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Regular Rational Homotopy Types

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§1. Regular Rational Homotopy Types

In general, spaces sharing the same integral cohomology ring need not be homotopy-equivalent. However we shall see that cohomology rings which satisfy an algebraic regularity condition may be shared by only a finite number of distinct homotopy types, [1], [2], [6].

The class of rings we shall consider are those associative, graded-commutative rings which, when tensored with the rational field, have a regular set of relations (see Definition 2.1).

THEOREM (3.1). *Let A be an associative, graded-commutative ring such that $A \otimes Q$ has a regular set of relations. Then there are only a finite number of homotopy types of finite, simply-connected polyhedra X for which $H^*(X; Z)$ is isomorphic to A, as graded rings.*

Within the category of finite, simply-connected CW complexes, all H-spaces, Riemannian symmetric spaces and homogeneous spaces which are compact Lie groups modulo a closed subgroup of maximal rank, all have integral cohomology appropriate to the above theorem.

Finally, we may view the above theorem as a generalization of the results of [1], because the set of relations $\{x_1^{n_1}, x_2^{n_2}, \ldots, x_m^{n_m}\}$ is regular.

§2. Regular Sequences of Relations, and the Construction of a Model Space

All rings under consideration will be associative and graded-commutative (i.e. $a \cdot b = (-1)^{(\dim a)(\dim b)} b \cdot a$).

A free algebra over the rational field Q will then be the tensor product of

1) polynomial algebras on each of the even-dimensional generators, and

2) exterior algebras on each of the odd-dimensional generators.

Denote such a free algebra as $F = F(\xi_1, \xi_2, \ldots, \xi_m)$ where $(\xi_1, \ldots, \xi_m)$ denotes a choice of generators.

DEFINITION 2.1. A sequence of elements $(\lambda_1, \ldots, \lambda_k)$ in F, a rational free algebra is said to be *regular* if F is a free-module over the free algebra $F(\lambda_1,\ldots,\lambda_k)$ (under the action of the inclusion of the generators λ_i in F).

From [5], under the condition that λ_i is of even degree, regularity is equivalent to the following condition on the sequence $(\lambda_1,\lambda_2,\ldots,\lambda_k)$:

$$\lambda_i \bmod (\lambda_1,\ldots,\lambda_{i-1}) \quad \text{in} \quad F/(\lambda_1,\ldots,\lambda_{i-1}) \quad \text{is not a zero divisor.}$$

If B, a rational algebra, has a regular set of relations, i.e. $B\simeq F(\xi_1,\xi_2,\ldots,\xi_m)/(\lambda_1,\ldots,\lambda_k)$, we may construct a CW complex M_B which has rational cohomology isomorphic to B, by the following procedure.

Let K_X and K_A be generalized Eilenberg-MacLane spaces.

$$K_X=\prod_{i=1}^{m} K(Z,\dim\xi_i)\,;\qquad K_A=\prod_{i=1}^{k} K(Z,\dim\lambda_i)\,.$$

The integral cohomology of K_X, modulo torsion, is generated by fundamental classes $x_1,x_2,\ldots,x_m$; that of K_A by generators $l_1,l_2\ldots l_k$, where $\dim\xi_i=\dim x_i$ and $\dim\lambda_i==\dim l_i$.

Without loss of generality we may consider $\lambda_i\in F(\xi_1,\ldots,\xi_m)$ to be a polynomial in $\xi_1,\ldots,\xi_m$ with coefficients which are integers, collectively having no common divisor.

Now define a map $\beta: K_X\to K_A$ (i.e. a sequence of integral cohomology classes of K_X) by requiring that $\beta^*(l_i)=\lambda_i(x_1,\ldots,x_m)$.

LEMMA 2.2. *The Fibre of β, denoted M_B, has rational cohomology $H^*(M_B;Q)\simeq B$.*

Proof. By induction on k, the number of relations $\lambda_1,\ldots,\lambda_k$. Suppose

$$B_k=\frac{F(x_1,\ldots,x_m)}{(\lambda_1,\ldots,\lambda_k)}\simeq H^*(\text{Fibre}\,\beta_k;Q)\,.$$

We have a commutative diagram

$$\begin{tikzcd}
 & M_{B_{k+1}} \arrow[d] & \\
M_{B_k} \arrow[r] \arrow[d, "l=l_{k+1}"] & K_X \arrow[r, "\beta_k"] \arrow[d, "\beta_{k+1}"] & K_{A_k} \arrow[d, equal] \\
K(Z,\dim\lambda_{k+1}) \arrow[r] & K_{A_{k+1}} \arrow[r] & K_{A_k}
\end{tikzcd}$$

in which the left hand square may be considered a pullback square. Hence $M_{B_{k+1}}$ is homotopy-equivalent to the fibre of l, and we may calculate $H^*(M_{B_{k+1}};Q)$ from the Serre spectral sequence with rational coefficients, of the fibration

$$\Omega K(Z,\dim\lambda_{k+1})\to M_{B_{k+1}}\to M_{B_k}$$

Let $s=\dim\lambda_{k+1}$. Let us first consider the case in which s is even. For dimensional reasons, the only possibly non-zero differential is d_s, and indeed the transgression of the fundamental class of the fibre is

$$\lambda_{k+1} \bmod (\lambda_1,, \lambda_k) \quad \text{in} \quad \frac{F(\xi_1, \dots \xi_m)}{(\lambda_1, \dots, \lambda_k)} \simeq H^*(M_{B_k}; Q).$$

Because this element is not a zero-divisor, $d_s^{*,s-1}$ is a monomorphism and $E_\infty^{*,*} \simeq \simeq H^*(M_{B_{k+1}}; Q) \simeq F(\xi_1, \dots, \xi_m)/(\lambda_1, \dots, \lambda_{k+1})$ as required.

Now assume that s is odd. The first possibly non-zero differential is again d_s; the transgression of the fundamental class of the fibre is $\tau(\iota_{s-1}) = \lambda_{k+1} \bmod (\lambda_1, \dots, \lambda_k)$. But $H^*(M_{B_k}; Q)$ is a free $F(\tau(\iota_{s-1}))$-module and the kernel of $d_s^{*,p(s-1)}$ for $p>0$ is exactly the $H^*(M_{B_k}; Q)$-module with basis $\iota_{s-1}^p \otimes \tau(\iota_{s-1})$. This coincides with the image of $d_s^{*,(p+1)(s-1)}$. Hence $E_{s+1}^{*,p} \simeq 0$, $p>0$ and again

$$E_{s+1}^{*,0} \simeq E_\infty^{*,*} \simeq H^*(M_{B_{k+1}}; Q) \simeq \frac{F(\xi_1, \dots, \xi_m)}{(\lambda_1, \dots, \lambda_{k+1})}.$$

§3. Distance Between Homotopy Types

Let A be a graded-commutative, associative ring such that $A \otimes Q \simeq F(\xi_1, \dots, \xi_m)/(\lambda_1, \dots \lambda_k) \simeq B$ is regular. If A is the integral cohomology ring of some finite CW complex, it has a highest non-vanishing dimension, say D. Every finite complex with cohomology A will then be homotopy-equivalent to some $(D+1)$-dimensional CW complex.

Denote a $(D+2)$-homology section of M_B by M_A [3]. Also let t denote the order of the torsion subgroup of A, considered as an abelian group. Finally let $q: H^*(\ ; Z) \to \to H^*(\ ; Q)$ be the coefficient homomorphism induced by the standard inclusion $Z \subsetneq Q$.

Given any finite, simply-connected CW complex X with $H^*(X; Z) \simeq A$, we shall construct a map $\phi: X \to M_A$ such that $\phi^*: H^*(M_A; Z) \to A$ has kernel and cokernel of orders bounded by some integer-valued function of the isomorphism class of A. By appealing to the results of [1], this will then be sufficient to deduce that there are at most a finite number of such homotopy types X.

First choose indivisible elements $y_1, y_2, \dots, y_m \in H^*(X; Z)$ such that $q(y_i) = \xi_i$. Let $\alpha: X \to K_X$ be defined by requiring $\alpha^*(x_i) = t^{d_i} \cdot y_i$, where $d_i = \text{degree}\, y_i$. Then $\alpha^*\beta^*(l_i) = \lambda_i(t^{d_1}y_1, t^{d_2}y_2,, t^{d_m}y_m) = t^{\deg \lambda_i} \lambda_i(y_1, y_2,, y_m)$ is a torsion element of $H^*(X; Z)$ and hence 0. $\beta \circ \alpha$ is null-homotopic and α lifts to the fibre of β, $\hat{\alpha}: X \to M_B$. Because X has dimension at most $(D+1)$, the cellular approximation theorem shows that $\hat{\alpha}$ factors through the $(D+1)$-skeleton of M_B, hence through M_A. It is trivial to verify that this map $\phi: X \to M_A$ induces an isomorphism on rational cohomology.

The kernel of $\phi^*: H^*(M_A) \to A$ has order bounded by the order of the torsion subgroup of $H^*(M_A; Z)$, a function of the isomorphism class of A.

The cokernel of ϕ^* is a quotient of the cokernel of α^*. The cokernel of α^* in turn is a quotient of the group $A/$(Algebra generated by $\alpha^* x_i$, $i=1, 2,, m$), a finite group whose order is a function of the isomorphism class of A.

Thus, according to the terminology of [1], each homotopy type X, with $H^*(X; Z) \simeq \simeq A$ is of bounded distance from M_A, and with the help of Theorem 3.2 of [1] there are only a finite number of such homotopy types. We have demonstrated

THEOREM 3.1. *Let A be a graded-commutative, associative ring such that $A \otimes Q$ has a regular set of relations. Let $HT(A)$ be the class of all homotopy types of simply-connected, finite* CW *complexes X such that, as graded algebras $H^*(X; Z) \simeq A$. Then $HT(A)$ is a finite set.*

It may be remarked that all such spaces X are 0-universal in the sense of Serre [4]. This may be demonstrated by noting that, for each integer r, there exists endomorphisms $\varrho_1: K_X \to K_X$ and $\varrho_2: K_A \to K_A$ defined by $\varrho_1^*(x_i) = r^{\dim x_i} x_i$ and $\varrho_2^*(l_i) = r^{\dim l_i} l_i$ inducing an endomorphism of the fibre of β, M_B, hence of M_A which satisfy Mimura, Toda and O'Neill's condition (b′) for 0-universality [4].

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