}^{(1)} \quad (\text{mod torsion}), \qquad (a_2)_*(\gamma_n) \equiv \varepsilon_n^{(2)} \quad (\text{mod torsion}),
$$
$$
\varepsilon_{n-1} = (b_2)_*(\varepsilon_{n-1}^{(2)}), \qquad \varepsilon_n = (b_1)_*(\varepsilon_n^{(1)})
$$

and hence

$$
b_2^*(\varepsilon_{n-1}) = 2\varepsilon_{n-1}^{(2)}, \qquad b_1^*(\varepsilon_n) = 2\varepsilon_n^{(1)}.
$$

(For (iii) see [8; (2.4), (2.7)].)

(2.13) *Remark.* The element $\gamma_n \in K^1(G_0)$ can be represented by the homomorphism $G_0 \xrightarrow{\Delta_3^-} U(2^{n-1}) \hookrightarrow U$ which factors through G_3, giving rise to a homomorphism $\Delta_3 : G_3 \to U$. The map Δ_3 represents an element in $K^1(G_3)$ which we denote by $\varepsilon_n^{(3)}$. The element $\varepsilon_n^{(1)} \in K^1(G_1)$ can not be represented by a group homomorphism. However, combining the two canonical Hopf multiplications on U, it is possible to write down explicitly a map $\Delta_1 : G_1 \to U$ representing $\varepsilon_n^{(1)}$.

3. Generators of Finite Order in $K^0(G)$.

Using the main result of [8] and reverting to (1.10) and (2.12) we first list the following two propositions.

(3.1) *There are elements* $v_1^{(1)}, \ldots, v_{n-2}^{(1)}$, $\varepsilon_{n-1}^{(1)}$, $\varepsilon_n^{(1)} \in K^1(G_1)$ *and* $\zeta_1 \in \tilde{K}^0(G_1)$ *which generate the ring* $K^*(G_1)$ *and such that*

(i) $K^*(G_1) \cong \{E_{\mathbb{Z}}(v_1^{(1)}, \ldots, v_{n-2}^{(1)}, \varepsilon_{n-1}^{(1)}, \varepsilon_n^{(1)}) \otimes T^0(G_1)\}/(\varepsilon_{n-1}^{(1)} \otimes \zeta_1)$ *where* $T^0(G_1)$ *is the subring of* $K^0(G_1)$ *generated by* 1 *and* ζ_1.

(ii) *The element* $1 + \zeta_1$ *is represented by the complex line bundle associated to the twofold covering* $G_0 \xrightarrow{a_1} G_1$; ζ_1 *is subject to the relations*

$$\zeta_1^2 + 2\zeta_1 = 0, \qquad 2^{k-1}\zeta_1 = 0, \qquad (k = v_2(n) + 2).$$

In particular $T^0(G_1) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{k-1}}$.

(3.2) *There are elements* $v_1^{(2)}, \ldots, v_{n-2}^{(2)}$, $\varepsilon_{n-1}^{(2)}$, $\varepsilon_n^{(2)} \in K^1(G_2)$ *and* $\zeta_2 \in \tilde{K}^0(G_2)$ *which generate the ring* $K^*(G_2)$ *and such that*

(i) $K^*(G_2) \cong \{E_{\mathbb{Z}}(v_1^{(2)}, \ldots, v_{n-2}^{(2)}, \varepsilon_{n-1}^{(2)}, \varepsilon_n^{(2)}) \otimes T^0(G_2)\}/(\varepsilon_n^{(2)} \otimes \zeta_2)$ *where* $T^0(G_2)$ *is the subring of* $K^0(G_2)$ *generated by* 1 *and* ζ_2.

(ii) *The element* $1 + \zeta_2$ *is represented by the complex line bundle associated to the twofold covering* $G_0 \xrightarrow{a_2} G_2$ *and* ζ_2 *is subject to the relations*

$$\zeta_2^2 + 2\zeta_2 = 0, \qquad 2^{n-1}\zeta_2 = 0.$$

In particular $T^0(G_2) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{n-1}}$.

Remark. The complex K-theory tells the homotopy types of G_1 and G_2 apart, a result which also appears in [4, (9.1)]. In [4] however the Steenrod algebra structure of the ordinary cohomology of G_1 and G_2 is used to distinguish the homotopy types of G_1 and G_2.

We now determine the image of the homomorphism induced by the map $c : G \to B_\pi$ classifying the universal covering of G.

(3.3) PROPOSITION. *Let* $T^0(G) = \mathrm{im}\,[K^0(B_\pi) \xrightarrow{c^*} K^0(G)]$. *Then* $T^0(G)$ *is a direct*

summand of $K^0(G)$ *and the homomorphism* $c^* \circ \alpha : R(\pi) \to K^0(G)$ *of* (2.4) *induces an isomorphism*

$$T^0(G) \cong R(\pi)/(j^*(I_{G_0})) \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}} \oplus \mathbf{Z}_{2^{n-1}} \oplus \mathbf{Z}_{2^{k-1}} ; \qquad (k = v_2(n) + 2).$$

Generators of the three finite cyclic summands of $T^0(G)$ *are given by* ξ_1, ξ_2 *and* $\xi_1 \cdot \xi_2$, *where the element* $1 + \xi_1$ *(respectively* $1 + \xi_2$) *is represented by the complex line bundle associated to the twofold covering* $b_2 : G_2 \to G$ *(respectively* $b_1 : G_1 \to G$). *The elements* ξ_1 *and* ξ_2 *are subject to the relations* $\xi_1^2 + 2\xi_1 = 0$, $\xi^2 + 2\xi_2^2 = 0$.

Proof. It follows from [2; (7.2)] that $c^* \circ \alpha$ maps $R(\pi)$ onto $\mathrm{im}\, c^* = T^0(G)$. Invoking (2.4) we infer that $c^* \circ \alpha$ induces an epimorphism

$$R(\pi)/(j^*(I_{G_0})) \twoheadrightarrow T^0(G).$$

Now consider the composite

$$G_1 \times G_2 \xrightarrow{b_1 \times b_2} G \times G \xrightarrow{m} G \xrightarrow{c} B_\pi$$

where m is the multiplication map on G, and set $t = m_0(b_1 \times b_2)$. Applying K^0 we get

$$R(\pi) \xrightarrow{\alpha} K^0(B_\pi) \xrightarrow{c^*} K^0(G) \xrightarrow{t^*} K^0(G_1 \times G_2). \tag{3.4}$$

Clearly, the elements $\sigma_i \in R(\pi)$ map onto $\xi_i \in K^0(G)$, $(i = 1, 2)$. Furthermore, looking at the Chern classes of the line bundles involved, one has $t^*(1 + \xi_1) = (1 + \zeta_1) \otimes 1$, $t^*(1 + \xi_2) = 1 \otimes (1 + \zeta_2) \in K^0(G_1) \otimes K^0(G_2) \subset K^0(G_1 \times G_2)$. With (3.1) and (3.2) we then obtain

$$t^* \circ c^* \circ \alpha(\sigma_1) = \zeta_1 \otimes 1 \in T^0(G_1) \otimes 1$$
$$t^* \circ c^* \circ \alpha(\sigma_2) = 1 \otimes \zeta_2 \in 1 \otimes T^0(G_2)$$

which implies that $t^* \circ c^* \circ \alpha$ maps $R(\pi)$ *onto* the direct summand $T^0(G_1) \otimes T^0(G_2)$ of $K^0(G_1 \times G_2)$. Hence there is an epimorphism

$$R(\pi)/j^*(I_{G_0})) \twoheadrightarrow T^0(G_1) \otimes T^0(G_2) \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}} \oplus \mathbf{Z}_{2^{n-1}} \oplus \mathbf{Z}_{2^{k-1}}$$

and the proposition is established.

4. A Basic Generator of Finite Order in $K^1(G)$.

The elements ξ_1, $\xi_2 \in K^0(G)$ and $v_1, \ldots, v_{n-2}$, ε_{n-1}, $\varepsilon_n \in K^1(G)$ do not yet generate the ring $K^*(G)$. In fact it can be shown, comparing the spectral sequences of the bundles $\Lambda = (G_0 \xrightarrow{u} G \xrightarrow{c} B_\pi)$ and $\Gamma_1 = (G_0 \xrightarrow{a_1} G_1 \xrightarrow{c_1} B_{\omega_1})$ that there must exist an element $\tau \in K^1(G)$ with $b_1^*(\tau) = \zeta_1 \cdot \varepsilon_n^{(1)} \in K^1(G_1)$. Such an element τ can not be expressed in terms of the elements in $K^*(G)$ described as yet. (Note $b_1^*(\varepsilon_n) = 2\varepsilon_n^{(1)}$.)

We are now going to define an element $\tau \in K^1(G)$ of finite order which together with the above elements will generate the ring $K^*(G)$.

To begin with let us consider $\varepsilon_n^{(1)}$, $\varepsilon_n^{(3)}$ and γ_n in $K^1(G_1)$, $K^1(G_3)$ and $K^1(G_0)$ respectively. By (2.12) and (2.13) these elements are related as follows.

$$a_1^*(\varepsilon_n^{(1)}) = \gamma_n = a_3^*(\varepsilon_n^{(3)}). \tag{4.1}$$

We now define

$$\tau = (b_3)_*(\varepsilon_n^{(3)}) - (b_1)_*(\varepsilon_n^{(1)}) \in K^1(G), \tag{4.2}$$

where $(b_i)_* : K^*(G_i) \to K^*(G)$, $(i=1, 3)$, is the Atiyah-transfer map associated to the twofold covering $b_i : G_i \to G$.

(4.3) PROPOSITION. *The element $\tau \in K^1(G)$ has the following properties*
  (i) $b_1^*(\tau) = \zeta_1 \varepsilon_n^{(1)} \in K^1(G_1)$
  (ii) $b_2^*(\tau) = 0 \in K^1(G_2)$.

Proof. For the basic properties of the transfer map $f_* : K^*(X) \to K^*(Y)$ associated to a finite covering projection $f : X \to Y$ we refer to [2] and [10]. In particular we point out the validity of the "Frobenius reciprocity law", i.e.

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x)$$

where $x \in K^*(X)$, $y \in K^*(Y)$ and $f^* : K^*(Y) \to K^*(X)$ the map induced by f. Consider the following morphisms of coverings

$$
\begin{array}{ccc}
G_0 & \xrightarrow{a_i} & G_i \\
{\scriptstyle a_j}\big\downarrow & & \big\downarrow{\scriptstyle b_i} \\
G_j & \xrightarrow{b_j} & G
\end{array}
$$

where $i \neq j$ and $i, j = 1, 2, 3$.

The transfer is natural with respect to such morphisms and with (4.1) we compute

$$b_2^* \circ (b_i)_*(\varepsilon_n^{(i)}) = (a_2)_* \circ a_i^*(\varepsilon_n^{(i)}) = (a_2)_*(\gamma_n), \qquad (i=1, 3),$$

thus establishing part (ii) of (4.3). On the trivial line bundle $1 \in K^0(G_0)$ the transfer $(a_1)_*$ is given by $(a_1)_*(1) = 2 + \zeta_1$; (see [2; p. 45]). Using the Frobenius law we then calculate

$$b_1^* \circ (b_3)_*(\varepsilon_n^{(3)}) = (a_1)_* \circ a_3^*(\varepsilon_n^{(3)}) = (a_1)_*(\gamma_n) = (a_1)_*(a_1^*(\varepsilon_n^{(1)}) \cdot 1) = \varepsilon_n^{(1)}(2 + \zeta_1).$$

Furthermore $b_1^* \circ (b_1)_*(\varepsilon_n^{(1)}) = 2\varepsilon_n^{(1)}$ and part (i) of (4.3) is verified.

(4.4) COROLLARY. *The following relations hold in* $K^0(G)$.

(i) $\xi_1\tau + 2\tau = 0$

(ii) $\xi_2\tau + 2\tau - \xi_1\varepsilon_n = 0$

(iii) $\tau\varepsilon_{n-1} = 0$, $\tau\varepsilon_n = 0$

(iv) $\tau^2 = 0$.

Proof. Recall that $\varepsilon_n = (b_1)_*(\varepsilon_n^{(1)})$ and $\varepsilon_{n-1} = (b_2)_*(\varepsilon_{n-1}^{(2)})$. Now observe that $(b_1)_*(1) = 2 + \xi_2$ and $(b_2)_*(1) = 2 + \xi_1$; (see definition of ξ_1, ξ_2 in (3.3)). Using (4.3) and the "Frobenius law" we get

$$(2 + \xi_1)\,\tau = (b_2)_*(1)\,\tau = (b_2)_*(1 \cdot b_2^*(\tau)) = 0$$

and analogously

$$(2 + \xi_2)\,\tau = (b_1)_*(1)\,\tau = (b_1)_*(1 \cdot b_1^*(\tau)) = (b_1)_*(\zeta_1 \cdot \varepsilon_n^{(1)}) = \xi_1 \cdot \varepsilon_n$$

thus establishing parts (i) and (ii) of (4.4). Next we verify

$$\tau\varepsilon_n = (b_1)_*(b_1^*(\tau) \cdot \varepsilon_n^{(1)}) = (b_1)_*(\zeta_1 \cdot \varepsilon_n^{(1)} \cdot \varepsilon_n^{(1)}) = 0$$

$$\tau\varepsilon_{n-1} = (b_2)_*(b_2^*(\tau) \cdot \varepsilon_{n-1}^{(2)}) = 0.$$

Eventually the fact that G is a finite CW complex and $\tau \in K^1(G)$ implies that $\tau^2 = 0$. This completes the proof of this corollary.

We now proceed to determine the order of τ.

(4.5) PROPOSITION. *The element* $\tau \in K^1(\mathrm{PSO}(2n))$ *is of order* 2^k *where* $k = v_2(n) + 2$.

Proof. The fact that $2^{k-1}\xi_1 = 0$, (see (3.3)), together with the relation $2\tau = -\xi_1\tau$, (see (4.4)), implies that $2^k\tau = 0$. It remains to show that $2^{k-1}\tau \neq 0$. This is done in the following way. The commutative square

$$
\begin{array}{ccc}
G_0 & \xrightarrow{\ a_2\ } & G_2 \\
{\scriptstyle a_1}\big\downarrow & & \big\downarrow{\scriptstyle b_2} \\
G_1 & \xrightarrow{\ b_1\ } & G
\end{array}
$$

gives rise to a map of pairs $j: (G_1, G_0) \to (G, G_2)$. (Replace the spaces in the bottom row by the mapping cylinders of a_1 and b_2 respectively.) We thus obtain a morphism of exact sequences

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & K^0(G_2) & \xrightarrow{\ \delta^{(2)}\ } & K^1(G, G_2) & \xrightarrow{\ i^*_2\ } & K^1(G) & \xrightarrow{\ b^*_2\ } & K^1(G_2) & \longrightarrow \cdots \\
& {\scriptstyle a^*_2}\big\downarrow & & {\scriptstyle j^*}\big\downarrow & & {\scriptstyle b^*_1}\big\downarrow & & {\scriptstyle a^*_2}\big\downarrow & \\
\cdots \longrightarrow & K^0(G_0) & \xrightarrow{\ \delta^{(1)}\ } & K^1(G_1, G_0) & \xrightarrow{\ i^*_1\ } & K^1(G_1) & \xrightarrow{\ a^*_1\ } & K^1(G_0) & \longrightarrow \cdots.
\end{array}
$$

Since $b_2^*(\tau) = 0$ there is an element $\omega \in K^1(G, G_2)$ such that $i_2^*(\omega) = \tau$. With $b_1^*(\tau) =$ $= \zeta_1 \varepsilon_n^{(1)}$ we infer $j^*(\omega) \equiv \zeta_1 \cdot \varepsilon_n^{(1)}$ $(\mathrm{mod\,im}\,\delta^{(1)})$, where in the latter expression the dot denotes the action of $K^*(G_1)$ on $K^*(G_1, G_0)$. Referring to (2.4), (2.9) (ii) and (2.12) we observe that $\delta^{(1)}(\gamma_{n-1}) = 2^{k-1}\zeta_1 \neq 0$ and thus $\delta^{(1)}(\gamma_{n-1}\gamma_n) = 2^{k-1}\zeta_1 \cdot \varepsilon_n^{(1)} \neq 0$. Hence

$$j^*(2^{k-1}\omega) = 2^{k-1}\zeta_1 \cdot \varepsilon_n^{(1)} = \delta^{(1)}(\gamma_{n-1}\gamma_n) \neq 0. \tag{4.6}$$

(Note, $2 \cdot \mathrm{im}\,\delta^{(1)} = 0$).

We show that $2^{k-1}\tau = 0$ leads to a contradiction. The assumption $2^{k-1}\tau = 0$ implies $i_2^*(2^{k-1}\omega) = 0$; hence there is an element in $K^0(G_2)$, say η, with $\delta^{(2)}(\eta) = 2^{k-1}\omega$. By (4.6) we then get

$$\delta^{(1)}a_2^*(\eta) = 2^{k-1}\zeta_1 \cdot \varepsilon_n^{(1)} = \delta^{(1)}(\gamma_{n-1}\gamma_n).$$

According to (2.12) we have $a_2^*(K^*(G_2)) = E_\mathbf{Z}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, 2\gamma_n) \subset K^*(G_0)$ and $\ker \delta^{(1)} = a_1^*(K^*(G_1)) = E_\mathbf{Z}(\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n)$. One now checks readily that

$$a_2^*(\eta) \not\equiv \gamma_{n-1}\gamma_n \quad (\text{modulo } \ker \delta^{(1)})$$

and the contradiction becomes evident. Hence the order of τ is indeed 2^k.

5. The Spectral Sequences.

In this section we compute all the differentials in the spectral sequence $(E_r(G), d_r^\Lambda)$ of the fibre bundle

$$\Lambda = (G_0 \underset{u}{\to} G \underset{c}{\to} B_\pi). \tag{5.1}$$

This will enable us to fully determine the target term $E_\infty(\Lambda)$. The additional information on $K^*(G)$ we get from $E_\infty(\Lambda)$ will then be sufficient to complete the description of the ring $K^*(G)$.

Basically we shall compare the spectral sequence of Λ with the "known" (see [8]) spectral sequences $(E_r(\Gamma_i), d_r^{\Gamma_i})$, where Γ_i is the fibre bundle

$$\Gamma_i = (G_0 \underset{a_i}{\to} G_i \underset{c_i}{\to} B_{\omega_i}), \quad (i = 1, 2). \tag{5.2}$$

For the E_2-term of the spectral sequence of Γ_i we have

$$E_2(\Gamma_i) \cong H^*(B\omega_i; \mathbf{Z}) \otimes K^*(G_0),$$

where $H^*(B_{\omega_i}; \mathbf{Z}) \cong \mathbf{Z}[w_i]/(2w_i)$, $w_i \in H^2(B_{\omega_i}; \mathbf{Z})$ and $K^*(G_0) = E_\mathbf{Z}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$, see (2.12). With (1.10) and [8] we obtain

(5.3) PROPOSITION. (i) *All differentials $d_r^{\Gamma_1}$ are trivial except for the differential $d_{2k}^{\Gamma_1}$, $(k = \nu_2(n) + 2)$, which, evaluated on the element $1 \otimes \gamma_{n-1}$, is given by*

$$d_{2k}^{\Gamma_1}(1 \otimes \gamma_{n-1}) = w_1^k \otimes 1.$$

The reduced E_∞-term, $\tilde{E}_\infty(\Gamma_1) = \bigoplus_{m > 0} E_\infty^{m;*}(\Gamma_1)$, is given by

$$\tilde{E}_\infty(\Gamma_1) \cong \{\tilde{H}^*(B_{\omega_1}; \mathbf{Z})/(w_1^k)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n) =$$
$$= \{(w_1)/(w_1^k)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n).$$

(ii) *All differentials $d_r^{\Gamma_2}$ are trivial except for the differential $d_{2n}^{\Gamma_2}$ which, evaluated on the element $1 \otimes \gamma_n$, is given by*

$$d_{2n}^{\Gamma_2}(1 \otimes \gamma_n) = w_2^n \otimes 1.$$

The reduced $E_\infty(\Gamma_2)$-term is given by $\tilde{E}_\infty(\Gamma_2) \cong \{(w_2)/(w_2^n)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1})$. We now focus on the following commutative diagram.

$$
\begin{array}{ccccc}
G_0 & \xleftarrow{\quad q_i \quad} & G_0 \times G_0 & \xrightarrow{\ m_0\ } & G_0 \\
{\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle a_1 \times a_2} & & \downarrow{\scriptstyle u} \\
G_i & \xleftarrow{\quad pr. \quad} & G_1 \times G_2 & \xrightarrow{\ t\ } & G \\
{\scriptstyle c_i}\downarrow & & \downarrow{\scriptstyle c_1 \times c_2} & & \downarrow{\scriptstyle c} \\
B_{\omega_i} & \xleftarrow{\quad p_i \quad} & B_{\omega_1} \times B_{\omega_2} & \xrightarrow[=]{\ h\ } & B_\pi
\end{array}
\qquad (i = 1, 2).
\tag{5.4}
$$

In (5.4) m_0 stands for the multiplication map, t is as in (3.4), p_i, q_i and $pr.$ are the canonical projections and h is the identification map induced by $\omega_1 \times \omega_2 = \pi$, (see 1). We denote the bundle in the middle of (5.4) by $\Gamma_1 \times \Gamma_2$ and the corresponding bundle homomorphisms by

$$\Gamma_i \xleftarrow{\ P_i\ } \Gamma_1 \times \Gamma_2 \xrightarrow{\ M\ } \Lambda.\tag{5.5}$$

For the E_2-terms of the spectral sequences of $\Gamma_1 \times \Gamma_2$ and Λ we have

$$E_2(\Gamma_1 \times \Gamma_2) \cong H^*(B_\pi; Z) \otimes K^*(G_0 \times G_0)$$
$$E_2(\Lambda) \cong H^*(B_\pi; Z) \otimes K^*(G_0).$$

We write $(E_r(B_\pi), d_r^{B_\pi})$ for the spectral sequence of the CW-complex $B_\pi = B_{\omega_1} \times B_{\omega_2}$ and make two basic observations.

(5.6) *Let $r \geqslant 2$. We have $E_{r+1}(\Gamma_1 \times \Gamma_2) \cong E_{r+1}(B_\pi) \otimes K^*(G_0 \times G_0)$ if, and only if, $E_r(\Gamma_1 \times \Gamma_2) \cong E_r(B_\pi) \otimes K^*(G_0 \times G_0)$ and $d_r(1 \otimes K^*(G_0 \times G_0)) = 0$. A similar remark can be made about the spectral sequence of Λ.*

This fact is easy to verify. Note, $E_r(B_\pi)$ is a differential subring of $E_r(B_\pi)\otimes$ $\otimes K^*(G_0 \times G_0)$ with $K^*(G_0 \times G_0)$ torsion free, and similarly for $E(\Lambda)$.

(5.7). *If* $E_r(\Gamma_1 \times \Gamma_2) \cong E_r(B_\pi) \otimes K^*(G_0 \times G_0)$ *for some* $r \geq 2$, *then* $E_r(\Lambda) \cong E^r(B_\pi)$ $\otimes K^*(G_0)$.

This is true for $r=2$ and it follows for $r>2$ by induction from (5.6) and the fact that the bundle map $M: \Gamma_1 \times \Gamma_2 \to \Lambda$ induces the monomorphism

$$E_{r-1}(B_\pi)\otimes K^*(G_0) \xrightarrow{\text{id.}\otimes m^*_0} E_{r-1}(B_\pi)\otimes K^*(G_0 \times G_0).$$

We then derive from that

(5.8) LEMMA. *For the bundles* $\Gamma_1 \times \Gamma_2$ *and* Λ *one has*

$$E_{2k}(\Gamma_1 \times \Gamma_2) \cong E_{2k}(B_\pi)\otimes K^*(G_0 \times G_0)$$
$$E_{2k}(\Lambda) \cong E_{2k}(B_\pi)\otimes K^*(G_0), \quad (k=v_2(n)+2).$$

Proof. Referring to (5.6) and (5.7) we have to show that

$$d_s^{\Gamma_1 \times \Gamma_2}(1\otimes K^*(G_0 \times G_0))=0, \quad (s=2, 3, ..., 2k-1), \tag{5.9}$$

By (5.3) the differentials $d_s^{\Gamma_i}$, $(s=2, 3, ..., 2k-1$ and $i=1, 2)$, are trivial (note that $k=v(n)+2<n$) and since $E_s^{0,*}(\Gamma_1 \times \Gamma_2) \cong 1\otimes K^*(G_0 \times G_0) \cong 1\otimes K^*(G_0)\otimes K^*(G_0)$ is generated by the images of the spectral sequence maps $E_s(P_i)$, $(i=1, 2)$, statement (5.9) follows.

We now list the relevant facts about the spectral sequence of $B_\pi = B_{\omega_1} \times B_{\omega_2}$. This spectral sequence is not trivial. However a computation of C. T. C. Wall (see [2; p. 61]) shows that

$$E_4(B_\pi) \cong E_\infty(B_\pi) \cong \text{Gr. } R(\pi) \cong Z[x, y]/(2x, 2y, x^2y-xy^2) \tag{5.10}$$

with

$$\text{Gr.}_{2s} R(\pi)=I_\pi^s/I_\pi^{s+1}, \quad \text{Gr.}_{\text{odd}} R(\pi)=0$$

where $x, y \in \text{Gr.}_2 R(\pi)=I_\pi/I_\pi^2$ are represented by σ_1, σ_2 respectively. We introduce the following notation

$$R_s=\text{Gr.}_{2s} R(\pi), \quad R=\bigoplus_{s=0}^{\infty} R_s=\text{Gr. } R(\pi), \quad \check{R}=\bigoplus_{s=1}^{\infty} R_s=\text{Gr. } I_\pi. \tag{5.11}$$

We then have $R_1 \cong Z_2 \oplus Z_2$, where x and y generate the two cyclic summands. For $s \geq 2$ the cyclic summands of $R_s \cong Z_2 \oplus Z_2 \oplus Z_2$ are generated by x^s, y^s and xy^{s-1} respectively.

For later use it is convenient to set

$$z_s = y^s + xy^{s-1} \in R_s, \qquad (s = 2, 3, \ldots)$$

and hence we have

$$x^r z_s = 0, \qquad y^r z_s = z_{r+s} = z_r z_s, \qquad x^r y^s = z_{r+s} - y^{r+s}. \tag{5.12}$$

We are now ready to give an explicit description of the $2k$-level of the spectral sequence of the bundle A.

(5.13) LEMMA. (i) $E_{2k}(A) = R \otimes K^*(G_0) \cong \{Z[x, y]/(2x, 2y, x^2 y - xy^2)\} \otimes E_Z(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$

(ii) $d^A_{2k}(R \otimes 1) = 0$, $\qquad d^A_{2k}(1 \otimes \beta_s) = 0$, $\qquad (s = 1, 2, \ldots, n-2)$,
$d^A_{2k}(1 \otimes \gamma_n) = 0$, $\qquad d^A_{2k}(1 \otimes \gamma_{n-1}) = x^k \otimes 1$.

Proof. Part (i) is a consequence of (5.8) and (5.10), since $2k > 4$. Also from (5.10) we infer that $d^A_{2k}(R \otimes 1) = 0$. Now the bundle maps of (5.4) induce homomorphisms of the corresponding spectral sequences, which on the $2k$-level are given as follows

$$H^*(B_{\omega_i} : Z) \otimes K^*(G_0) \xrightarrow[p^*_i \otimes q^*_i]{} R \otimes K^*(G_0 \times G_0) \xleftarrow[\text{id.} \otimes m^*_0]{} R \otimes K^*(G_0)$$

$$\|\wr \qquad\qquad\qquad\qquad \|\wr \qquad\qquad\qquad\qquad \|\wr$$

$$E_{2k}(\Gamma_i) \xrightarrow{\hspace{3cm}} E_{2k}(\Gamma_1 \times \Gamma_2) \xleftarrow{\hspace{3cm}} E_{2k}(A).$$

Using (5.3), the fact that $p^*_1(w^k_1) = x^k \otimes 1$ and the primitivity of the elements $\beta_1, \ldots, \beta_{n-2}$, γ_{n-1}, γ_n with respect to m^*_0 we immediately complete the proof of this lemma. (Again note that $k < n$.)

A short computation involving (5.12) and (5.13) shows that

$$E^{0;*}_{2k+1}(A) \cong Z \otimes E_Z(\beta_1, \overset{\cdots}{\ldots}, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n)$$

and

$$\tilde{E}_{2k+1}(A) \cong \tilde{R}/(x^k) \otimes E_Z(\beta_1, \ldots, \beta_{n-2}, \gamma_n)$$
$$\oplus (z_2) \otimes E_Z(\beta_1, \ldots, \beta_{n-2}, \gamma_n) \cdot \gamma_{n-1}.$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \tag{5.14}$$

(Here (v) stands for the ideal generated by $v \in R$.)

To get a hold on the differentials d^A_r, for $r > 2k$, we consider the bundle maps

$$F_i : \Gamma_i \to A, \qquad (i = 1, 2) \tag{5.15}$$

which are given by the commutative diagrams

$$\begin{array}{ccc}
G_0 \longrightarrow & G_i \xrightarrow{\ c_i\ } & B_{\omega_i} \\
\downarrow{\scriptstyle 1} & \downarrow{\scriptstyle b_i} & \downarrow{\scriptstyle s_i} \\
G_0 \longrightarrow & G \xrightarrow{\ c\ } & B_\pi
\end{array} \qquad (i = 1, 2).$$

(5.16) LEMMA. (i) *The homomorphism*

$$E_{2k+1}(F_2): E_{2k+1}^{0;\,*}(A) \cong E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n)$$

$$\to E_{2k+1}^{0;\,*}(\Gamma_2) \cong E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$$

is the canonical inclusion.

(ii) $E_{2k+1}(F_2)$ *maps* $(z_2) \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n) \cdot \gamma_{n-1} \subset E_{2k+1}(A)$ *isomorphically onto* $(w^2) \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n) \cdot \gamma_{n-1} \subset E_{2k+1}(\Gamma_2)$.

(iii) $E_{2k+1}(F_2): E_{2k+1}^{2p;\,*}(A) \to E_{2k+1}^{2p;\,*}(\Gamma_2)$ *is an isomorphism for* $2p \geqslant 2k + 2$.
(Note, $E_{2k+1}^{\mathrm{odd};\,*}(A) = 0 = E_{2k+1}^{\mathrm{odd};\,*}(\Gamma_2)$.)

Proof. Part (i) is clear. For parts (ii) and (iii) we observe that

$$E_{2k}(F_2): R \otimes K^*(G_0) \to H^*(B_{\omega_2}; \mathbf{Z}) \otimes K^*(G_0)$$

is given by $E_{2k}(F_2)(x \otimes 1) = 0$, $E_{2k}(F_2)(y \otimes 1) = w_2 \otimes 1$, hence $E_{2k}(F_2)(z_s \otimes 1) = w_2^s \otimes 1$. To complete the proof look at the induced map on the $(2k+1)$-level.

It follows from (5.16) that d_r^A, $(r \geqslant 2k+1)$, is trivial as long as $d_r^{\Gamma_2} = 0$, and with (5.3) (ii) we get immediately

(5.17) LEMMA. (i) $d_r^A = 0$ *for* $r = 2k+1, \ldots, 2n-1$, *i.e.* $E_{2k+1}(A) \cong E_{2n}(A)$

(ii) $d_{2n}^A(1 \otimes \gamma_n) = \bar{y}^n \otimes 1$; *(where* $\bar{y} \in \tilde{R}/(x^k)$ *is the element represented by* $y \in \tilde{R}$). d_{2n}^A *is zero on the elements* $1 \otimes \beta_1, \ldots, 1 \otimes \beta_{n-2}, 1 \otimes 2\gamma_{n-1}, \bar{x} \otimes 1, \bar{y} \otimes 1, z_2 \otimes \gamma_{n-1}$; *(where* $\bar{x}$ *is the element represented by* x). *In particular,* $d_{2n}^A(z_2 \otimes \gamma_{n-1}\gamma_n) = z_{n+2} \otimes \gamma_{n-1}$.

An explicit calculation resting on (5.12), (5.14) and (5.17) then gives

(5.18) $E_{2n+1}^{0;\,*}(A) = E_{2n+2}^{0;\,*}(A) = 1 \otimes A$, where A is the subalgebra of $E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$ generated by $\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, 2\gamma_n$ and $2\gamma_{n-1}\gamma_n$. Moreover we have

$$\tilde{E}_{2n+1}(A) \cong \tilde{E}_{2n+2}(A) \cong \{\tilde{R}/(x^k, y^n)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2})$$

$$\oplus \{(x)/(x^k)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}) \gamma_n$$

$$\oplus \{(z_2)/(z_{n+2})\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}) \gamma_{n-1}.$$

Since $E_{2n+2}^{p;\,*}(A) = 0$ for $p > 2n + 3$, we conclude that $d_r = 0$ for $r \geqslant 2n + 3$ and $d_{2n+2}^A(E_{2n+2}^{q;\,*}(A)) = 0$ for $q > 0$. On the other hand elements of the form $2\gamma_{n-1}\gamma_n \alpha \in K^*(G_0)$, where $\alpha = \beta_{i_1}\beta_{i_2} \ldots \beta_{i_\ell}$ are not in the image of $u^*: K^*(G) \to K^*(G_0)$, (see (2.12)), i.e. these elements can not "survive" in the spectral sequence of A. Hence for $1 \otimes 2\gamma_{n-1}\gamma_n \alpha \in E_{2n+2}^{0;\,*}(A)$ we must have

$$d_{2n+2}^A(1 \otimes 2\gamma_{n-1}\gamma_n \alpha) = \bar{z}_{n+1} \otimes \gamma_{n-1}\alpha$$

and thus we get

$$E^{0,\,*}_\infty\,(\Lambda)\cong Z\otimes E_Z\,(\beta_1,\,...,\,\beta_{n-2},\,2\gamma_{n-1},\,2\gamma_n)$$

$$\tilde{E}_\infty\,(\Lambda)\cong \tilde{R}/(x^k,\,y^n)\otimes E_Z\,(\beta_1,\,...,\,\beta_{n-2})$$

$$\oplus(x)/(x^k)\otimes E_Z\,(\beta_1,\,...,\,\beta_{n-2})\,\gamma_n \qquad\qquad (5.19)$$

$$\oplus(z_2)/(z_{n+1})\otimes E_Z\,(\beta_1,\,...,\,\beta_{n-2})\,\gamma_{n-1}\,.$$

In particular $E^{\text{odd},\,*}_\infty\,(\Lambda)=0$, $E^{p,\,*}_\infty\,(\Lambda)=0$ for $p\geqslant 2n+2$.

The ringstructure on the right hand side of (5.19) is the one inherited from $R\otimes E_Z(\beta_1,\,...,\,\beta_{n-2},\,\gamma_{n-1},\,\gamma_n)$.

Note that – as abelian groups – the "quotients" in $\tilde{E}_\infty\,(\Lambda)$ can be exhibited as follows (the elements under the Z_2-summands indicate the respective generators):

$$\tilde{R}/(x^k,\,y^n)\cong(Z_2\oplus Z_2)\oplus(Z_2\oplus Z_2\oplus Z_2)\oplus...\oplus(Z_2\oplus Z_2\oplus Z_2)\oplus(Z_2\oplus Z_2)\oplus$$
$$\bar{x}\quad \bar{y}\qquad \bar{x}^2\quad \bar{y}^2\quad \bar{x}\bar{y}\qquad ...\qquad \bar{x}^{k-1}\bar{y}^{k-1}\,\bar{x}\bar{y}^{k-2}\bar{y}^k\quad \bar{x}\bar{y}^{k-1}$$

$$\oplus Z_2\oplus...\oplus Z_2$$
$$\bar{y}^{k+1}\,...\,\,\bar{y}^{n-1}$$

$$(x)/(x^k)\cong Z_2\oplus(Z_2\oplus Z_2)\oplus\,........\,\oplus(Z_2\oplus Z_2)\oplus Z_2$$
$$\bar{x}\qquad \bar{x}^2\quad \bar{x}\bar{y}\qquad ....\qquad \bar{x}^{k-1}\,\bar{x}\bar{y}^{k-2}\bar{x}\bar{y}^{k-1}$$

$$(z_2)/(z_{n+1})\cong Z_2\oplus Z_2\oplus\,........\,\oplus Z_2$$
$$\bar{z}_2\quad \bar{z}_3\quad ....\quad \bar{z}_n \qquad\qquad (5.20)$$

We are now going to extract as much information from the structure of $E_\infty\,(\Lambda)$ as we need in order to be able to complete the description of the ring $K^*(\text{PSO}(2n))$. In this sense the following corollaries rest basically on (5.19).

Since the total space G of the fibre bundle Λ is of the homotopy type of a finite CW-complex the spectral sequence converges, i.e.

$$E_\infty\,(\Lambda)\cong \text{Gr.}\,K^*\,(G),$$

where $\text{Gr.}\,K^*\,(G)$ is the graded ring associated to the usual filtration (see [2; p. 29]) of $K^*\,(G)$. There are no elements of finite order in $E^{0,\,*}_\infty\,(\Lambda)$ and no elements of infinite order in $\tilde{E}_\infty\,(\Lambda)$. Hence

$$|\text{Tors.}\,K^*\,(G)|=|\tilde{E}_\infty\,(\Lambda)|\,.$$

(5.21) COROLLARY. *The number of elements of finite order in* $K^*\,(G)$ *is given by*
$$|\text{Tors.}\,K^*\,(G)|=2^{(2n+4k-6)2^{n-2}}$$
where $k=v_2\,(n)+2$.

Proof. Use (5.19) and (5.20).

(5.22). According to (5.19) the elements $1\otimes\beta_1,\,...,\,1\otimes\beta_{n-2}$, $1\oplus 2\gamma_{n-1}$, $1\otimes 2\gamma_n$, $\bar{x}\otimes 1$, $\bar{y}\otimes 1$, $\bar{x}\otimes\gamma_n$, $\bar{z}_2\otimes\gamma_{n-1}$ form a system of generators of the graded ring $E_\infty\,(G)\cong$ $\cong\text{Gr.}\,K^*(G)$. (Recall that $(\bar{y}\otimes 1)\,(z_2\otimes\gamma_{n-1})=\bar{z}_{2+r}\otimes\gamma_{n-1}.$)

In the following table we record which elements of $K^*(G)$ represent the above generators of $E_\infty(A)$.

$K^*(G)$	$s=1,2,...,n-2$ v_s	ε_{n-1}	ε_n	ξ_1	ξ_2	τ	$\xi_2\varepsilon_{n-1}$
$E_\infty(G)$	$1\otimes\beta_s$	$1\otimes 2\gamma_{n-1}$	$1\otimes 2\gamma_n$	$\bar{x}\otimes 1$	$\bar{y}\otimes 1$	$\bar{x}\otimes\gamma_n$	$\bar{z}_2\otimes\gamma_{n-1}+v$

$$(5.23)$$

where in the right hand corner $v\in E_\infty^{4;\,*}(A)$ is an element of the form $v=\bar{x}\bar{y}\otimes\alpha_1+(\bar{x}\otimes\gamma_n)\cdot(\bar{y}\otimes\alpha_2)$; $\alpha_1,\alpha_2\in E_Z(\beta_1,...,\beta_{n-2})$.

Only the last two entries of this table require some comment. By (4.3) one has $b_1^*(\tau)=\zeta_1\varepsilon_n^{(1)}\in K^*(G_1)$ and $b_2^*(\tau)=0$. The element $\zeta_1\varepsilon_n^{(1)}$ has exact filtration 2 and represents $w_1\otimes\gamma_n\in E_\infty(\Gamma_1)$. Hence the torsion element τ has also exact filtration 2. Looking at the homomorphisms $E_\infty^{2;\,*}(F_1)$ and $E_\infty^{2;\,*}(F_2)$ we then see that τ represents $\bar{x}\otimes\gamma_n$; (use (5.3) and (5.19)).

The filtration of $\xi_2\varepsilon_{n-1}$ is greater than 2, the reason being $(\bar{y}\otimes 1)\cdot(1\otimes 2\gamma_{n-1})=0$ in $E_\infty^{2;\,*}(A)$. On the other hand we have $b_2^*(\xi_2\varepsilon_{n-1})=b_2^*(\xi_2\cdot(b_{2*})\varepsilon_{n-1}^{(2)}))=\zeta_2\cdot 2\varepsilon_{n-1}^{(2)}$. Since $2\zeta_2\varepsilon_{n-1}^{(2)}=-\zeta_2^2\varepsilon_{n-1}^{(2)}$ has exact filtration 4, the same now holds for $\xi_2\varepsilon_{n-1}$. Hence $\xi_2\varepsilon_{n-1}$ represents an element $w\in E_\infty^{4;\,*}(A)$ such that $E_\infty(F_2)(w)=w_1^2\otimes\gamma_{n-1}$ and $E_\infty(F_1)(w)=0$ (recall that $b_1^*(\xi_2\varepsilon_{n-1})=0$) and the result again follows by looking at the homomorphisms $E_\infty^{4;\,*}(F_1)$ and $E_\infty^{4;\,*}(F_2)$.

(5.24) *Remark.* Note that in $E_\infty(A)$ we have $(\bar{y}\otimes 1)^{k-1}\cdot(\bar{x}\otimes\gamma_n)\neq 0$ and hence $\xi_2^{k-1}\tau\neq 0$. By (3.3), (4.4) and (4.5) we then conclude that the order of $\xi_2\tau$ is 2^{k-1}.

Since $K^*(G)$ has finite filtration we derive from (5.22) and (5.23):

(5.25) COROLLARY. *The elements* $v_1,...,v_{n-2},\varepsilon_{n-1},\varepsilon_n,\xi_1,\xi_2$ *and* τ *generate the ring* $K^*(G)$.

By (5.19) we have $E_\infty^{p;\,*}(A)=0$ for $p>2n$ and hence we can identify $E_\infty^{2n;\,*}(A)$ with $K_{2n}^*(G)$, the subgroup of elements of filtration $2n$. Elements of $E_\infty^{2n;\,*}(A)$ are of the form $\bar{z}_n\otimes\gamma_{n-1}\beta=(\bar{y}^{n-2}\otimes 1)(\bar{z}_2\otimes\gamma_{n-1}+v)(1\otimes\beta)$, where $\beta\in E_Z(\beta_1,...,\beta_{n-2})$ and v is as in (5.23). (Note that $(\bar{y}^{n-2}\otimes 1)\cdot v=0$.) The latter element is represented by $\xi_2^{n-2}(\xi_2\varepsilon_{n-1})v=2^{n-2}\xi_2\varepsilon_{n-1}v$, where $v\in E_Z(v_1,...,v_{n-2})$. Consequently we may remark:

(5.26). Any element $\mu\in K^*(G)$ of filtration $2n$ is of the form

$$\mu=2^{n-2}\xi_2\varepsilon_{n-1}v,$$

where $v\in E_Z(v_1,...,v_{n-2})$.

Finally we derive from $E_\infty(A)$ the following relation involving the (non-zero) element $2^{k-1}\tau \in K^1(G)$.

(5.27) COROLLARY. *There is an element* $v \in E_Z(v_1, ..., v_{n-2}) \subset K^*(G)$ *such that*

$$2^{k-1}\tau = 2^{n-2}\xi_2\varepsilon_{n-1}v.$$

Proof. Note that $2^{k-1}\tau = \xi_1^{k-1}\tau$ (see (4.4) and (4.5)). In $E_\infty(A)$ we have $(\bar{x} \otimes 1)^{k-1} \cdot (\bar{x} \otimes \gamma_n) = 0 \in E_\infty^{2k,*}(A)$ and we conclude that $\xi_1^{k-1}\tau \in K^1(G)$ has filtration greater than $2k$. This in turn implies that $\xi_1^{k-1}\tau$ represents a non-zero element $t \in E_\infty^{2s,*}(A)$ for some s with $k+1 \leqslant s \leqslant n$. Since $b_2^*(\xi_1^{k-1}\tau) = 0$ we infer that $E_\infty^{2s,*}(F_2)(t) = 0$. But $E_\infty^{2s,*}(F_2)$ is an isomorphism for $k+1 \leqslant s \leqslant n-1$; (see (5.3) and (5.19)). Hence $t \in E_\infty^{2n,*}(A)$, i.e. $\xi_1^{k-1}\tau$ has exact filtration $2n$, and the corollary follows from (5.26).

6. The Ring $K^*(\mathrm{PSO}(2n))$; n even.

In this section we state the main theorem – for the "non cyclic" case – and complete its proof.

For this purpose define the $\mathbf{Z}_2$-graded commutative ring $T^*(G) = T^0(G) \oplus T^1(G)$ to be the subring of $K^*(G)$ generated by 1, ξ_1, ξ_2 and $\tau \in K^*(G)$.

Referring to (3.3), (4.4), (4.5) and (5.24) we get:

(6.1) *The subring* $T^*(G) \subset K^*(G)$ *is subject to the following relations*

(i) *The elements* ξ_1, $\xi_1\xi_2$ *and* $\tau\xi_2$ *are of order* 2^{k-1}, *the element* τ *is of order* 2^k, *where* $k = v_2(n) + 2$. *The element* ξ_2 *is of order* 2^{n-1}.

(ii) $\xi_i^2 + 2\xi_i = 0$, $(i = 1, 2)$, $\tau^2 = 0$ *and* $\xi_1\tau + 2\tau = 0$.

(6.2) THEOREM (Non-cyclic case). *Let* $G = \mathrm{PSO}(2n)$, *where* $n \geqslant 6$ *is an even integer. Then the canonical homomorphism*

$$E_Z(v_1, ..., v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G) \to K^*(G)$$

induces a ring isomorphism

$$\{E_Z(v_1, ..., v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G)\}/S(G) \cong K^*(G),$$

where $S(G)$ *is the ideal generated by the elements*

$$\varepsilon_{n-1} \otimes \xi_1, \ \varepsilon_n \otimes \xi_2, \ \varepsilon_{n-1} \otimes \tau, \ \varepsilon_n \otimes \tau, \ \varepsilon_{n-1} \otimes 2^{n-2}\xi_2 - 1 \otimes 2^{k-1}\tau$$

and

$$1 \otimes \tau\xi_2 - \varepsilon_n \otimes \xi_1 + 1 \otimes 2\tau.$$

Proof. Let us first establish the relation $2^{k-1}\tau = 2^{n-2}\xi_2\varepsilon_{n-1}$ in $K^*(G)$. Reverting to (5.27) we recall that we have already shown $2^{k-1}\tau = 2^{n-2}\xi_2\varepsilon_{n-1}v$, for some $v \in E_{\mathbb{Z}}(v_1, \ldots, v_{n-2})$. In order to verify that actually $v \equiv 1 \pmod 2$ and hence $2^{n-2}\xi_2\varepsilon_{n-1}v = 2^{n-2}\xi_2\varepsilon_{n-1}$, we propose to look at the homomorphism $g^*: K^*(G) \to$ $\to K^*(G) \otimes K^*(G_0)$ which is induced by the obvious action map $g: G \times G_0 \to G$. We then easily calculate that

$$g^*(2^{k-1}\tau) = 2^{k-1}\tau \otimes 1.$$

On the other hand – since v_s, $(s=1, \ldots, n-2)$, is primitive modulo torsion and since $2^{n-2}\xi_2 \cdot \mathrm{Tors}.K^*(G) = 0$ – it is not hard to show that

$$g^*(2^{n-2}\xi_2\varepsilon_{n-1}v) = 2^{n-2}\xi_2\varepsilon_{n-1}v \otimes 1 + \alpha(v),$$

where $\alpha(v) \neq 0$ unless $v \equiv 1 \pmod 2$. Hence the relation $2^{k-1}\tau = 2^{n-1}\xi_2\varepsilon_{n-1}$ is established.

Next we observe that we have $\varepsilon_{n-1}\xi_1 = 0$ and $\varepsilon_n\xi_2 = 0$. (Use the fact that ε_{n-1} $= (b_2)_* (\varepsilon_{n-1}^{(2)})$, $\varepsilon_n = (b_1)_* (\varepsilon_n^{(1)})$, (see (2.12)), $b^*(\xi_1) = 0$, $b_1^*(\xi_2) = 0$, (see (3.3)), and the 'Frobenius law'.) The validity of the above relations together with (3.3), (4.4) and (4.5) then imply that the canonical homomorphism $h: E_{\mathbb{Z}}(v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G)$ $\to K^*(G)$ factors through $\{E_{\mathbb{Z}}(v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G)\}/S(G)$. On the other hand h is an epimorphism by (5.25) and the order of the torsion subgroup of $\{E_{\mathbb{Z}}(v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G)\}/S(G)$ is the same as $|\mathrm{Tors}.K^*(G)|$ (see (5.21)). Therefore h is an isomorphism and the theorem is proved.

II. THE CYCLIC CASE; $\pi_1(\mathrm{PSO}(2n)) \cong \mathbb{Z}_4$

7. The Ring $K^*(\mathrm{PSO}(2n))$; n odd.

If $n \geq 5$ is an *odd* integer then the centre π of $G_0 = \mathrm{Spin}(2n)$ is isomorphic to $\mathbb{Z}_4$. In order to determine the ring structure of $K^*(G)$, where $G = G_0/\pi$, one has to analyze the spectral sequence of the fibration

$$\Lambda = (G_0 \xrightarrow{u} G \xrightarrow{c} B_\pi)$$

where $\pi \cong \mathbb{Z}_4$, $G_0 \xrightarrow{u} G$ the universal 4-fold covering of G and c is its classifying map. The structure of the spectral sequence of Λ can be worked out essentially along the lines of [8]. It turns out that the only non-trivial differentials are d_6^Λ and d_{2n}^Λ. The reason for that may be indicated as follows.

Let $j: \pi \hookrightarrow G_0$ be the inclusion of the centre. Then $R(\pi)/J \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{n-1}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where J is the ideal generated by $j^*(I_{G_0})$ and the cyclic summands of $R(\pi)/J$ are

generated by 1, $\bar{\sigma}$, $\bar{\sigma}^2 + 2\bar{\sigma}$ and $\bar{\sigma}^3 + 2\bar{\sigma}^2$, with $1 + \sigma$ being the canonical representation of π.

The fact that $J \subset I_\pi^3$ but $J \not\subset I_\pi^4$ together with [8; (5.5)] implies that $d_6^A \neq 0$.

The non-triviality of d_{2n}^A then is worked out by comparing the spectral sequence of Λ with the spectral sequence of $\Gamma_2 = (G_0 \xrightarrow{a_2} SO(2n) \xrightarrow{c_2} B_{\mathbf{Z}_2})$.

From the $E_\infty(\Lambda)$ term we derive that

$$T^*(G) = T^0(G) = \text{im} \{K^*(B_\pi) \xrightarrow{c^*} K^*(G)\} \cong R(\pi)/J. \tag{7.1}$$

Let $1 + \xi \in K^0(G)$ represent the line bundle associated to the (cyclic) covering $G_0 \xrightarrow{u} G$. Clearly $\xi \in T^0(G)$ and moreover it corresponds to the generator $\bar{\sigma}$ under the above isomorphism $T^0(G) \cong R(\pi)/J$. In particular ξ generates $\tilde{T}^0(G)$ and it is subject to the relations

$$2^{n-1}\xi = 0,\ (1 + \xi)^4 = 1 \quad \text{and} \quad 2(\xi^2 + 2\xi) = 0.$$

As in the "non-cyclic" case there are elements $v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n \in K^1(G)$ generating an exterior algebra in $K^*(G)$ which is isomorphic to $K^*(G)/\text{Tors}.K^*(G)$.

Summarizing all the information we get from the spectral sequence of Λ and from the transfer maps of the coverings involved, we arrive at the following description of the ring $K^*(G)$.

(7.2) THEOREM (Cyclic case). *Let* $G = PSO(2n)$, *where* $n \geqslant 5$ *is an odd integer. Then the canonical homomorphism*

$$E_{\mathbf{Z}}(v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G) \to K^*(G)$$

induces a ring isomorphism

$$\{E_{\mathbf{Z}}(v_1, \ldots, v_{n-2}, \varepsilon_{n-1}, \varepsilon_n) \otimes T^*(G)\}/S(G) \cong K^*(G),$$

where $T^*(G) = T^0(G) \cong R(\pi)/(j^*(I_{G_0}))$ *and* $S(G)$ *is the ideal generated by* $\varepsilon_n \otimes 2\xi$, $\varepsilon_{n-1}\varepsilon_n \otimes \xi$, $\varepsilon_n \otimes \xi^3$ *and* $\varepsilon_{n-1} \otimes (\xi^2 + 2\xi)$.

BIBLIOGRAPHY

[1] ATIYAH, M. F., *Thom Complexes*, Proc. London Math. Soc. (1961), pp. 291–310.

[2] ——, *Characters and cohomology of finite groups*, Publ. Math. Inst. Hautes Etudes Scientifiques, (1961).

[3] ——, *K-theory*, (Benjamin, 1967).

[4] BAUM, P. F. and BROWDER, W. *The cohomology of quotients of classical groups*, Topology *3* (1965), 305–336.

[5] BOREL, A., *Seminar on transformation groups*, Princeton University Press 1960.

[6] GITLER, S. and LAM, K. Y., *The K-theory of Stiefel manifolds*, Springer Lecture Notes in Mathematics, *168*, pp. 35–66.

[7] HELD, R. P. und SUTER, U., *Die Bestimmung der unitären K-Theorie von* SO(n) *mit Hilfe der Atiyah-Hirzebruch Spektralreihe*, Math. Z. *122* (1971), 33–52.

[8] ——, *On the unitary K-theory of compact Lie groups with finite fundamental group*, Quart. J. Math. Oxford (2), *24* (1973), 343–356.

[9] HODGKIN, L., *On the K-theory of Lie groups*, Topology, *6* (1967), 1–36.

[10] KAHN, D. S. and PRIDDY, S. B., *Applications of the transfer to stable homotopy theory* (preprint, Northwestern University, Evanston, Illinois).

[11] TITS, J., *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Springer Lecture Notes in Mathematics, *40* (1967).

Department of Mathematics
The University of B.C.
Vancouver 8, B.C.
Canada