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Regular Rational Homotopy Types

by Richard Body

§1. Regular Rational Homotopy Types

In general, spaces sharing the same integral cohomology ring need not be homotopy-equivalent. However we shall see that cohomology rings which satisfy an algebraic regularity condition may be shared by only a finite number of distinct homotopy types, [1], [2], [6].

The class of rings we shall consider are those associative, graded-commutative rings which, when tensored with the rational field, have a regular set of relations (see Definition 2.1).

THEOREM (3.1). Let A be an associative, graded-commutative ring such that $A \otimes Q$ has a regular set of relations. Then there are only a finite number of homotopy types of finite, simply-connected polyhedra X for which $H^*(X; Z)$ is isomorphic to A, as graded rings.

Within the category of finite, simply-connected CW complexes, all *H*-spaces, Riemannian symmetric spaces and homogeneous spaces which are compact Lie groups modulo a closed subgroup of maximal rank, all have integral cohomology appropriate to the above theorem.

Finally, we may view the above theorem as a generalization of the results of [1], because the set of relations $\{x_1^{n_1}, x_2^{n_2}, ..., x_m^{n_m}\}$ is regular.

§2. Regular Sequences of Relations, and the Construction of a Model Space

All rings under consideration will be associative and graded-commutative (i.e. $a \cdot b = (-1)^{(\dim a) \cdot (\dim b)} b \cdot a$).

A free algebra over the rational field Q will then be the tensor product of

- 1) polynomial algebras on each of the even-dimensional generators, and
- 2) exterior algebras on each of the odd-dimensional generators.

Denote such a free algebra as $\overline{F} = F(\xi_1, \xi_2, ..., \xi_m)$ where $(\xi_1, ..., \xi_m)$ denotes a choice of generators.

DEFINITION 2.1. A sequence of elements $(\lambda_1, ..., \lambda_k)$ in \vec{F} , a rational free algebra

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is said to be *regular* if F is a free-module over the free algebra $F(\lambda_1, ..., \lambda_k)$ (under the action of the inclusion of the generators λ_i in \overline{F}).

From [5], under the condition that λ_i is of even degree, regularity is equivalent to the following condition on the sequence $(\lambda_1, \lambda_2, ..., \lambda_k)$:

$$\lambda_i \mod (\lambda_1, ..., \lambda_{i-1})$$
 in $F/(\lambda_1, ..., \lambda_{i-1})$ is not a zero divisor.

If B, a rational algebra, has a regular set of relations, i.e. $B \simeq F(\xi_1, \xi_2, ..., \xi_m)/(\lambda_1, ..., \lambda_k)$, we may construct a CW complex M_B which has rational cohomology isomorphic to B, by the following procedure.

Let K_X and K_A be generalized Eilenberg-MacLane spaces.

$$K_X = \prod_{i=1}^m K(Z, \dim \xi_i); \qquad K_A = \prod_{i=1}^k K(Z, \dim \lambda_i).$$

The integral cohomology of K_X , modulo torsion, is generated by fundamental classes $x_1, x_2, ..., x_m$; that of K_A by generators $l_1, l_2 ... l_k$, where $\dim \xi_i = \dim x_i$ and $\dim \lambda_i = \dim l_i$.

Without loss of generality we may consider $\lambda_i \in F(\xi_1, ..., \xi_m)$ to be a polynomial in $\xi_1, ..., \xi_m$ with coefficients which are integers, collectively having no common divisor.

Now define a map $\beta: K_X \to K_A$ (i.e. a sequence of integral cohomology classes of K_X) by requiring that $\beta*(l_i) = \lambda_i(x_1, ..., x_m)$.

LEMMA 2.2. The Fibre of β , denoted M_B , has rational cohomology $H^*(M_B; Q) \simeq B$. Proof. By induction on k, the number of relations $\lambda_1, ..., \lambda_k$. Suppose

$$B_k = \frac{F(x_1, ..., x_m)}{(\lambda_1, ..., \lambda_k)} \simeq H^* (\text{Fibre } \beta_k; Q).$$

We have a commutative diagram

$$M_{B_{k}} \xrightarrow{M_{B_{k+1}}} K_{X} \xrightarrow{\beta_{k}} K_{A_{k}}$$

$$\downarrow^{l=l_{k+1}} \qquad \downarrow^{\beta_{k+1}} \qquad \parallel$$

$$K(Z, \dim \lambda_{k+1}) \xrightarrow{K_{A_{k+1}}} K_{A_{k}}$$

in which the left hand square may be considered a pullback square. Hence $M_{B_{k+1}}$ is homotopy-equivalent to the fibre of l, and we may calculate $H^*(M_{B_{k+1}}; Q)$ from the Serre spectral sequence with rational coefficients, of the fibration

$$\Omega K(Z, \dim \lambda_{k+1}) \rightarrow M_{B_{k+1}} \rightarrow M_{B_k}$$

Let $s = \dim \lambda_{k+1}$. Let us first consider the case in which s is even. For dimensional reasons, the only possibly non-zero differential is d_s , and indeed the transgression of the fundamental class of the fibre is

$$\lambda_{k+1} \mod (\lambda_1, \lambda_k)$$
 in $\frac{F(\xi_1, \dots \xi_m)}{(\lambda_1, \dots, \lambda_k)} \simeq H^*(M_{B_k}; Q)$.

Because this element is not a zero-divisor, $d_s^{*,s-1}$ is a monomorphism and $E_{\infty}^{*,*} \simeq H^*(M_{B_{k+1}}; Q) \simeq F(\xi_1, ..., \xi_m)/(\lambda_1, ..., \lambda_{k+1})$ as required.

Now assume that s is odd. The first possibly non-zero differential is again d_s ; the transgression of the fundamental class of the fibre is $\tau(\iota_{s-1}) = \lambda_{k+1} \mod (\lambda_1, ..., \lambda_k)$. But $H^*(M_{B_k}; Q)$ is a free $F(\tau(\iota_{s-1}))$ -module and the kernel of $d_s^{*, p(s-1)}$ for p>0 is exactly the $H^*(M_{B_k}; Q)$ -module with basis $\iota_{s-1}^p \otimes \tau(\iota_{s-1})$. This coincides with the image of $d_s^{*, (p+1)(s-1)}$. Hence $E_{s+1}^{*, p} \simeq 0$, p>0 and again

$$E_{s+1}^{*,0} \simeq E_{\infty}^{*,*} \simeq H^*(M_{B_{k+1}}; Q) \simeq \frac{F(\xi_1,...,\xi_m)}{(\lambda_1,...,\lambda_{k+1})}.$$

§3. Distance Between Homotopy Types

Let A be a graded-commutative, associative ring such that $A \otimes Q \simeq F(\xi_1, ..., \xi_m)/(\lambda_1, ..., \lambda_k) \simeq B$ is regular. If A is the integral cohomology ring of some finite CW complex, it has a highest non-vanishing dimension, say D. Every finite complex with cohomology A will then be homotopy-equivalent to some (D+1)-dimensional CW complex.

Denote a (D+2)-homology section of M_B by M_A [3]. Also let t denote the order of the torsion subgroup of A, considered as an abelian group. Finally let $q: H^*(\ ; Z) \to H^*(\ ; Q)$ be the coefficient homomorphism induced by the standard inclusion $Z \subseteq Q$.

Given any finite, simply-connected CW complex X with $H^*(X; Z) \simeq A$, we shall construct a map $\phi: X \to M_A$ such that $\phi^*: H^*(M_A; Z) \to A$ has kernel and cokernel of orders bounded by some integer-valued function of the isomorphism class of A. By appealing to the results of [1], this will then be sufficient to deduce that there are at most a finite number of such homotopy types X.

First choose indivisible elements $y_1, y_2, ..., y_m \in H^*(X; Z)$ such that $q(y_i) = \xi_i$. Let $\alpha: X \to K_X$ be defined by requiring $\alpha^*(x_i) = t^{d_i} \cdot y_i$, where $d_i = \text{degree } y_i$. Then $\alpha^*\beta^*(l_i) = \lambda_i(t^{d_1}y_1, t^{d_2}y_2, t^{d_m}y_m) = t^{\text{deg}\lambda_i} \lambda_i(y_1, y_2, y_m)$ is a torsion element of $H^*(X; Z)$ and hence $0. \beta \circ \alpha$ is null-homotopic and α lifts to the fibre of β , $\alpha: X \to M_B$. Because X has dimension at most (D+1), the cellular approximation theorem shows that α factors through the (D+1)-skeleton of M_B , hence through M_A . It is trivial to verify that this map $\phi: X \to M_A$ induces an isomorphism on rational cohomology.

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The kernel of $\phi^*: H^*(M_A) \to A$ has order bounded by the order of the torsion subgroup of $H^*(M_A; \mathbb{Z})$, a function of the isomorphism class of A.

The cokernel of ϕ^* is a quotient of the cokernel of α^* . The cokernel of α^* in turn is a quotient of the group $A/(Algebra generated by <math>\alpha^*x_i$, i=1, 2, m), a finite group whose order is a function of the isomorphism class of A.

Thus, according to the terminology of [1], each homotopy type X, with $H^*(X; Z) \simeq A$ is of bounded distance from M_A , and with the help of Theorem 3.2 of [1] there are only a finite number of such homotopy types. We have demonstrated

THEOREM 3.1. Let A be a graded-commutative, associative ring such that $A \otimes Q$ has a regular set of relations. Let HT(A) be the class of all homotopy types of simply-connected, finite CW complexes X such that, as graded algebras $H^*(X; Z) \simeq A$. Then HT(A) is a finite set.

It may be remarked that all such spaces X are 0-universal in the sense of Serre [4]. This may be demonstrated by noting that, for each integer r, there exists endomorphisms $\varrho_1: K_X \to K_X$ and $\varrho_2: K_A \to K_A$ defined by $\varrho_1^*(x_i) = r^{\dim x_i} x_i$ and $\varrho_2^*(l_i) = r^{\dim l_i} l_i$ inducing an endomorphism of the fibre of β , M_B , hence of M_A which satisfy Mimura, Toda and O'Neill's condition (b') for 0-universality [4].

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