Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 50 (1975)

Artikel: Free Cyclic Actions on Manifolds.

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DOI: https://doi.org/10.5169/seals-38793

Comment. Math. Helvetici 50 (1975) 59–80 Birkhäuser Verlag, Basel

Free Cyclic Actions on Manifolds

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0. Introduction

This paper presents a geometric description and partial classification of smooth or PL oriented closed $(n-1)$ connected $2n$ manifolds M, $n\geqslant 3$, which support free actions of the cyclic group $Z_p = Z/pZ$ (where $p\geqslant 3$ is prime) with preferred generator T_M and orbit manifold M/Z_p. If $L(n)$ is an n-dimensional lens space (or the n-skeleton of the usual CW decomposition of $L(n+1)$ if $n=2k$) with $\pi_1(L(n))=Z_p$, universal cover $E(n)$, and fixed generator T_E of the induced Z_p action on $E(n)$, we define a standard model to be a $2n$ dimensional smooth or PL thickening N/Z_p of $L(n)$ with generator T_N of the Z_p action on N corresponding to T_E. Given an equivariant isomorphism $f:(\partial N, T_N)\to(\partial N, T_N)$ (i.e. $fT_N = T_N f$, so there is an induced isomorphism f/Z_p of $\partial N/Z_p$) and a closed $(n-1)$ connected $2n$ manifold K, the universal cover of $(N/Z_p \bigcup_{f/Z_p})\# K$ (written $(N\bigcup_f N)\#_{Z_p} K$) is such a manifold M. The underlying idea of this work is that most of the manifolds M can be obtained in this way.

A standard model N/Z_p is untwisted if the homology intersection form on $H_n(N)$ is identically zero, and it is simple if in addition $N\cong S^n\times D^n$ (not necessarily equivariantly) when $n=2k+1$. The homomorphism $(T_M)_*$ makes $H_n(M)$ a $\Lambda = ZZ_p$ module, and Wall ([35], §5) defines a Λ valued intersection form λ on $H_n(M)$. We say that M has hyperbolic rank $\geqslant d$ if the λ form orthogonally splits off d hyperbolic planes (see 2.9).

THEOREM A. *Suppose M has hyperbolic rank $\geqslant 2$, $n\neq 4, 8$, and $n+1<p$ if M is smooth. Then for some untwisted model N/Z_p (simple if M is smooth), isomorphism $f:(\partial N, T_N)\to(\partial N, T_N)$, and closed $(n-1)$ connected $2n$ manifold K, there is an orientation preserving equivariant isomorphism between (M, T_M) and $(N\bigcup_f N)\#_{Z_p} K$ (with Z_p generator corresponding to T_N).*

This is proved in several stages. We first show in §2 that untwisted models have bundle structures determined by homotopy classes $[L(n), BH_n]$ when $n=2k+1$ and $[L(n), BH_{n-1}]$ when $n=2k$ ($H=0$ or PL depending on the category), and M has an equivariant decomposition $M=N_1\cup L\cup N_2$ where N_1, N_2 are isomorphic to the same untwisted model N and L is a cobordism with boundary components $\partial_1=\partial N_1$,

[1] Partially supported by NSF grant GP 28938A.

$\partial_2 = \partial N_2$ such that $H_n(L, \partial_1)$ is a free Λ module and $H_j(L, \partial_1) = 0, j \neq n$. In §3 we define a unimodular intersection form λ on $H_n(L, \partial_1)$ such that, for $n \neq 4, 8$ and appropriate Λ basis $\underline{b}_L \subseteq H_n(L, \partial_1)$, $(H_n(L, \partial_1), \lambda, \underline{b}_L)$ represents algebraically an element x of the Wall group $L_{2n}(Z_p)$. Using a result of Petrie [25] x is in $\text{im}(L_{2n}(0) \to L_{2n}(Z_p))$ and so, by a straightforward geometric argument, L is equivariantly isomorphic to $(\partial_1 \times I) \#_{Z_p} K$.

In §4 surgery theory is used to study the effect of the gluing f. Let s_j denote the order of the torsion subgroup of $\pi_j(G/O)$ (see 4.2).

THEOREM B. *For fixed choice of (N, T_N) and K and orientations on them, up to orientation preserving equivariant isomorphism there are at most finitely many manifolds $(N \bigcup_f N) \#_{Z_p} K$. For suitable p, a specific upper bound b for the number of such manifolds is given by:*

a) *$b = \text{order}\ (\pi_{2n-1}(S^n))$ when M is PL, $n = 2k+1 < 2p-1$.*

b) *$b = s_n\, s_{2n}\ \text{order}\ (\pi_{2n-1}(S^n))$ when M is smooth, $n = 2k+1 < p-1$.*

c) *$b = 2$ when M is PL, $n = 2k < 2p-2$.*

d) *$b = 2s_{2n}$ when M is smooth, $n = 2k < p-1$.*

The general classification problem is complicated by the fact that the decomposition of Theorem A is not unique. But if M is smooth, $n \not\equiv 0, 4 \pmod 8$ and $n+3 < 2p$, then any two simple models in M^{2n} are equivariantly isomorphic, determined by a unique homotopy class $\varrho(M) \in [L(n), BO_{n-1}]$ if $n = 2k$ and $\varrho(M) \in \ker([L(n), BO_n] \to [E(n), BO_n])$ if $n = 2k+1$.

THEOREM C. *Suppose the smooth manifold M^{2n} (as above) has hyperbolic rank $\geqslant 2$, $n \equiv 3, 5, 6, 7, \pmod 8$, and $n+1 < p$. Then M is classified up to orientation preserving equivariant isomorphism and equivariant connected sum with a homotopy sphere $\Sigma \in \theta_{2n}$ by the model invariant $\varrho(M)$, the Z rank of $H_n(M)$, and the Arf invariant $\sigma(M) \in Z_2$ (the signature $\sigma(M) \in 8p\,Z$ if $n \equiv 6$). All values of $\varrho(M)$, $\sigma(M)$ occur independently (except that $\sigma(M) = 0$ when $n = 3, 7$). Furthermore, $\theta_{2n} = p \cdot \theta_{2n}$ has no p-torsion, and θ_{2n} acts freely.*

Explicit computations can be made in other cases also. For example, any smooth M^6 with $H_n(M) = Z+Z$ is equivariantly diffeomorphic to $S^3 \times S^3$ with quotient $L(3) \times S^3$. There are precisely $2p$ smooth oriented manifolds M^{14} with $H_n(M) = Z+Z$ ($\varrho(M) \in Z_p$ and $\theta_{14} = Z_2$ acts freely).

The analogous problems for $p = 2$ have been studied by S. Lopez de Medrano [18]; R. Wells [36], [37] and I. Hambleton [40]. Although several of the results are similar, their methods are of necessity quite different.

Many of these results first appeared in my Princeton Ph.D. thesis written under the direction of Professoı J. L. Shaneson. Many thanks are due to him and to Professors W. Browder, E. Brown, and J. Morgan for helpful discussions.

1. Notation and Preliminary Results

1.1. Throughout this paper M will denote a smooth or PL oriented compact $(n-1)$ connected manifold of dimension $2n$, $n \geqslant 3$, and $T_M: M \to M$ is a smooth or PL isomorphism such that $(T_M)^p = 1$ for some odd prime p and $T_M(x) \neq x$ for all $x \in M$. Then T_M is orientation preserving since p is odd and $(T_M)^j\,(x) \neq x$ for $0 < j < p$ since p is prime and we have a free Z_p action on M with preferred generator T_M.

More generally, given simply connected manifolds K and L supporting free Z_p actions generated by T_K and T_L, respectively, we denote the associated orbit space by K/Z_p, a manifold of the same dimension, and let $\pi_K: K \to K/Z_p$ be the quotient map, a covering projection of degree p. Then $(\pi_K)_{\#}: \pi_i(K) \to \pi_i(K/Z_p)$ is an isomorphism, $i > 1$, and we identify T_K with the homotopy class $[\pi_K \circ \omega] \in \pi_1(K/Z_p) = Z_p$ where $\omega: [0, 1] \to K$ is a path from a base point x_0 to $T_K(x_0)$. A map $f: K \to L$ is equivariant (written $f: (K, T_K) \to (L, T_L)$) provided $fT_K = T_L f$. Given such a map there is an induced map $f/Z_p: K/Z_p \to L/Z_p$ such that $\pi_L \circ f = f/Z_p \circ \pi_K$. Furthermore, f/Z_p is an immersion, embedding, or isomorphism in the appropriate category if and only if f is. We call f/Z_p the quotient of f and call f a lift of f/Z_p. Since K and L are simply connected, any map $f/Z_p: K/Z_p \to L/Z_p$ such that $(f/Z_p)_{\#}(T_K) = T_L$ has a lift $f: (K, T_K) \to (L, T_L)$, and all possible lifts are $f, T_L \circ f, \ldots, T_L^{p-1} \circ f$.

1.2. The spaces $L(2k+1)$, $L(2k)$, and $\overline{L(2k)}$. Let $E(2k+1)$ denote the sphere $S^{2k+1} = \{(Z_0, \ldots, Z_k) \mid Z_i \in \check{C},\ \Sigma|Z_i|^2 = 1\}$ ($\check{C}$ = complex numbers) together with the free Z_p action generated by $T_E(Z_0, \ldots, Z_k) = (\xi Z_0, \ldots, \xi Z_k)$ where $\xi = e^{2i\pi/p}$, and set $L(2k+1) = E(2k+1)/Z_p$, a $2k+1$ dimensional lens space. For each $0 \leqslant r \leqslant k$ and $0 \leqslant s \leqslant p-1$ define

$$\sigma_s^{2r+1} = \left\{(Z_0, \ldots, Z_r, 0, \ldots, 0) \in S^{2k+1} \mid Z_r = 0 \quad \text{or} \quad \frac{2\pi s}{p} < \operatorname{Arg} Z_r < \frac{2\pi(s+1)}{p}\right\},$$

$$\sigma_s^{2r} = \left\{(Z_0, \ldots, Z_r, 0, \ldots, 0) \in S^{2k+1} \mid Z_r = 0 \quad \text{or} \quad \operatorname{Arg} Z_r = \frac{2\pi s}{p}\right\}.$$

Then $T_E: \sigma_s^j \to \sigma_{s+1}^j$ is a bijection and we have an equivariant CW decomposition of $E(2k+1)$ inducing a CW decomposition of $L(2k+1)$ (see [19] for details). The $2k$ skeleton $E(2k)$ inherits a free Z_p action with generator T_E, and $L(2k) = E(2k)/Z_p$ is the $2k$ skeleton of $L(2k+1)$. Since $L(2k)$ is not even homotopy equivalent to a $2k$ manifold, we define the $2k+1$ manifold $\overline{L(2k)}$ with boundary S^{2k} as a smooth regular neighborhood of $L(2k)$ in $L(2k+1)$. Thus $\overline{L(2k)}$ is obtained from $L(2k+1)$ by deleting the interior of a disk. Collapsing defines a deformation retraction r/Z_p: $\overline{L(2k)} \to L(2k)$. Let $\overline{E(2k)}$ denote the universal cover of $\overline{L(2k)}$ and let $r: \overline{E(2k)} \to (2k)$ be some fixed lift of r/Z_p.

Finally, we remark that we could as easily have considered the generalized $(2k+1)$ dimensional lens spaces $L(r_0,\ldots,r_k)$ without affecting our results. An explicit comparison between the resulting theories can be made by noting that there is an equivariant degree m map $E(2k+1)\to E(r_0,\ldots,r_k)$ whenever $m\equiv r_0\cdot\cdots\cdot r_k\ (\mathrm{mod}\,p)$.

1.3. A standard model N/Z_p is a smooth or PL $2n$ dimensional thickening (see [34]) of $L(n)$. Thus we have a simple homotopy equivalence $\phi/Z_p:L(n)\to N/Z_p$, and let $\phi:(E(n),T_E)\to(N,T_N)$ (where $T_N=(\phi/Z_p)_{\#}(T_E)$) be some lift to the universal covers, again a simple equivalence. We say that N/Z_p is untwisted if for any $x,y\in H_n(N)$ we have intersection $x\cdot y=0$. Otherwise the model is twisted.

Clearly any $4k+2$ dimensional model is untwisted, as is $\overline{L(2k)}\times D^{k-1}$. To obtain a twisted model, simply attach a $2k$ handle h to $L(2k-1)\times D^{2k+1}$ (killing the $(2k-1)$ homotopy) with a twisted framing, or attach h so that the resulting left hand $(2k-1)$-spheres in $E(2k-1)\times D^{2k+1}$ are linked.

1.4. By considering the cell decomposition of $L(n)$ one can easily give N/Z_p a standard handle decomposition (see [12]) $D^{2n}=N_0/Z_p\subseteq N_1/Z_p\subseteq\cdots\subseteq N_n/Z_p=N/Z_p$ in which N_i/Z_p is obtained from N_{i-1}/Z_p by attaching a single i-handle h^i. Let $h_1^i,\ldots,h_p^i$ be the i-handles of N covering h^i with $T_E h_j^i=h_{j+1}^i$. But h_j^i corresponds canonically to a generator of $C_i(N/Z_p)=H_i(N_i/Z_p,N_{i-1}/Z_p)=Z$, and $h_1^i,\ldots,h_p^i$ generate $C_i(N)=$ $=H_i(N_i,N_{i-1})$ freely over Z. Then the handle decomposition can be so chosen that the boundary maps $\partial:C_i(N/Z_p)\to C_{i-1}(N/Z_p)$ and $\partial:C_i(N)\to C_{i-1}(N)$ are given by $\partial h^{2j}=p\cdot h^{2j-1}$, $\partial h^{2j-1}=0$, $\partial h_r^{2j}=h_1^{2j-1}+\cdots+h_p^{2j-1}$, and $\partial h_r^{2j-1}=h_{r+1}^{2j-2}-h_r^{2j-2}$. In particular,

$$H_i(N/Z_p)=H_i(C_*(N/Z_p))=\begin{cases}Z_p & i=2j+1,\quad 0<i<n\\ 0 & i=2j,\quad 0<i<n.\end{cases}$$

2. The Decomposition Theorem

Let M^{2n} be as in 1.1 A simple obstruction theory argument shows that there must exist mappings $\phi:(E(n),T_E)\to(M,T_M)$. By [12], 12.1 we may, after adjusting ϕ/Z_p by a homotopy, find a subcomplex $K/Z_p\subseteq M/Z_p$ with $\phi:L(n)\to K/Z_p$ a simple homotopy equivalence. Then any (smooth) regular neighborhood N/Z_p of K/Z_p defines a standard model induced by ϕ. We begin the proof of the decomposition theorem by studying the geometry of these induced models (and in particular their uniqueness). For convenience we consider separately the cases $n=2k+1$ and $n=2k$.

2.1. PROPOSITION. *Let M^{2n} be as in* 1.1 *with $n=2k+1$. Then $\phi:(E(n),T_E)\to$ $\to(M,T_M)$ deforms equivariantly to an embedding $\psi:(E(n),T_E)\to(M,T_M)$, and the* *induced model N/Z_p is a (disk or block) bundle over $\psi/Z_p(L(n))$. There exist embeddings $f_1/Z_p, f_2/Z_p: N/Z_p \to \operatorname{int}(N/Z_p)$ homotopic to the identity and with disjoint images. Furthermore, for fixed choice of T_E, T_M (and $n>3$ in the smooth category) the spaces N and $M-\operatorname{int}(f_1(N)\cup f_2(N))$ are determined up to equivariant isomorphism by the homotopy class of ϕ.*

Proof. $L(n)$ is a manifold and ϕ/Z_p is $(n-1)$ connected and thus deforms to an embedding ψ/Z_p by [11] or [6]. Then N/Z_p is the normal disk or block bundle of $\psi/Z_p(L(n))$ in M/Z_p. To find $f_1/Z_p, f_2/Z_p$ it suffices, by uniqueness of regular neighborhoods, to push the 0 section of N/Z_p off itself. But this follows by the Whitney method since $\psi(E(n))\cdot\psi(E(n))=0$ in M and $(\psi/Z_p)_\#(\pi_1(L(n))=\pi_1(N/Z_p)$. Equivalently, one could check that the only obstruction to a non-zero section, the 2-torsion Euler class, must vanish. The final statement will follow by uniqueness of regular neighborhoods once we show that, given homotopic embeddings $f_1\circ\psi, f_2\circ\psi, f_1'\circ\psi'$, $f_2'\circ\psi':(E(n), T_E)\to(M, T_M)$, we can first push $(f_1\circ\psi)/Z_p$ to $(f_1'\circ\psi')/Z_p$ by a homotopy (and thus an isotopy by [10] or [6] and then push $(f_2\circ\psi)/Z_p$ to $(f_2'\circ\psi')/Z_p$ in $M/Z_p-(f_1\circ\psi)/Z_p(L(n))$. We defer the proof, which uses obstruction theory with local coefficients, to the end of the section (2.12). □

In particular we have the following (see [26] for the PL case).

2.2 COROLLARY. *The $4k+2$ dimensional standard models are classified by $[L(2k+1), BO_{2k+1}]$ in the smooth category and $[L(2k+1), B\widetilde{PL}_{2k+1}]$ in the PL category.*

2.3. PROPOSITION. *Let M^{2n} be as in 1.1 with $n=2k$ (and $n\neq 4$ in the smooth category). For any map $\phi:(E(n), T_E)\to(M, T_M)$ the following are equivalent:*

1) *A model N/Z_p induced by ϕ/Z_p is untwisted,*

2) *$\phi/Z_p\circ r/Z_p:\overline{L(n)}\to M/Z_p$ deforms to an embedding ψ/Z_p.*

3) *There are embeddings $f_1, f_2:(N, T_N)\to\operatorname{int}(N, T_N)$ equivariantly homotopic to the identity and with disjoint images.*

If the above conditions hold, then N/Z_p is a disk or block bundle over $\psi/Z_p(\overline{L(n)})$. For fixed choice of T_E, T_M, the spaces N and $M-\operatorname{int}(f_1(N)\cup f_2(N))$ are determined up to equivariant isomorphism by the homotopy class of ϕ.

Proof. We work in the PL category and apply approximation theorems of [6] for the smooth case. A direct proof is also possible. We can construct $\overline{L(n)}$ by attaching a single n-handle $h=D^n\times[-1, 1]$ (killing the $(n-1)$ homotopy) to a regular neighborhood J/Z_p of $L(n-1)$ in $L(n+1)$. By [12], 8.3 and general position we can push $\phi/Z_p\circ r/Z_p$ to a map θ/Z_p embedding J/Z_p and sending $\operatorname{int}(D^n)\times[-1, 1]$ to M/Z_p- $-\theta/Z_p(J/Z_p)$. If λ is the ZZ_p valued intersection form of [35], §5 and $x=\theta/Z_p(\partial\overline{L(n)})$, then 1) implies that $\lambda(x, x)=0$ and thus $\mu(x)=0$ (see 3.1). It follows that $\lambda(x, y)=0$

where $y=\theta/Z_p(D^n\times 0)$ and we take intersections with ∂y held fixed. Thus by the Whitney method we may first assume $\theta/Z_p(\partial\overline{L(n)})\cap\theta/Z_p(L(n))=\phi$ and then that θ/Z_p embeds $\partial\overline{L(n)}$ disjointly from $L(n)$ (J/Z_p might not be embedded now). If U is a regular neighborhood of $\theta/Z_p(\partial\overline{L(n)})$, a simple homotopy argument pushes θ/Z_p to a map embedding $\partial\overline{L(n)}$ in ∂U and $\theta/Z_p(\operatorname{int}\overline{L(n)})\subseteq M/Z_p-\operatorname{int}U$. By [11], this deforms to an embedding ψ/Z_p (rel $\partial\overline{L(n)}$).

Suppose $\psi:(\overline{E(n)}, T_E)\to(M, T_M)$ is an embedding. Then $\psi/Z_p(J/Z_p)$ has a normal disk bundle in M/Z_p which splits a line subbundle (see [26] or [7]) and we obtain homotopic embeddings ψ_1/Z_p, ψ_2/Z_p with $\psi_1/Z_p(J/Z_p)\cap\psi_2/Z_p(J/Z_p)=\phi$. By a simple intersection number argument we can push $\psi_1/Z_p(h)$ off $\psi_2/Z_p(h)$ and 3) follows by taking disjoint regular neighborhoods of $\psi_1/Z_p(L(n))$ and $\psi_2/Z_p(L(n))$. Since 3) clearly implies 1), the first part of the proof is complete.

Uniqueness preceeds much as in 2.1. For if $f_1\circ\psi, f_1'\circ\psi, f_1\circ\psi', f_2'\circ\psi'$: $(E(n),T_E)\to(M, T_M)$ are homotopic embeddings, then $g/Z_p=(f_1\circ\psi)/Z_p$ and $g'/Z_p=(f_1'\circ\psi')/Z_p$ are homotopic by 2.12. By [10] we may assume $g/Z_p|_{L(n-1)}=g'/Z_p|_{L(n-1)}$. Let K/Z_p be some regular neighborhood of the image of $L(n-1)$ with $g/Z_p(L(n)-L(n-1))$ and $g'/Z_p(L(n)-L(n-1))$ meeting $\partial K/Z_p$ nicely. Then an application of the Hurewicz theorem, 2.12, and [12] provides an ambient isotopy throwing $K/Z_p\cup g/Z_p(L(n))$ onto $K/Z_p\cup g'/Z_p(L(n))$, and uniqueness of N/Z_p follows by uniqueness of regular neighborhoods. Again by 2.12, $f_2/Z_p\circ\psi/Z_p$ and $f_2'/Z_p\circ\psi'/Z_p$ are homotopic as maps into the complement, and the uniqueness proof is completed by applying the above argument again. □

2.4. COROLLARY. *The* $4k$ *dimensional untwisted standard models are classified by* $[L(2k), BO_{2k-1}]$ *in the smooth category* ($k\neq 2$), *and* $[L(2k), B\widetilde{PL}_{2k-1}]$ *in the PL category.*

2.5. We consider now the algebra of the decomposition theorem. The integral group ring $\Lambda=ZZ_p$ of Z_p can be described as the direct sum of p copies of Z together with a Z_p action generated by $Tx_i=x_{i+1}$ (for some Z basis $x_1,\dots, x_p$ of Λ) by defining multiplication in Λ by $x_1\cdot v=Tv$, $v\in\Lambda$. With this description a Λ module is an abelian group with a Z_p action and a Λ module homomorphism is an equivariant group homomorphism. Two Λ modules are of particular interest. Let $\mathbf{Z}$ denote the integers with generator z and Z_p action defined by $Tz=z$. Let Δ denote the direct sum of $(p-1)$ copies of Z with basis $y_1,\dots, y_{p-1}$ and Z_p action generated by $Ty_1=y_2,\dots, Ty_{p-2}=y_{p-1}$, $Ty_{p-1}=-y_1-\cdots-y_{p-1}$. We then have exact sequences of Λ modules $O\to\mathbf{Z}\xrightarrow{f_1}\Lambda\xrightarrow{f_2}\Delta\to O$ and $O\to\Delta\xrightarrow{g_1}\Lambda\xrightarrow{g_1}\mathbf{Z}\to O$ where f_i, g_i are defined by equivariance and the formulas $f_1(z)=x_1+\cdots+x_p$, $f_2(x_1)=y_1$, $g_1(y_1)=x_p-x_1$, and $g_2(x_1)=z$. Neither sequence splits so $\mathbf{Z}$ and Δ are not projective.

2.6. We can define homology and cohomology with coefficients in a Λ module. Let X be a simply connected space supporting a free (topological) Z_p action and suppose we have some fixed CW (respectively, handle) decomposition $V/Z_p=(V_0/Z_p \subseteq \cdots \subseteq V_k/Z_p = X/Z_p)$ in which V_0/Z_p is a disjoint union of points (disks) and V_i/Z_p is obtained from V_{i-1}/Z_p by attaching i cells (i handles). Let V denote the induced equivariant decomposition of X. Then $C_i(V)=H_i(V_i, V_{i-1})$ together with the induced Z_p action is a free Λ module. A specific basis $\underset{\sim}{c}_i(V)$ can be represented by choosing some lift to X of each i cell (respectively, the left hand disk of each i handle) of V/Z_p. The usual boundary map $\partial: C_i(V) \to C_{i-1}(V)$ is equivariant, and for any Λ module R we define $H_*(X/Z_p, R)=H_*(C_*(V)\otimes_\Lambda R)$ and $H^*(X/Z_p, R)=H_*(\mathrm{Hom}_\Lambda(C_*(V), R))$. In particular, we have canonical isomorphisms $\phi_V: H_*(C_*(V)) \to H_*(X)$ and $H_*(C_*(V) \otimes Z) \to H_*(X/Z_p)$ where the image groups are the ordinary singular groups with integer coefficients. Similar isomorphisms exist in cohomology.

If Y is another such space with equivariant cell or handle decomposition W, then any map $f:(X, T_X) \to (Y, T_Y)$ can be pushed equivariantly to a skeleton preserving map and induces maps $(f/Z_p)_*: H_*(X/Z_p, R) \to H_*(Y/Z_p, R)$ and $(f/Z_p)^*: H^*(Y/Z_p, R) \to H^*(X/Z_p, R)$. Any short exact sequence of Λ modules induces Bocksteins and long exact homology and cohomology sequences. There are canonical isomorphisms between this theory and the local coefficient theory of [28]. See [35], §2 for more details.

2.7. LEMMA. *Given M^{2n} as in* 1.1 *and $\phi:(E(n), T_E) \to (M, T_M)$, then ϕ is not null homotopic. Furthermore,*

1) *If $n=2k+1$, then $H_n(E(n))=Z$ with generator z. There exists $\psi:(E(n), T_E) \to (M, T_M)$ with $\psi_*(z)=w \in H_n(M)$ if and only if $w-\phi_*(z)=x+(T_M)_*x+\cdots+(T_M)_*^{p-1}x$ for some $x \in H_n(M)$.*

2) *If $n=2k$, then $H_n(E(n))=\Delta$ with basis $y_1, \ldots, y_{p-1}$ (and $(T_E)_*$ acting on the basis as in* 2.5*). There exists $\psi:(E(n), T_E) \to (M, T_M)$ with $\psi_*(y_i)=w_i$ if and only if $w_i-\phi_*(y_i)=(T_M)_*^i x-(T_M)_*^{i-1} x$, $i=1, \ldots, p-1$, for some $x \in H_n(M)$.*

Proof. For the first part it suffices to show that $\phi^*: H^n(M) \to H^n(E(n))$ is nonzero. To that end, the map ϕ/Z_p together with the Bocksteins of the sequences $0 \to Z \to \Lambda \to \Delta \to 0$ and $0 \to \Delta \to \Lambda \to Z \to 0$ yield commutative diagrams

$$\begin{array}{ccc} H^i(M/Z_p, \Delta) & \to & H^{i+1}(M/Z_p, Z) \\ \downarrow & & \downarrow \\ H^i(L(n), \Delta) & \to & H^{i+1}(L(n), Z) \end{array} \qquad \begin{array}{ccc} H^i(M/Z_p, Z) & \to & H^{i+1}(M/Z_p, \Delta) \\ \downarrow & & \downarrow \\ H^i(L(n), Z) & \to & H^{i+1}(L(n), \Delta). \end{array}$$

The horizontal maps are isomorphisms for $0<i<n-1$ and $n<i<2n-1$ and injections if $i=n-1$, $2n-1$ since $H^i(M/Z_p, \Lambda)=H^i(L(n), \Lambda)=0$ if $i\neq 0$, n, $2n$. Then since $H^2(M/Z_p, Z) \to H^2(L(n), Z)=Z_p$ is an isomorphism (by universal coefficients, for example), we have that $H^n(M/Z_p, Z) \to H^n(L(n), Z)$ is non-zero if $n=2k$ and $H^n(M/Z_p, \Delta) \to H^n(L(n), \Delta)$ is non-zero if $n=2k+1$. From the above diagrams we also see that $H^{n+1}(M/Z_p, \Delta)=0$ if $n=2k$ and $H^{n+1}(M/Z_p, Z)=0$ if $n=2k+1$, and it now follows easily that $(\phi/Z_p)^*: H^n(M/Z_p, \Lambda) \to H^n(L(n), \Lambda)$ is non-zero, and thus so is $\phi^*: H^n(M) \to H^n(E(n))$.

The groups $H_n(E(n)) = H_n(L(n), \Lambda)$ can be computed using 1.4. For the rest, we may first assume by obstruction theory that $\phi, \psi: (E(n), T_E) \to (M, T_M)$ agree on $E(n-1)$. If σ^n is an n-cell of $E(n)$ (see 1.2) then x is represented by the mapping $S^n \to M$ with upper hemisphere mapped onto $\psi(\sigma^n)$ and lower hemisphere mapped to $\phi(\sigma^n)$. Then 1) and 2) follow by writing down generators for $H_n(E(n))$. □

2.8. COROLLARY. *There is a map $\phi: (E(n), T_E) \to (M, T_M)$ such that $\phi_*(H_n(E(n)))$ is a direct summand (over Z) of $H_n(M)$.*

Proof. Let $n=2k+1$, $\psi: (E(n), T_E) \to (M, T_M)$, and suppose $\psi_*(z)=ry$ where z generates $H_n(E(n))$ and $y \in H_n(M)$ generates a Z summand. For any m there exists $\psi': (E(n), T_E) \to (M, T_M)$ with $\psi'(z)=ry+p\cdot my \neq 0$ by 2.7 and thus $r \not\equiv 0 \pmod p$. It is easy to find an element $y' \in H_n(M)$ with $(T_E)_* y'=y'$ such that $\{y, y'\}$ freely generate a Z summand of $H_n(M)$. By 2.7 there exists $\phi: (E(n), T_E) \to (M, T_M)$ with $\phi_*(z)=ry+py'$, a generator of a Z summand since $r \not\equiv 0 \pmod p$.

We omit the more involved proof when $n=2k$ as it is not used in the sequel. □

2.9. When $n=2k$ the above result is unsatisfactory since we need to have untwisted induced models. To consider this problem we need some preliminary definitions. If M is as in 1.1, we say that M (or M/Z_p) has hyperbolic rank $\geqslant 1$ if and only if we have a decomposition of Λ modules $H_n(M)=\Lambda+\Lambda+H$ for some Λ module H such that

a) $\Lambda+\Lambda$ is orthogonal to H in the usual intersection form on H.

b) We have Λ generators x_1, x_2 for $\Lambda+\Lambda$ such that $(T_M)^i_* x_l \cdot (T_M)^j_* x_l=0$, $l=1, 2, i, j=0, \ldots, p-1$, and $(T_M)^i_* x_1 \cdot (T_M)^j_* x_2=\delta_{ij}$. Thus, in the terminology of [35], §5, the λ form on $H_n(M)$ splits off a hyperbolic plane. If it splits off d orthogonal hyperbolic planes, we say that M has hyperbolic rank $\geqslant d$.

If L is any $(n-1)$ connected $2n$ manifold, then $\pi_1(M/Z_p \# L)=Z_p$ and the universal covering space, written $M \#_{Z_p} L$, the equivariant connected sum of M and L, satisfies the conditions of 1.1. Note that $M \#_{Z_p} (S^n \times S^n)$ has hyperbolic rank $\geqslant 1$.

Finally, suppose $n=2k$ and $\phi: (E(n), T_E) \to (M, T_M)$. Then $\phi_*: H_n(E(n)) \to H_n(M)$ is non-zero by 2.7 and by a simple linear algebra argument is therefore an injection. Let $y, (T_M)_* y, \ldots, (T_M)^{p-2}_* y \in H_n(M)$ generate $\phi_*(H_n(E(n)))$ freely over Z. We say that $x \in H_n(M)$ is a complementary element for ϕ if and only if $(T_M)^i_* x \cdot (T_M)^j_* x=0$, all i, j, $x \cdot y=1$, and $x \cdot (T_M)^j_* y=0$, $1 \leqslant j \leqslant p-2$.

2.10. LEMMA. *Let M^{2n} be as in* 1.1 *with $n=2k$, and suppose $\phi: (E(n), T_E) \to (M, T_M)$ has complementary element x and $\phi_*(H_n(E(n)))$ is a Z summand of $H_n(M)$. Then there* *exists $\psi:(E(n), T_E)\to(M, T_M)$ with $\psi_*(H_n(E(n)))$ also a summand and with the induced standard model untwisted. In particular, any M with hyperbolic rank $\geqslant 1$ has such a map ψ.*

Proof. Let y, x be as above, and set $z=(T_M)_* x-x$. If $a_i=y\cdot(T_M)_*^i y$, then $a_i=a_{p-i}$ since intersection numbers are invariant under the Z_p action. In particular, from $(1+(T_M)_*+\cdots+(T_M)_*^{p-1})\,y=0$ it follows that $a_0=-2a_1-\cdots-2a_m$ is even (where $m=\frac{1}{2}(p-1)$). For $j=1,\ldots,m$ set $b_j=-j(\frac{1}{2}a_0)-(j-1)\,a_1-\cdots-2a_{j-2}-a_{j-1}$. By 2.7 there exists $\psi:(E(n), T_E)\to(M, T_M)$ with $\psi_*(H_n(E(n)))$ generated by $w=y+b_1(T_M)_* z+\cdots+b_m(T_M)_*^m z$, $(T_M)_* w,\ldots,(T_M)_*^{p-2} w$. A tedious computation verifies that $(T_M)_*^i w\cdot(T_M)_*^j w=0$, all i,j. But $\phi_*(H_n(E(n)))$ and the span of $\{(T_M)_*^i z\}$ are both isomorphic to Λ and have intersection of rank $<(p-1)$, and so by linear algebra they have intersection 0. Since ϕ_* maps to a summand, so does ψ_*. Finally, suppose M has hyperbolic rank $\geqslant 2$, so that $H_n(M)=\Lambda+\Lambda+H$ with generators x_1, x_2 for $\Lambda+\Lambda$ as in 2.9. Using 2.7 one can find $\phi':(E(n), T_E)\to(M, T_M)$ with $(\phi')_*(H_n(E(n)))$ generated by $y',\ldots,(T_M)_*^{p-2} y'\in H$. By 2.7 again there exists ϕ: $(E(n), T_E)\to(M, T_M)$ with generators $y=y'+x_1-(T_M)_* x_1,\ldots,(T_M)_*^{p-2} y$. Then $\operatorname{im}\phi_*$ is a Z summand and x_2 is a complimentary element. □

2.11. DECOMPOSITION THEOREM. *Let M^{2n} be as in* 1.1 *where M has hyperbolic rank $\geqslant 1$ if $n=2k$ and $n\neq 4$ if M is smooth. Then there exists $\phi:(E(n), T_E)\to(M, T_M)$ with induced standard model N untwisted and disjoint embeddings f_1, $f_2:(N, T_N)\to(\operatorname{int}N, T_N)$ homotopic to the identity such that the following hold. If $N_i=f_i(N)$* and *$L=M-\operatorname{int}(N_1\cup N_2)$ with boundary components $\partial_i=\partial N_i$, $i=1,2$, then $H_j(L/Z_p, \partial_1/Z_p)=0=H_j(L, \partial_1)$ unless $j=n$, $H_n(L/Z_p, \partial_1/Z_p)$ is free abelian, and $H_n(L, \partial_1)$ is a free Λ module. Furthermore, we have canonical isomorphisms*

1) $H_n(M)=H_n(N_1)+H_n(L, \partial_1)+H_n(N_2, \partial_2)$ *(as Λ modules)*

2) $H_n(M/Z_p)=H_n(N_1/Z_p)+H_n(L/Z_p, \partial_1/Z_p)+H_n(N_2/Z_p, \partial_2/Z_p)$.

Finally, this decomposition is uniquely determined by the homotopy class of ϕ ($n\neq 3$, 4 in the smooth category).

We call the above the standard decomposition of M induced by ϕ and call L the splitting space (for reasons to appear later) of the decomposition (or of ϕ).

Proof. By 2.1, 2.3, 2.8, and 2,10 we can choose an embedding $\phi:(E(n), T_E)\to(M, T_M)$ with $\phi_*(H_n(E(n)))$ a Z summand, the induced model untwisted, and disjoint embeddings $f_i:(N, T_N)\to(\operatorname{int}N, T_N)$. By 2.7 and linear algebra, $\phi_*:H_n(E(n))\to H_n(M)$ is an isomorphism onto a Z summand. Then from the exact homology sequence of (M, N_i) we obtain the split (over Z) sequence

a) $0\to H_n(N_i)\to H_n(M)\to H_n(M, N_i)\to 0$, $i=1,2$. In addition, $H_n(M, N_2)$ is free abelian from which it follows that $H^{n+1}(M, N_2)=0$ and so (by excision and the exact sequence of $(M-\operatorname{int}N_2, N_1)$) $H_j(L, \partial_1)=0$ unless $j=n$. Consequently, $H_j(L/Z_p, \partial_1/Z_p)=0$ unless $j=n$ and $H_n(L/Z_p, \partial_1/Z_p)$ is free abelian.

By [12], p. 247, L has an equivariant handle decomposition $\partial_1 \times I = V_{-1} \subseteq V_{n-1} \subseteq V_n = L$ (see 2.5). From the exact sequence $0 \to \ker\partial \to C_n(V) \to C_{n-1}(V) \to 0$ and the Λ module isomorphism $\phi_V : \ker\partial \to H_n(L, \partial_1)$ it follows that $H_n(L, \partial_1)$ is a stably free Λ module and thus is free by [32]. From the exact sequence of the triple $(M, M-\operatorname{int} N_2, N_1)$ we obtain the split (over Z) exact sequence

b) $0 \to H_n(M-\operatorname{int} N_2, N_1) \to H_n(M, N_1) \to H_n(M, M-\operatorname{int} N_2) \to 0$. By excision we obtain the isomorphism 1) once we check that the sequences a), b) split over Λ. But this follows by a routine algebraic argument. For example, suppose $n=2k+1$ in b). We thus have an exact sequence $0 \to F \to R \to \mathbf{Z} \to 0$ where R is some Λ module, F is a free Λ module with Z basis $\{x_j^i \mid i=1,\dots,p, j=1,\dots,m\}$ where $(T_M)_* x_j^i = x_j^{i+1}$, and $\mathbf{Z}$ has generator z. Suppose $x_j^i \to \tilde{x}_j^i \in R$ and choose $\tilde{z} \in R$ with $\tilde{z} \to z$. Then $\{\tilde{x}_j^i \, \tilde{z}\}$ generate R freely over Z, and $(T_M)_* \tilde{z} = \tilde{z} + \sum_{i,j} a_j^i \tilde{x}_j^i$ where $\sum_i a_j^i = 0$ for each j since $(T_M)_*^p \tilde{z} = \tilde{z}$. An equivariant splitting map $\mathbf{Z} \to R$ is defined by sending z to

$$\tilde{z} + \sum_j a_j^2 \tilde{x}_j^2 + \sum_j (a_j^2 + a_j^3)\, \tilde{x}_j^3 + \cdots + \sum_j (a_j^2 + \cdots + a_j^p)\, \tilde{x}_j^p .$$

In the quotient space, by separately checking the cases $n=2k+1$ and $n=2k$ we easily obtain exact sequences

$$\begin{aligned} &\mathrm{a}/Z_p) \quad 0 \to H_n(N_i/Z_p) \to H_n(M/Z_p) \to H_n(M/Z_p, N_i/Z_p) \to 0 \\ &\mathrm{b}/Z_p) \quad 0 \to H_n((M-\operatorname{int} N_2)/Z_p, N_1/Z_p) \to H_n(M/Z_p, N_1/Z_p) \\ &\qquad\qquad \to H_n(M/Z_p, (M-\operatorname{int} N_2)/Z_p) \to 0. \end{aligned}$$

When $n=2k+1$ both sequences split since the right hand groups are free abelian, and when $n=2k$ we obtain splitting by an application of the Hurewicz theorem to sequences b), b/Z_p).

Finally, uniqueness follows from 2.1, 2.3. □

2.12. Completion of the proofs of 2.1, 2.3. Suppose we have homotopic maps $f_1 \circ \phi = g_1$, $f_2 \circ \phi = g_2$, $f_1' \circ \phi' = g_1'$, $f_2' \circ \phi' = g_2' : (E(n), T_E) \to (M, T_M)$. By 2.11 and the Hurewicz theorem there is a free Λ module F such that $\pi_n(M/Z_p) = \pi_n(M)$ equals $\mathbf{Z} + \mathbf{Z} + F$ when $n=2k+1$ and equals $\Delta + \Delta + F$ when $n=2k$. Since $(g_1/Z_p)_\# = (g_1'/Z_p)_\#$ on $\pi_1(L(n))$, the only obstruction to a homotopy lies in $H^n(L(n), \mathbf{Z}+\mathbf{Z}+F)$ when $n=2k+1$ and in $H^n(L(n), \Delta+\Delta+F)$ when $n=2k$ (see [24]). From the proof of 2.7 the above groups are free abelian, and the first part of the result follows by checking (on the cochain level) that the pullback of the above obstructions to the obstructions for the homotopic maps $\pi_M \circ g_1$, $\pi_M \circ g_1'$, which lie in the ordinary singular groups $H^n(E(n), \pi_n(M/Z_p))$, is an injection.

From the proofs of 2.1, 2.3 we may thus assume that $\phi_1 = \phi_1'$, $f_1 = f_1'$, and let $K = M$-int $(f_1(N))$. From the proof of 2.11, the homology sequences of $(M, M-\operatorname{int}$

$(f_1(N)))$ and $(M-\operatorname{int}f_1(N), f_2(N))$, and the Hurewicz theorem it follows that $\pi_n(K)=\pi_n(K/Z_p)$ equals $Z+F$ or $\Delta+F$, and this group injects into $\pi_n(M)$. Consequently, g_2, g_2' are homotopic as maps into K and we apply the above argument to g_2, g_2' to complete the proof. □

3. Splitting

3.1. Throughout this section $(L^{2,n}\, \partial_1, \partial_2)$ will denote the splitting space of some standard decomposition of a manifold M induced by ϕ as in 2.11. If $v=\sum_{i=0}^{p-1} n_i\cdot(T_L)^i \in \Lambda = ZZ_p$, let $\bar{v}=\sum_{i=0}^{p-1} n_i(T_L)^{-i}$ and define $Q_k=\Lambda/I_k$ where $I_k=\{v-(-1)^k\,\bar{v} \mid v\in\Lambda\}$. Then given immersions $\alpha, \beta: S^n\to\operatorname{int}(L/Z_p)$, Wall ([35], §5) defines the intersection $\lambda(\alpha,\beta)\in\Lambda$ and self intersection $\mu(\alpha)\in Q_n$. If $v=\sum_{i=0}^{p-1} n_i(T_L)^i$ represents $\mu(\alpha)$ and $\lambda(\alpha,\alpha)=0$, then by [35], 5.2, iii), $v=n_0+\zeta-(-1)^n\,\zeta$ where $\zeta=\sum_{i=1}^{\frac{1}{2}(p-1)} n_i(T_L)^i$. It follows that α is homotopic (not regularly unless $n_0=0$) to an embedding. Because of this argument we ignore μ from now on, noting only that it can be recovered from λ, a fixed regular homotopy class of α, and some knowledge of the normal bundle of α.

3.2. LEMMA. *Any $x\in H_n(L,\partial_1)$ is represented by a map $\alpha_x: S^n\to\operatorname{int}L$, and the equation $\lambda(x,y)=\lambda(\pi_L\circ\alpha_x, \pi_L\circ\alpha_y)$ well defines a unimodular λ form on $H_n(L,\partial_1)$.*

Proof. If N is the induced standard model of ϕ as in 2.11 (so that $N_1, N_2\subseteq\operatorname{int}N$), let K denote the quotient space of M in which $M-\operatorname{int}N$ is collapsed to a point. Then by duality and excision we have $H_n(K-\operatorname{int}N_2)=H_n(N-\operatorname{int}N_2, \partial N)=H^n(N, N_2)=0$. From the exact sequence of $(K-\operatorname{int}N_2, N_1)$ it follows that $H_n(K-\operatorname{int}(N_1\cup N_2), \partial N_1)$ $=H_n(K-\operatorname{int}N_2, N_1)=0$ and thus the composite $H_{n-1}(\partial N_1)\to H_{n-1}(L)\to H_{n-1}$ $(K-\operatorname{int}(N_1\cup N_2))$ is an injection. Consequently $H_n(L,\partial_1)\to H_{n-1}(\partial_1)$ is the zero map and thus (since ∂_1 is $(n-2)$ connected) so is $\pi_n(L,\partial_1)\to\pi_{n-1}(\partial_1)$. But each $x\in H_n(L,\partial_1)$ can be represented by $f:(D^n, S^{n-1})\to(L,\partial_1)$ and the existence (non-unique) of α_x follows since $f|_{S^{n-1}}: S^{n-1}\to\partial_1$ is null homotopic. That $\lambda(\pi_L\circ\alpha_x, \pi_L\circ\alpha_y)$ depends only on x, y is a consequence of the definition of λ and the fact that ∂_1 bounds an untwisted model.

We check that λ is unimodular when $n=2k$, leaving the easier $n=2k+1$ case as an exercise. Using sequences a), b) of 2.11 we can choose elements $y_1, y_2, x_1, \ldots, x_m\in H_n(M)$ so that $y_1,\ldots,(T_M)_*^{p-2}\,y_1$ generate $\operatorname{im}(H_n(N_1)\to H_n(M))$, $y_2,\ldots, (T_M)_*^{p-2}$ generate a summand Δ mapping onto $H_n(M, M-\operatorname{int}N_2)=H_n(N_2,\partial_2)$, and $x_1,\ldots,x_m$ correspond to a Λ basis of $H_n(L,\partial_1)$. From the proof of 2.11 we can clearly choose y_1, y_2 so that $y_1\cdot y_2=1$ and $(T_M)_*^i\, y_1\cdot y_2=0$ if $1<i<p-1$. By the above argument $(T_M)_*^i\, x_j\cdot y_1$ $=0$. Let $y_2\cdot(T_M)_*^i\, x_j=m_{(p-1)j}$ so that $m_{0j}+\cdots+m_{(p-1)j}=0$. If $\alpha_i=-m_{0j}-\cdots-m_{ij}$ and $x_j'=x_j+\alpha_0\, y_1+\cdots+\alpha_{(p-2)}(T_M)_*^{p-2}\, y_1$, then $\{x_1',\ldots,x_m'\}=\underline{b}$ corresponds to a Λ basis of $H_n(L,\partial_1)$, and the Λ submodule of $H_n(M)$ generated by $\underline{b}$ is an orthogonal summand of $H_n(M)$ (equipped with the usual Z valued intersection form). By

Poincaré duality the induced map $H_n(L, \partial_1) \to \mathrm{Hom}_\Lambda(H_n(L, \partial_1), \Lambda)$ is an isomorphism and λ is unimodular ([35], §5). $\square$

3.3. Suppose Λ-rank $(H_n(L, \partial_1)) = r > 0$ and let $\underline{b}$ be some Λ basis. We can then define a torsion invariant in $Wh(Z_p)$ (see [20] or [12]), and a different choice of $\underline{b}$ changes the torsion by an element of image $(GL_r(\Lambda) \to Wh(Z_p))$. We say that $\underline{b}_L$ is a splitting basis of $H_n(L, \partial_1)$ if the associated torsion vanishes. But $Wh(Z_p) = Z + \cdots + Z$ ($(p-3)/2$ copies) generated by the units of Λ, so such bases always exist and are unique up to stabilization of $H_n(L, \partial_1)$ and elementary basis changes on the resulting model.

3.4. THEOREM. *For any splitting space L and splitting basis $\underline{b}_L \subseteq H_n(L, \partial_1)$, there is an equivariant handle decomposition $\partial_1 \times I = V_{-1} \subseteq V_n = L$ such that $\phi_V^{-1}(\underline{b}_L) = \underline{c}_n(V)$ for appropriate choices of lifts and orientations of the handles of V/Z_p (see 2.6). Given another splitting space L' and an orientation preserving (reversing) isomorphism $\psi: (\partial_1, T_L) \to (\partial_1', T_{L'})$, ψ extends to an orientation preserving (reversing) isomorphism $\tilde{\psi}: (L, T_L) \to (L', T_{L'})$ if and only if there exists an isomorphism $\gamma (= \tilde{\psi}_*): (H_n(L, \partial_1), (T_L)_*) \to (H_n(L', \partial_1'), (T_{L'})_*)$ satisfying:*

1) *$\lambda(\gamma(x), \gamma(y)) = \varepsilon\lambda(x, y)$ for $x, y \in H_n(L, \partial_1)$ where $\varepsilon = +1$* or -1 *depending on whether ψ preserves orientation or not.*

2) *$\gamma(\underline{b}_L)$ is a splitting basis of $H_n(L', \partial_1')$*

3) *If $x \in H_n(L, \partial_1)$, then there exist embeddings $\alpha_x: S^n \to \mathrm{int}\, L$ and $\alpha_{\gamma(x)}: S^n \to \mathrm{int}\, L'$ with isomorphic normal (disk or block) bundles. In particular, if $H_n(L, \partial_1) = 0$ then $L/Z_p = \partial_1/Z_p \times I$.*

Proof. First suppose $H_n(L, \partial_1) \neq 0$. The existence of an n-handle decomposition V/Z_p of L/Z_p with $\phi_V^{-1}(\underline{b}_L) = \underline{c}_n(V)$ is given in [12], p. 269.

If the equivariant isomorphism $\tilde{\psi}$ exists, then $(\tilde{\psi})_*$ is the desired map γ. Thus suppose γ satisfies the above properties and let $\underline{b}_L$ be some fixed splitting basis (clearly 2) is independent of the choice of $\underline{b}_L$). For each $b \in \underline{b}_L$ choose embeddings $\alpha_b: S^n \to \mathrm{int}\, L$ and $\alpha_{\gamma(b)}: S^n \to \mathrm{int}\, L'$ as in 3) so that $\pi_L \circ \alpha_b$ and $\pi_{L'} \circ \alpha_{\gamma(b)}$ are immersions. Then $\mu(\pi_L \circ \alpha_b) = \varepsilon\mu(\pi_{L'} \circ \alpha_{\gamma(b)})$ by 1), 3.1, and the fact that α_b, $\alpha_{\gamma(b)}$ are embeddings. Choose small disjoint disks $D_b^n \subseteq \partial L$ for each $b \in \underline{b}_L$ and connect α_b to $\partial(D_b^n)$ by a thin tube, thus defining an embedding $f_b: (D^n, S^{n-1}) \to (L, \partial_1)$. Define $f_{\gamma(b)}: (D^n, S^{n-1}) \to (L', \partial')$ by connecting a tube to $\psi(\partial(D_b^n))$. Then by 1) we can pipe the intersections and self intersections of the immersions $\pi_L \circ f_b$ and $\pi_{L'} \circ f_{\gamma(b)}$ across ∂_1/Z_p and ∂_1'/Z_p in such a way that the resulting embeddings $g_b: (D^n, S^{n-1}) \to (L/Z_p, \partial_1/Z_p)$ and $g_{\gamma}(b): (D^n, S^{n-1}) \to (L'/Z_p, \partial_1^1/Z_p)$ satisfy $\psi/Z_p \circ g_b|_{S^{n-1}} = g_{\gamma(b)}|_{S^{n-1}}$. Thus ψ/Z_p extends to an isomorphism

$$K/Z_p = \partial_1/Z_p \bigcup_{b \in \underline{b}_L} g_b(D^n) \to \partial_1'/Z_p \bigcup_{\gamma(b) \in \gamma(\underline{b}_L)} g_{\gamma(b)}(D^n) = K'/Z_p.$$

But by 3) this extends to an isomorphism of regular neighborhoods $J/Z_p \to J'/Z_p$ of K/Z_p and K'/Z_p in L/Z_p and L'/Z_p, respectively. Finally, by 2) and the s-cobordism theorem,

$L/Z_p - \mathrm{int}(J/Z_p) = \partial_2/Z_p \times I$ and $L'/Z_p - \mathrm{int}(J'/Z_p) = \partial_2'/Z_p \times I$.

If $H_n(L, \partial_1)=0$ and $n \geqslant 4$, by the uniqueness part of 2.1 and 2.3 we have a PL isomorphism $(L/Z_p, \partial_1/Z_p, \partial_2/Z_p) \to (L/Z_p, \partial_2/Z_p, \partial_1/Z_p)$. In particular we have torsion $\tau_0 = \tau(L/Z_p, \partial_1/Z_p) = \tau(L/Z_p, \partial_2/Z_p) = (-1)^{2n-1}\, \bar{\tau}_0$ (see [20]). But $Wh(Z_p)$ is free abelian and $\tau_0 = \bar{\tau}_0$ ([20], 6.7), so $\tau_0 = 0$ and $L/Z_p = \partial_1/Z_p \times I$. If $H_n(L, \partial_1)=0$ and $n=3$, we have the same conclusion by a Reidemeister torsion argument ([20], 12.8). $\square$

3.5. COROLLARY. *Let L be a splitting space of some PL manifold M as in* 2.11. *Then any isomorphism $\phi/Z_p : \partial_1/Z_p \to \partial_1/Z_p$ inducing the identity on $\pi_1(\partial_1/Z_p)$ extends to an isomorphism $\phi : L/Z_p \to (\partial_1/Z_p \times I) \# K$ (orientation preserving if ϕ/Z_p is) for some $(n-1)$ connected $2n$ manifold K (and thus L and $(\partial_1 \times I) \#_{Z_p} K$ are equivariantly isomorphic) if and only if there is a Λ basis $\underline{b}_L$ for $H_n(L, \partial_1)$ such that $\lambda(b_i, b_j) \in Z \subseteq \Lambda$ for all $b_i, b_j \in \underline{b}_L$. Then $\underline{b}_L$, called an integral basis, is necessarily a splitting basis.*

Proof. If $L/Z_p = (\partial_1/Z_p \times I) \# K$ and $b_1, \ldots, b_m$ is a Z basis for $H_n(K)$, it clearly defines an integral basis for $H_n(L, \partial_1)$. Conversely, if $\underline{b}_L = \{b_1, \ldots, b_m\}$ is integral, then the (Z valued) intersection from on $H_n(L, \partial_1)$ is the orthogonal sum of p copies of the form on the Z-span of $\underline{b}_L$, so the latter is unimodular. Choose embeddings $\alpha_{b_1} : S^n \to \to \mathrm{int} L$ representing each $b_i \in \underline{b}_L$. Attach n-handles $h_{b_1}, \ldots, h_{b_m}$ to D^{2n} so that if K^0 is the resulting handlebody, the Z basis $b_1', \ldots, b_m'$ of $H_n(K^0)$ corresponding to the handles satisfies $\lambda(b_i, b_j) = \varepsilon b_i' \cdot b_j'$ ($\varepsilon = \pm 1$ depending on orientation) and $\alpha_{b_i}, \alpha_{b_i'}$ have isomorphic normal bundles. As in [33], $\partial K^0 = S^{2n-1}$ and we let K denote the resulting closed $(n-1)$ connected PL manifold (suitably oriented).

Following the proof of 3.4 we may extend $\phi/Z_p : \partial_1/Z_p \to \partial_1/Z_p$ to an isomorphism $\hat{\phi}/Z_p$ of L/Z_p into $(\partial_1/Z_p \times I) \# K$. Then $(\partial_1/Z_p \times I) \# K - \mathrm{int}\,\phi/Z_p(L/Z_p)$ is an h-cobordism J/Z_p and $M/Z_p = N_1/Z_p \cup (J/Z_p \# K) \cup N_2/Z_p$. If we set $M'/Z_p = N_1/Z_p \cup J/Z_p \cup N_2/Z_p$, then M' satisfies the conditions of 1.1 and the above is a standard decomposition of M'/Z_p. By 3.4, J/Z_p is trivial so $\underline{b}_L$ is a splitting basis and L/Z_p and $(\partial_1 Z_p \times I) \# K$ are isomorphic. $\square$

3.6. *PL* SPLITTING THEOREM. *Let L be a splitting space* (2.11) *of M^{2n} with hyperbolic rank* $\geqslant 2$ (2.9) *and $n \neq 4, 8$. Then L has an integral basis and hence is equivariantly PL isomorphic to $(\partial_1 \times I) \#_{Z_p} K$ for some closed $(n-1)$ connected PL manifold K^{2n}.*

Proof. Let $L_{2n}(Z_p)$ denote the $2n$th Wall group of Z_p ([35], §5) and let $\underline{b}_L$ be some splitting basis. We want to construct a manifold L'/Z_p with lower boundary $\partial_1'/Z_p = \partial_1/Z_p$ representing an element in $L_{2n}(Z_p)$ as in [35], 5.8 such that the associated based λ form on $H_n(L', \partial_1')$ is isomorphic to $(H_n(L,\partial_1), \lambda, \underline{b}_L)$. Since λ is unimodular this can be done provided the intersections $b_i \cdot b_j$ are even (L' must have trivial stable normal bundle). But for $n \geqslant 3$ this always holds unless $n=4,8$ since the mod 2 reduction of $b_i \cdot b_j$ vanishes by [1]. Note next that ∂_2'/Z_p is isomorphic to ∂_1'/Z_p. For if $L''/Z_p = L/Z_p \cup (-L'/Z_p)$ glued along the common boundary ∂_1/Z_p, then $H_n(L''/Z_p, \partial_2/Z_p)$ with the induced λ form is a sum of hyperbolic planes by [35], 5.4. By the first part of the proof, $L''/Z_p = (\partial_2/Z_p \times I) \# K$ for suitable K. Thus $\partial_2'/Z_p \cong \partial_2/Z_p \cong \partial_1/Z_p \cong \partial_1'/Z_p$.

Let $L_{2n}^0(Z_p)$ denote the kernel of the canonical map $L_{2n}(Z_p) \to L_{2n}(0)$, so $L_{2n}(Z_p) = L_{2n}^0(Z_p) + L_{2n}(0)$. But $L_{2n}(0)$ acts trivially on the homotopy triangulations ([35], §10) of ∂_1'/Z_p by the first part of the theorem and $L_{2n}^0 = Z^{1/2(p-1)}$ acts freely ([25]). It follows that L'' represents an element of image $(L_{2n}(0) \to L_{2n}(Z_p))$ and thus (algebraically) so does $(H_n(L, \partial_1), \lambda, \underline{b}_L)$. Consequently $\underline{b}_L$ is stably (i.e., in $H_n(L \#_{Z_p} (S^n \times S^n \# \cdots \# S^n \times S^n), \partial_1))$ equivalent to an integral basis. But since Krull dimension $(\Lambda) = \dim\max(\Lambda) = 1$, it follows from [2] that we need stabilize at most $1 + \dim\max(\Lambda) = 2$ times. In particular, if M has hyperbolic rank $\geqslant 2$ then $H_n(L, \partial_1)$ splits two hyperbolic planes (regardless of the standard decomposition) and has an integral basis. Uniqueness of the resulting form (generated over Z by $\underline{b}_L$) follows easily from [21] if $n = 2k$ and by considering a symplectic basis if $n = 2k+1$. □

If L has hyperbolic rank $\geqslant 2$ and $n = 4$, 8 we can still find an integral basis for $H_n(L \#_{Z_p} M', \partial_1)$ where M' is the double of some non-stably trivial n disk bundle over S^n. If we try to extend 3.5 and 3.6 to the smooth category we have one problem – ∂K^0 might not be the standard sphere. This can be avoided by an assumption about N_1, N_2. We say that a standard decomposition is simple if $n = 2k$ or if $n = 2k+1$ and $N_1 = S^n \times D^n$ (N_1/Z_p might not be trivial). Although it is not the best possible result, the following is sufficient for our purposes.

3.7. LEMMA. *Let M^{2n} be as in* 1.1 *with $n+3<2p$ if M is PL. M^{4k+2} has a simple decomposition provided there is an embedding $\alpha: S^n \to M$ with trivial normal bundle which represents a generator x of a summand Λ of $H_n(M)$ such that $\lambda(x, x) = 0$. Two simple decompositions of M^{2n} have equivariantly isomorphic models if $n \not\equiv 0$, 4* (mod 8).

Proof. Note first that $\pi_i(BH_n)$, $i \leqslant n+1$, $H = 0$ or $\widetilde{PL}$, has no p-torsion if $n+3<2p$ (see 4.2). If $n = 2k+1$, using Poincaré duality and 2.7 one can easily find $\phi: (E(n), T_E) \to (M, T_M)$ so that $\lambda(\phi_*(z), x) = 0$ (where z generates $H_n(E(n))$) and $\phi_*(z)$ lies outside the span of x. We can choose an embedding $\psi_*: (E(n), T_E) \to (M, T_M)$ with trivial normal bundle and $\psi_*(z) = (1 + jp) \cdot \phi_*(z)$. Then if $\zeta: (E(n), T_E) \to (M, T_M)$ satisfies $\zeta_*(z) = \psi_*(z) + x + \cdots + (T_M)^{p-1} x$, the resulting decomposition is simple.

Next suppose $n = 2k+1$ and $\phi_1/Z_p, \phi_2/Z_p: L(n) \to M/Z_p$ induce simple decompositions with maps $\gamma_1, \gamma_2: L(n) \to BH_n$ ($H = 0$ or $\widetilde{PL}$) classifying the resulting models. By [12], 10.3 (and [6], 6.1 in the smooth case) we may assume $\phi_1/Z_p|_{\overline{L(n-1)}} =$

$=\phi_2/Z_p|\overline{L(n-1)}$ and thus $\gamma_1|_{L(n-1)}$ and $\gamma_2|_{L(n-1)}$ are homotopic. But $\gamma_1\circ\pi_E$, $\gamma_2\circ\pi_E$ are homotopic since the decompositions are simple, so γ_1, γ_2 are homotopic since $\pi_n(BH_n)$ has no p-torsion. If $n=2k$ and ϕ_1/Z_p, $\phi_2/Z_p: L(n)\to M/Z_p$ are embeddings with classifying maps $\gamma_1, \gamma_2: L(n)\to BH_{n-1}$, one shows as before that $\gamma_1|_{L(n-2)}$ and $\gamma_2|_{L(n-2)}$ are homotopic. This improves to a homotopy on $L(n-1)$ since $\pi_{n-1}(BH_{n-1})$ has no p-torsion, and we have $\gamma_1\simeq\gamma_2$ on all $L(n)$ if $n\equiv 2, 6 \pmod 8$ since in these cases $\pi_n(BH_{n-1})$ has no p-torsion or fiee part (see 4.2). □

3.8. SMOOTH SPLITTING THEOREM. *Let L be the splitting space of a simple decomposition of a smooth manifold M^{2n}, $n+1<p$, and let $\underline{b}_L\subseteq H_n(L, \partial_1)$ be an integral basis. Then any diffeomorphism $\phi/Z_p:\partial_1/Z_p\to\partial_1/Z_p$ which induces the identity on $\pi_1(\partial_1/Z_p)$ extends to a diffeomorphism (orientation preserving if ϕ/Z_p is) $\hat{\phi}/Z_p: L/Z_p \to(\partial_1/Z_p\times I)\# K$ for some smooth closed $(n-1)$ connected manifold K^{2n}. Any smooth M with hyperbolic rank $\geqslant 2$ has a simple decomposition and an associated integral basis.*

Proof. Let $K^0\subseteq \operatorname{int} L/Z_p$ be as in 3.5. Taking a smooth regular neighborhood [9] we may assume K^0 is smooth and set $J^0/Z_p=L/Z_p-\operatorname{int}K^0$. In the total space, form $\hat{J}$ from J^0 by removing the interiors of $(p-1)$ small tubes connecting the boundary components. If $\hat{M}=N_1\cup\hat{J}\cup N_2$, then since $\partial\hat{M}=p\cdot\partial K^0$ in θ_{2n-1} and θ_{2n-1} has no p-torsion if $n+1<p$ (see 4.2) it suffices to show that $\partial\hat{M}=S^{2n-1}$.

Suppose $n=2k$. Since N_1 is untwisted and $(1+\cdots+(T_{N_1})_*^{p-1})$ annihilates $H_n(N_1)$, it follows that N_1 is parallelizable. Let $\gamma: S^n\to\operatorname{int}\hat{M}$ be an embedding which intersects N_1 in the right hand disk of one of the n-handles. Using a Λ generator of $H_n(N_1)$ as complementary element and following the proof of 2.10 we can replace γ by a map $\gamma': S^n\to\operatorname{int}\hat{M}$ so that $\pi_M\circ\gamma'$ is an embedding with $H_n(N_1)$ and image (γ_*) generating $H_n(\hat{M})$. But $(1+(T_M)_*+\cdots+(T_M)_*^{p-1})$ annihilates image $(\gamma')_*$. It follows that γ' has trivial normal bundle and $\hat{M}$ is parallelizable. But clearly index $(H_n(\hat{M}))=0$, so $\hat{M}$ is cobordant (rel $\partial\hat{M}$) to a disk and $\partial\hat{M}=S^{2n-1}$.

If $n=2k+1$, then $\hat{M}$ is a handlebody with two n-handles with linking number ± 1 and one handle attached with trivial thickening (since the decomposition is simple). But by attaching $S^n\times D^n$ to itself by a diffeomorphism of $S^n\times S^{n-1}$ sending $(u, v)\to(\alpha(v)\cdot u, v)$ where $\alpha\in\pi_{n-1}(SO_{n+1})$ one can obtain all possible invariants for such a handlebody $\hat{M}$ [33] in a closed manifold, so $\partial\hat{M}=S^{2n-1}$.

Finally, if $n=2k+1$ and $H\subseteq H_n(M)$ is the orthogonal sum of two hyperbolic planes, then by immersion theory [8] and the fact that $\pi_{n-1}(SO_n)$ is Z_2 or Z_2+Z_2 one can find $x\in H$ as in 3.7. The last statement follows from 3.6, 3.7. □

4. Gluing and the Classification Problem

So far we have considered conditions under which M has a standard decomposition $M=N_1\bigcup_{f_1} L\bigcup_{f_2} N_2$ where $f_i:(\partial N_i, T_{N_i})\to(\partial_i, T_L)$ is an isomorphism, $i=1, 2$.

We now study the effect of the gluings f_i. If $M'=N_1 \bigcup_{g_1} L \bigcup_{g_2} N_2$, then by 3.4 the identity $N_1 \to N_1$ extends to an isomorphism $\phi:(N_1 \bigcup_{f_1} L, T_M) \to (N_1 \bigcup_{g_1} L, T_{M'})$, and thus (M, T_M) and $(M', T_{M'})$ are isomorphic provided $g_2^{-1} \circ \phi \circ f_2:(\partial N_2, T_{N_2}) \to (\partial N_2, T_{N_2})$ extends to an equivariant isomorphism of N_2. We study the latter problem.

4.1. First recall some standard definitions and results from surgery theory. Given an oriented compact PL (respectively, smooth) manifold K with boundary ∂K (possibly empty), the set $HT(K, \partial K)$ of homotopy triangulations of K (rel∂K) (respectively, the set $HS(K, \partial K)$ of homotopy smoothings) consists of pairs (N, ϕ) where $(N, \partial N)$ is a compact oriented PL (respectively, smooth) manifold and $\phi:(N, \partial N) \to (K, \partial K)$ is a simple homotopy equivalence such that $\phi \mid \partial N$ is an isomorphism onto ∂K. Two pairs (N_1, ϕ_1) and (N_2, ϕ_2) are equivalent if and only if there is an isomorphism $\psi:N_1 \to N_2$ so that $\phi_2\psi$ and ϕ_1 are homotopic (rel∂N_1). The sets $HT(K, \partial K)$ and $HS(K, \partial K)$ have as base point the class of $(K, 1)$. Let $L_j(\pi)$ denote the jth Wall group of π. Since $L_{2j+1}(Z_p)=0$ ([15] and [3]) the surgery exact sequence ([35], §10) for a PL (smooth) standard model N^{2n}/Z_p becomes

$$0 \to HT(N/Z_p, \partial N/Z_p) \xrightarrow{\zeta} [N/Z_p, \partial N/Z_p; G/PL, *] \xrightarrow{\zeta} L_{2n}(Z_p)$$
$$(0 \to HS(N/Z_p, \partial N/Z_p) \xrightarrow{\zeta} [N/Z_p, \partial N/Z_p; G/O, *] \xrightarrow{\zeta} L_{2n}(Z_p)).$$

In particular, ζ is an injection ([35], 10.5).

4.2. For $g \geqslant 3$ there is an exact braid of groups ([16], [26])

$$\begin{tikzcd}
\pi_i(O_q) \arrow[dr] \arrow[rr, bend left] & & \pi_i(G_q) \arrow[dr] \arrow[rr, bend left] & & \pi_i(G/PL) \\
& \pi_i(\widetilde{PL}_q) \arrow[ur] \arrow[dr] & & \pi_i(G_q, O_q) \arrow[ur, "\alpha_q"'] \arrow[dr] & \\
\pi_{i+1}(G/PL) \arrow[ur] \arrow[rr, bend right] & & \Gamma_i^q \arrow[ur] \arrow[rr, bend right] & & \pi_{i-1}(O_q).
\end{tikzcd}$$

The groups $\pi_i(G/PL)$ form the period four sequence Z, O, Z_2, O for $i \equiv 0, 1, 2, 3$ (mod4) [29]. Since $G_q=\{f:S^{q-1} \to S^{q-1} \mid \deg f=\pm 1\}$ with the compact open topology and $\pi_i(G_q)=\pi_{i+q-1}(S^{q-1})$ has no p-torsion for $i+3<2p$ ([27]), neither does $\pi_{i+1}(B\widetilde{PL}_q)=\pi_i(\widetilde{PL}_q)$. For $i \leqslant q+4$ the groups $\pi_i(O_q)$ are well known ([4], [13]) and in particular have no odd torsion. If $q \gg i$ we have the stable braid of Kervaire and Milnor in which $\pi_i(G_q, O_q)=\pi_i(G/O)$ and $\Gamma_i^q=\Gamma_i=\theta_i$ is the group of homotopy i-spheres. Thus θ_i, $\pi_i(G/O)$ have no p-torsion if $i+3<2p$.

4.3. LEMMA. *If N/Z_p is a standard model and $f:((N, \partial N), T_N) \to ((N, \partial N), T_N)$ is an isomorphism on ∂N, then f/Z_p is a simple homotopy equivalence of the pair $(N/Z_p, \partial N/Z_p)$.*

Proof. By the relative Whitehead theorem ([19], IV. 3.3) and standard torsion arguments ([20]) it suffices to show that $f/Z_p: N/Z_p \to N/Z_p$ is a simple equivalence. By equivariance, $(f/Z_p)_\#$ is the identity on $\pi_1(N/Z_p)$. Since $f/\partial N$ is an isomorphism, f is a degree ± 1 map of $(N, \partial N)$ and so, by a standard duality argument ([5], I. 2.5), f_* is a split surjection and thus a bijection of the finitely generated groups $H_i(N)$. It follows that $(f/Z_p)_\#$ is an isomorphism and f/Z_p is a homotopy equivalence. If $n=2k+1$ we are finished since f/Z_p must be homotopic to the identity (not rel $\partial N/Z_p$) by 2.7 and the uniqueness part of 2.1.

If $n=2k$, let K/Z_p be a regular neighborhood of $L(n-1) \subseteq N/Z_p$. By [31], Theorem A, there is a $2n$ submanifold K'/Z_p containing $L(n-1)$ such that, after a deformation mod $\partial N/Z_p$, f/Z_p induces a homotopy equivalence of K/Z_p into K'/Z_p and a PL isomorphism of the complements of their interiors. We may assume $K/Z_p \subseteq \operatorname{int} K'/Z_p$, so $J/Z_p = K'/Z_p - \operatorname{int} K/Z_p$ is an h-cobordism from $\partial K/Z_p$ to itself ($f/Z_p|_{\partial K/Z_p}$ is an isomorphism). By a Reidemeister torsion argument ([20], 12.8) $J/Z_p = \partial K/Z_p \times I$, and it follows as in the $n=2k+1$ case that f/Z_p is a simple equivalence.

Our main application of surgery to the gluing problem is the following.

4.4. THEOREM. *For any PL standard model N/Z_p, $HT(N/Z_p, \partial N/Z_p)$ consists of the base point only. If N/Z_p is a smooth model, $HS(N/Z_p, \partial N/Z_p)$ is finite. If in addition $n<p-1$ and s_n denotes the order of the torsion subgroup of $\pi_n(G/O)$, then $HS(N/Z_p, \partial N/Z_p)$ has at most $s_n \cdot s_{2n}$ elements when $n=2k+1$ and at most s_{2n} elements when $n=2k$.*

Proof. If $x \in HT(N/Z_p, \partial N/Z_p)$, the only possible non-zero obstruction to a null homotopy for $\zeta(x): (N/Z_p, \partial N/Z_p) \to (G/PL, *)$ lies in $H^{2n}(N/Z_p, \partial N/Z_p; \pi_{2n}(G/PL)) = \pi_{2n}(G/PL) = Z$ or Z_2 (when $n=2k$ or $n=2k+1$, respectively). But $\theta: [S^{2n}, G/PL] \to L_{2n}(O)$ is a bijection [29], and if follows that this top obstruction vanishes since $\theta(\zeta(x))$ must vanish by exactness. Hence x is the base point.

In the smooth case, note first that $\pi_i(G/O)$ is finite unless $i=4j$ and $\pi_{4j}(G/O)=Z+$(finite group) (4.2). Then $HS(N/Z_p, \partial N/Z_p)$ is finite when $n=2k+1$ since the homotopy obstruction groups $H^i(N/Z_p, \partial N/Z_p; \pi_i(G/O))$ are all finite and ζ is an injection. If $n=2k$, then $H^{2n}(N/Z_p, \partial N/Z_p; \pi_{2n}(G/O)) = \pi_{2n}(G/O)$ has a component Z. Since the canonical map $\alpha: G/O \to G/PL$ satisfies $\alpha_\# = \lim \alpha_q$, by 4.2 $\alpha_\#$ is an isomorphism modulo torsion. But for any $x \in HS(N/Z_p, \partial N/Z_p)$, $\alpha_\# \zeta(x) \in [N/Z_p, \partial N/Z_p; G/PL, *]$ is the normal invariant for the homotopy triangulation induced by x and so must vanish. Thus given $x, y \in HS(N/Z_p, \partial N/Z_p)$ such that the restrictions of $\zeta(x)$, $\zeta(y)$ to $N/Z_p - D^{2n}$ are homotopic (rel $\partial N/Z_p$), the remaining homotopy obstruction lies in the torsion subgroup of $\pi_{2n}(G/O)$.

Finally, if $n<p-1$ then $\pi_i(G/O)$ has no p-torsion for $i \leqslant 2n-1$. Thus all but the top obstruction group vanishes when $n=2k$, and all but the nth and $2n$th vanish when $n=2k+1$. $\square$

4.5. In any standard decomposition $M = N_1 \cup_{f_1} L \cup_{f_2} N_2$, the canonical isomorphism $N_1 \to N_2$ of 2.1 and 2.3 is orientation preserving and the gluings $f_i : \partial N_i \to \partial_i$ reverse the "inward normal" orientations. If N_1 is smooth it is an odd dimensional disk bundle over a manifold (it is untwisted) and multiplying the fiber by -1 yields an orientation reversing isomorphism $(N_1, T_{N_1}) \to (N_1, T_{N_1})$.

4.6. THEOREM. (Finiteness of Gluings). *For fixed choice of models $(N_1, T_{N_1}) \cong (N_2, T_{N_2})$ and splitting space (L, T_L), up to equivariant isomorphism (orientation preserving for fixed orientations of N_1, L) there are at most finitely many manifolds M with standard decomposition $M = N_1 \bigcup_{f_1} L \bigcup_{f_2} N_2$. For suitable p, a specific bound b for the number of such manifolds is given by:*

a) *$b = \text{order } (\pi_{2n-1}(S^n))$ when M is PL, $n = 2k+1 < 2p-1$.*

b) *$b = s_n s_{2n}$ order $(\pi_{2n-1}(S^n))$ when M is smooth, $n = 2k+1 < p-1$.*

c) *$b = 2$ when M is PL, $n = 2k < 2p-2$.*

d) *$b = 2s_{2n}$ when M is smooth, $n = 2k < p-1$.*

Proof. We prove only the second part. The first follows by a similar argument since the relevant obstruction groups are finite. If $M' = N_1 \bigcup_{g_1} L \bigcup_{g_2} N_2)$ is another gluing, by 3.4 there is an (orientation preserving) isomorphism $\phi : (N_1 \bigcup_{f_1} L, T_M) \to (N_1 \bigcup_{g_1} L, T_{M'})$. By 4.3, 4.4 this extends to a *PL* isomorphism $\phi : (M, T_M) \to (M', T'_M)$ if and only if $(g_2^{-1} \circ \phi \circ f_2)/Z_p : \partial N_2/Z_p \to \partial N_2/Z_p$ extends to a map $N_2/Z_p \to N_2/Z_p$. We check that if N/Z_p is an untwisted *PL* model, for suitable p there are at most b homotopy classes (in N/Z_p) of maps $f/Z_p : \partial N/Z_p \to \partial N/Z_p$ covered by an equivariant isomorphism. The smooth case follows from this and 4.4.

If $n = 2k+1$, N/Z_p is a block bundle over $L(n)$ and there is an embedding ϕ/Z_p: $L(n) \to \partial N/Z_p$ homotopic to the O section. By 2.1, two isomorphisms $f, f' : (\partial N, T_N) \to (\partial N, T_N)$ can be equivariantly deformed in N so that $f|_{\phi(E(n))} = f'|_{\phi(E(n))}$. The homotopy obstructions $(\text{rel}\,\phi/Z_p(L(n)))$ for $f/Z_p, f'/Z_p$ lie in $H^i(\partial N/Z_p, \phi/Z_p(L(n)), \pi_i(N/Z_p)) = H_{2n-i-1}(L(n), \pi_i(S^n))$, $i > 1$. For $i \leqslant 2n-2 < n+2p-3$, $\pi_i(S^n)$ has no p-torsion so only the top obstruction group $\pi_{2n-1}(S^n)$ is non-zero.

If $n = 2k$, following the proof of 3.8 we can find $\gamma' : S^n \to N \bigcup_f N = N_f$ so that $S^n \xrightarrow{\gamma'} N_f \to N_f/Z_p$ is an embedding with trivial normal bundle, and $\text{im}(\gamma'_*)$ and $H_n(N)$ generate $H_n(N_f)$. (We use the fact that $\pi_{n-1}(\widetilde{PL}_n)$ has no p-torsion. For the general finiteness result one compares separately gluings which, for suitable γ', define the same element of the p-torsion of $\pi_{n-1}(\widetilde{PL}_n)$.). If K/Z_p is the normal bundle of $L(n-1)$ in N/Z_p, it follows that f/Z_p extends to an isomorphism of $M/Z_p - \text{int}(K/Z_p)$ to itself. Following the proof for $n = 2k+1$ we compare the homotopy classes for f/Z_p: $\partial K/Z_p \to \partial K/Z_p \subseteq K/Z_p$ $(\text{rel}\,\phi/Z_p(L(n-1)))$, the relevant obstruction groups now being $H_{2n-i-1}(L(n-1), \pi_i(S^{n-1}))$. The first non-vanishing group (when $i = n$) is Z_2. But if the $i = n$ homotopy obstruction for two maps vanishes, their restrictions to $\partial K/Z_p - D^{2n-1}$ are homotopic in K/Z_p since $n < 2p-2$. Since K is contained in a disk $D^{2n} \subseteq M$, the top obstruction must also vanish.

EXAMPLE. Suppose $n \equiv 6 \pmod 8$, $n<p-1$, and $f:(\partial N, T_N) \to (\partial N, T_N)$ is a diffeomorphism for some untwisted smooth model N. I claim that f/Z_p extends across N/Z_p. As in 4.6 this reduces to extending an induced diffeomorphism $f/Z_p:\partial K/Z_p \to \partial K/Z_p$. The first extension obstruction lies in $H^{n+1}(K/Z_p, \partial K/Z_p', \pi_n(K/Z_p))=Z_2$ and vanishes if and only if the corresponding obstruction for a lift $f:S^{n-1}\times S^n \to S^{n-1}\times S^n$ vanishes. According to [17], the diffeomorphism f is concordant to a composite of maps of the form

i) $(x,y)\to(\beta(y)\cdot x, y)$ for $\beta:S^n\to O_n$

ii) $(x,y)\to(x, \gamma(x)\cdot y)$ for $\gamma:S^{n-1}\to O_{n+1}$

iii) g such that $g=1$ outside some disk.

Maps of types ii), iii) clearly extend across $S^{n-1}\times D^{n+1}$, and type i) has no Z_2 obstruction since the canonical map $\pi_n(O_n)\to\pi_n(S^{n-1})$ is trivial when $n\equiv 6 \pmod 8$ ([13]). Thus the Z_2 obstruction for f/Z_p vanishes. Following 4.6, $f/Z_p|_{\partial K/Z_p}$ extends to a map $K/Z_p \to N/Z_p$.

Suppose M^{2n} as in 1.1 is smooth, has hyperbolic rank $\geqslant 2$, $n\equiv 6 \pmod 8$, and $n<p-1$. By 3.7 any two untwisted models in M are isomorphic, determined by a unique homotopy class $\varrho(M)\in[L(n), BO_{n-1}]$. If $M=\varepsilon(N_1 \cup_{f_1} L \cup_{f_2} N_2)$, $\varepsilon=\pm 1$, is some standard decomposition, by 3.8 and 4.5 there is an orientation preserving diffeomorphism $\phi_\varepsilon:(\varepsilon(N_1\cup_{f_1} L), T_{N_1})\to(N_1\cup_{f_1}((\partial_1\times I)\#_{Z_p}K_\varepsilon), T_{N_1})$ for some oriented smooth closed $(n-1)$ connected K_ε^{2n} with uniquely determined intersection form. Since $\pi_{n-1}(SO)=0$, the intersection form determines the normal bundles, so K is determined up to the action of θ_{2n} by M. The orthogonal complement of $H_n(L,\partial_1)$ in $H_n(M)$ has signature O since L is untwisted, and the form on $H_n(M)$ is even (see the proof of 3.6). By 4.3, 4.4, and the work above, a change in gluings changes M/Z_p by addition of a homotopy sphere (the only obstruction to a trivial normal invariant in $\pi_{2n}(G/O)$ maps to O in $\pi_{2n}(G/PL)$). Note also that θ_{2n} has no p-torsion since $n<p-1$. Applying [21] or [33] we have the following.

4.7. THEOREM. If $n\equiv 6 \pmod 8$ *and* $n<p-1$, *then the smooth manifolds* M^{2n} *as in* 1.1 *with hyperbolic rank* $\geqslant 2$ *are classified up to orientation preserving equivariant diffeomorphism and the action of* $p\cdot\theta_{2n}=\theta_{2n}$ *by the model invariant* $\varrho(M)\in[L(n), BO_{n-1}]$, *the rank of* $H_n(M)$, *and the signature* $\sigma(M)\in 8pZ$. *All values of* $\varrho(M)$, $\sigma(M)$ *occur independently.*

EXAMPLE. Suppose $n\equiv 3, 5, 7 \pmod 8$, $n<p-1$, and $f:(\partial N, T_N)\to(\partial N, T_N)$ is a diffeomorphism for some simple smooth model N. I claim that f/Z_p extends to $f/Z_p:N/Z_p\to N/Z_p$ in such a way that the first obstruction to trivial normal invariant (in $\pi_n(G/O)$ - see 4.4) vanishes. Let $g:(D^n, S^{n-1})\to(N/Z_p, \partial N/Z_p)$ be an embedding such that a lift $\tilde g$ generates $H_n(N,\partial N)$. Since $\pi_{n-1}(SO)=0$, by immersion theory ([8]) and the fact that N is simple there is an embedding $h:S^n\to N\cup_f N=N_f$ with

trivial normal bundle so that the lower hemisphere maps to the first copy of N via $\tilde{g}$ and the upper hemisphere maps to the second copy of N. If h represents $x \in H_n(N_f)$, then $(T_{N_f})_* x = x$ and $x \cdot x = 0$. It follows as in 3.1 that $\mu(\pi_{N_f} \circ h) = 0$ so h can be pushed by an ambient isotopy of the second copy of N leaving ∂N fixed so that $\pi_{N_f} \circ h$ is an embedding. Thus f/Z_p extends to an embedding of the normal bundle J of image (g). By obstruction theory this extends to a map $N/Z_p - \operatorname{int}(D^{2n}) \to N/Z_p$, and f/Z_p extends if $\alpha = f/Z_p|_{\partial D^{2n}}: S^{2n-1} \to N/Z_p$ is null homotopic. Connecting the disks $D^{2n} \subseteq N$ which cover D^{2n} by $(p-1)$ small tubes we obtain a disk D_0^{2n} such that $\partial D_0^{2n} \xrightarrow{f} N \xrightarrow{\pi_N} N/Z_p$ represents $p \cdot \alpha$. Since $\pi_{2n-1}(N/Z_p) = \pi_{2n-1}(S^n)$ has no p-torsion, f/Z_p extends provided $f|_{\partial D_0^{2n}}$ extends. If $\tilde{J}$ is a regular neighborhood of $\operatorname{im}(\tilde{g})$ covering J, then any two extensions of $f|_{\partial N \cup \tilde{J}}$ to $N - D_0^{2n}$ are homotopic (rel ∂N), so f/Z_p extends if $f|_{\partial N \cup \tilde{J}}$ extends to N. But this is trivially true since closure $(N - \tilde{J})$ is a disk. Finally, note that we thus have an extension f/Z_p covered by a map f which is homotopic (rel ∂N) to a map $\hat{f}$ which embeds $\tilde{J}$ and is a homotopy equivalence of the complements. It follows that the first normal invariant obstruction in $\pi_n(G/O)$ vanishes (see also [29], [30]).

If M^{2n} as in 1.1 has hyperbolic rank $\geqslant 2$, then by 3.7, 3.8 it has a simple decomposition $M = \varepsilon(N_1 \bigcup_{f_1} L \bigcup_{f_2} N_2)$, $\varepsilon = \pm 1$, where the models N_1, N_2 are determined by a unique homotopy class $\varrho(M) \in \ker\ (\pi_E^{\#}: [L(n), BO_n] \to \pi_n(BO_n) = Z_2)$. By 3.8, 4.5, there is an orientation preserving diffeomorphism $\phi_\varepsilon: (\varepsilon(N_1 \bigcup_{f_1} L), T_{N_1}) \to (N_1 \bigcup_{f_1} ((\partial_1 \times I) \#_{Z_p} K_\varepsilon, T_{N_1})$ for $(n-1)$ connected oriented K_ε^{2n} with unique intersection form. Since $n \equiv 3, 5, 7, (\operatorname{mod} 8)$, K is uniquely determined (up to the action of θ_{2n}) by the rank of $H_n(K)$ and the Arf invariant $\sigma(K) \in Z_2$ (the rank classifies when $n = 3, 7$ – see [33]). Since the decomposition is simple, $\sigma(M) = p \cdot \sigma(K) = \sigma(K)$. From the first part of the argument, a change in gluings alters M/Z_p by addition of a homotopy sphere.

4.8. THEOREM. *If $n \equiv 3, 5, 7$ (mod 8) and $n < p-1$, the smooth manifolds M^{2n} as in 1.1 of hyperbolic rank $\geqslant 2$ are determined up to orientation preserving equivariant diffeomorphism and the action of $p \cdot \theta_{2n} = \theta_{2n}$ by the model invariant $\varrho(M) \in \ker((\pi_E)^{\#} [L(n), BO_n] \to \pi_n(BO_n))$, the rank of $H_n(M)$, and the Arf invariant $\sigma(M) \in Z_2$. (If $n = 3, 7$, the rank and $\varrho(M)$ classify.) All values of $\varrho(M)$, $\sigma(M)$ occur independently.*

Remark. If Z rank $(H_n(M)) = 2$, $n < p-1$, and $n \equiv 3, 5, 7$, then by immersion theory it has a simple decomposition. An argument similar to the above shows that M is classified up to the action of θ_{2n} by $\varrho(M)$. For example, there is only one such manifold if $n = 3$ $(M/Z_p = L(3) \times S^3)$ and there are only $2p$ such manifolds when $n = 7$ $(\varrho(M) \in Z_p$ and $\theta_{14} = Z_2$ acts freely – see [33]).

In the above examples the action of θ_{2n} can be determined exactly.

4.9. THEOREM. *If $n \equiv 3, 5, 6, 7$ (mod 8) then θ_{2n} acts freely on the orientation preserving diffeomorphism classes of $(n-1)$ connected smooth $2n$ manifolds M.*

Proof. Suppose $\Sigma \in \theta_{2n}$ is such that there exists an orientation preserving diffeomorphism $\phi: M \# \Sigma \to M$. Let J denote the obvious cobordism with boundary components M, Σ, and $-M \# \Sigma$, and form J_ϕ from J by gluing M to $M \# \Sigma$ using ϕ. Since $n \equiv 3$, 5, 6, 7 (mod 8), J_ϕ is n-parallizable and thus cobordant (rel $\partial J_\phi = \Sigma$) to a disk by [38]. Thus $\Sigma = S^{2n}$. (See also [39]). □

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Received June 21, 1973.